Group structures in a class of control systems

Andrew D. Lewis*

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Abstract

We investigate two classes of control systems, one of Brockett and one of Murray and Sastry. We are able to show that these two systems may be formulated in the language of principle fibre bundles. Controllability of these systems is then shown to be a direct consequence of the Ambrose-Singer holonomy theorem.

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1. Control systems on principal fibre bundles

In this section we will examine control systems where the control vector fields are invariant under the action of a group on a principal fibre bundle. Our notions regarding principal fibre bundles and notation are largely from [Kobayashi and Nomizu 1963].

1.1. *G*-invariant distributions. Let $P(B,G)$ be a principal fibre bundle. Thus P is an n-dimensional differentiable manifold which is acted upon freely by an r-dimensional Lie group G and $B = P/G$ is the manifold of orbits. We denote the canonical projection by $\pi\colon P \to B$. We will suppose that the action of G on P is from the right and we will denote it by R_g for each $g \in G$. We will assume here that P is connected which implies that B is connected. The subbundle $VP = \ker(T\pi)$ is the ***vertical subbundle*** and at each $p \in P$, V_pP is simply the subspace of T_pP spanned by the infinitesimal generators of the action of G on P.

We wish to introduce distributions which are compatible with the group structure, and which have no vertical component. A connection on $P(B,G)$ is an example of such a distribution.

1.1 Definition: Let $P(B,G)$ be a principal fibre bundle. A distribution $\mathscr{D}$ on P is said to be ***G-invariant*** if $TR_g(\mathscr{D}(p)) = \mathscr{D}(R_g(p))$ for each $p \in P$ and $g \in G$, and if $V_pP \cap \mathscr{D}(p) = \{0\}$ for each $p \in P$. A ***connection*** on $P(B,G)$ is a maximal G-invariant distribution in the sense that if $\mathscr{D}$ is a connection and if $\mathscr{D}'$ is a G-invariant distribution such that $\mathscr{D} \subset \mathscr{D}'$ then $\mathscr{D}' = \mathscr{D}$. If $\mathscr{D}$ is a G-invariant distribution on $P(B,G)$ and if $c\colon I \subset \mathbb{R} \to P$ is a curve on P we say that c is ***$\mathscr{D}$-compatible*** if $c'(t) \in \mathscr{D}(c(t))$ for each $t \in I$. Similarly, a vector field X on P is said to be ***$\mathscr{D}$-compatible*** if $X(p) \in \mathscr{D}(p)$ for each $p \in P$. □

*Assistant Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew@mast.queensu.ca, URL: http://www.mast.queensu.ca/~andrew/
Work performed while a Graduate Research Assistant at the California Institute of Technology.

1.2 Remarks: 1. Note that we have the standard G-invariance condition *plus* we add the restriction that no vectors in the distribution be vertical.

2. It is possible to discuss more general distributions on a principal fibre bundle than those which are G-invariant. San Martin and Crouch [1984] consider ***projectable*** distributions. These distributions are more general than G-invariant ones in the sense that they allow the control vector fields to have some vertical component.

3. If $\mathscr{D}$ is a distribution then $\mathscr{D}$-compatible curves and vector fields are ***horizontal*** in the manner usual for connections. □

The following result shows how G-invariant distributions are generalisations of connections.

1.3 Lemma: *If $\mathscr{D}$ is a G-invariant distribution on a principal fibre bundle $P(B,G)$, then there exists a connection $\overline{\mathscr{D}}$ which contains $\mathscr{D}$.*

Proof: One may partially order the set of G-invariant distributions by saying that $\mathscr{D} \subset \mathscr{D}'$ if $\mathscr{D}(p) \subset \mathscr{D}'(p)$ for every $p \in P$. The result then follows from Zorn's lemma. ■

Note that $\overline{\mathscr{D}}$ is not uniquely defined unless $\mathscr{D}$ is itself a connection in which case $\overline{\mathscr{D}} = \mathscr{D}$.

Note that a G-invariant distribution $\mathscr{D}$ induces a distribution $\mathscr{D}_\pi$ on B defined by $\mathscr{D}_\pi(b) = T\pi(\mathscr{D}(p))$ where $p \in \pi^{-1}(b)$. That this definition is independent of the choice of p is a consequence of $\mathscr{D}$ being G-invariant. Note that $T\pi \mid \mathscr{D}(p)\colon \mathscr{D}(p) \to \mathscr{D}_\pi(b)$ is an isomorphism for each $b \in B$ and $p \in \pi^{-1}(b)$. We wish to consider G-invariant distributions of which have a certain property.

1.4 Definition: Let $\mathscr{D}$ be a G-invariant distribution on a principal fibre bundle $P(B,G)$. The induced distribution $\mathscr{D}_\pi$ on B is called the ***base distribution***. The distribution $\mathscr{D}$ will be called ***base-controllable*** if for each $b_1, b_2 \in B$ there exists a curve $c\colon I \subset \mathbb{R} \to B$ connecting b_1 and b_2 and such that $c'(t) \in \mathscr{D}_\pi(c(t))$ for each $t \in \mathbb{R}$. Let $c\colon I \subset \mathbb{R} \to$ be a curve on B which lies in $\mathscr{D}_\pi$ and let $p \in \pi^{-1}(c(\tilde{t}))$ for some $\tilde{t} \in I$. We define the ***lift of c through p*** to be the unique curve c_l on P such that $T\pi(c_l'(t)) = c'(t)$ for each $t \in I$, and such that $c_l(\tilde{t}) = p$. (Note that c_l is $\mathscr{D}$-compatible.) □

Note that if $\mathscr{D}$ is a connection then it is base-controllable since $\mathscr{D}_\pi(b) = T_bB$ for each $b \in B$. We wish to think of base-controllable distributions as generalisations of connections where we do not specify every possible horizontal direction as an admissible control direction, but the directions which are specified are enough to generate all horizontal directions via Lie brackets.

1.2. Controllability for *G*-invariant distributions. Let $\mathscr{D}$ be a G-invariant distribution on $P(B,G)$. For $p_1, p_2 \in P$ we will write $p_1 \sim p_2$ if there exists a $\mathscr{D}$-compatible curve from p_1 to p_2. Now fix $p \in P$ and let $b = \pi(p)$. We define

$$\Phi(p) = \{g \in G \mid \ R_g(p) \sim p\}$$

to be the ***holonomy group of $\mathscr{D}$ with reference point p***. It is a Lie subgroup of G. Associated with it is the submanifold of $\pi^{-1}(b)$ given by

$$M_p = \{p' \in P \mid \ p' = R_g(p) \text{ for some } g \in \Phi(p)\}.$$

Thus M_p is the set of all points in the same fibre as p which may be connected to p with a $\mathscr{D}$-compatible curve. Note that $M_p = \pi^{-1}(b)$ if and only if $\Phi(p) = G$ if and only if the Lie algebra of $\Phi(p)$ is $\mathfrak{g}$, the Lie algebra of G.

For the sake of notation, let's suppose that there is a basis of vector fields for $\mathscr{D}$ over $C^\infty(P)$. Thus there exists $\mathscr{D}$-compatible vector fields $X_1, \dots, X_s$ such that

$$\mathscr{D}(p) = \langle X_1(p), \dots, X_s(p) \rangle$$

for each $p \in P$. Here $s \leq n - r$. We may define a control system on P by

$$\dot{p} = \sum_{i=1}^{s} u_i X_i. \tag{1.1}$$

We will say the system (1.1) is ***controllable*** if for each $p_1, p_2 \in P$ we may find controls $u_1(t), \dots, u_s(t)$ which steer the system (1.1) from p_1 to p_2.

We may now state the following:

1.5 PROPOSITION: *If system (1.1) is controllable then $M_p = \pi^{-1}(\pi(p))$ for each $p \in P$. Conversely, if $\mathscr{D}$ is base-controllable and if $M_p = \pi^{-1}(\pi(p))$ for each $p \in P$, then (1.1) is controllable.*

Proof: If (1.1) is controllable then $M_p = \pi^{-1}(\pi(p))$ for each $p \in P$. Indeed, if (1.1) is controllable then every two points in P may be connected with a $\mathscr{D}$-compatible curve. In particular, two points in the same fibre may be connected with a $\mathscr{D}$-compatible curve. But this means exactly that $M_p = \pi^{-1}(\pi(p))$ for each $p \in P$.

Now suppose that $\mathscr{D}$ is base-controllable and that $M_p = \pi^{-1}(\pi(p))$ for each $p \in P$. Let $p_1, p_2 \in P$ and denote $b_1 = \pi(p_1)$ and $b_2 = \pi(p_2)$. Since $\mathscr{D}$ is base-controllable we may find a curve c which connects b_1 with b_2 and which is compatible with $\mathscr{D}_\pi$. Let c_l be the lift of c through p_1. The curve c_l is a $\mathscr{D}$-compatible curve which connects p_1 with a point $p_2' \in \pi^{-1}(b_2)$. Now, since $M_p = \pi^{-1}(\pi(p))$, we may find a $\mathscr{D}$-compatible curve d which connects p_2' with p_2. The concatenated curve $d * c_l$ is a $\mathscr{D}$-compatible curve which connects p_1 with p_2 which shows that (1.1) is controllable. ■

San Martin and Crouch [1984] prove a generalisation of the following result.

1.6 PROPOSITION: (San Martin and Crouch) *Let $P(B, G)$ be a principal fibre bundle with compact structure group G, and let $\mathscr{D}$ be an accessible G-invariant distribution on P. Then the system (1.1) is controllable if and only if $\mathscr{D}$ is base-controllable.*

The generalisation in [San Martin and Crouch 1984] is to systems with distributions which are more general than G-invariant ones. In this instance the compactness of the structure group is essential because the system may contain a drift term. However, if there is no drift term, the above proposition is true even when G is not compact.

Now we present a theorem by Ambrose and Singer [1953]. This gives necessary and sufficient conditions for a connection on a principal fibre bundle to be controllable in terms of the Lie algebra of the holonomy group.

1.7 THEOREM: (Ambrose and Singer) *Let $P(B, G)$ be a principal fibre bundle with connection $\mathscr{D}$ and curvature form Ω. Let $p \in P$, and let $\Phi(p)$ be the holonomy group of $\mathscr{D}$ with reference point p and let $P(p)$ be the holonomy bundle through p. Then the Lie algebra of $\Phi(p)$ is the subspace of $\mathfrak{g}$ generated by the subset*

$$\big\{ \operatorname{range}(\Omega_{p'} \mid H_{p'} P) \mid \quad p' \in P(p) \big\}$$

As an easy consequence of this we have the following corollary.

1.8 COROLLARY: *Let $P(B, G)$ be a principal fibre bundle with connection $\mathscr{D}$ and curvature form Ω. Then $\mathscr{D}$ is accessible if and only if*

$$\mathfrak{g} = \operatorname{span}\big\{ \operatorname{range}(\Omega_{p'} \mid H_{p'} P) \mid \quad p' \in P(p) \big\}$$

2. The principal bundle structure in Brockett's normal form

In this section we wish to further explore Remark 3) made in Section 1. Namely, we wish to investigate a system which attains its control Lie algebra after one level of bracketing of the control vector fields.

2.1. Brockett's normal form as a principal bundle. In [Brockett 1982] a normal form is given for systems where the control distribution has a certain dimension with respect to the system, and where the involutive closure of the control distribution is spanned by first order brackets. The control system studied there is on $\mathbb{R}^{m(m+1)/2}$ and is written as

$$\begin{aligned} \dot{x}^i &= u^i, \qquad i = 1, \dots, m \\ \dot{y}^{ij} &= u^i x^j - u^j x^i, \qquad i, j = 1, \dots, m. \end{aligned}$$

Note that we may think of this as being a control system on $\mathbb{R}^m \times o(m)$ where $o(m)$ is the set of skew-symmetric $m \times m$ matrices. If we write $(x, Y) \in \mathbb{R}^m \times o(m)$ we get the equations

$$\begin{aligned} \dot{x} &= u \\ \dot{Y} &= x u^T - u x^T. \end{aligned} \tag{2.1}$$

We wish to cast this system as a control system on a principal bundle as in Section 1.

Let $P \triangleq \mathbb{R}^m \times o(m)$. We may consider $G \triangleq o(m)$ as an Abelian Lie group via addition of matrices. We may also identify $\mathfrak{g} \simeq o(m)$ and the Lie bracket on $\mathfrak{g}$ is the trivial one. The group G then acts freely on P by translation in the second factor. Thus, if $A \in G$ we have

$$R_A(x, Y) = (x, Y + A).$$

We may identify $M = P/G \simeq \mathbb{R}^m$ in the natural way. Since $P = \mathbb{R}^m \times o(m)$ is a vector space we have a canonical identification of $TP \simeq P \times P$. It is easy to verify that if $\xi \in \mathfrak{g} \simeq o(m)$ and if ξ_P denotes the infinitesimal generator for ξ then we have

$$\xi_P(x, Y) = (x, Y, 0, \xi).$$

Thus the vertical subbundle of the principal fibre bundle $P(M, G)$ is given by

$$V_{(x,Y)} P = \big\{ (a, B) \in T_{(x,Y)} P \mid \quad a = 0 \big\}.$$

2.1 PROPOSITION: *The distribution $\mathscr{D}$ on P defined by*

$$\mathscr{D}(x,Y) = \big\{(a,B) \in T_{(x,Y)}P \mid \ a = u,\ B = xu^T - ux^T \text{ for } u \in \mathbb{R}^m\big\}$$

is the horizontal distribution of a connection on $P(M,G)$.

Proof: We first show that $T_{(x,Y)}P = V_{(x,Y)}P \oplus \mathscr{D}(x,Y)$. For $(a,B) \in T_{(x,Y)}P$ we may write

$$(a,B) = (a, xa^T - ax^T) + (0, B - xa^T + ax^T)$$

where $(a, xa^T - ax^T) \in \mathscr{D}(x,Y)$ and $(0, B - xa^T + ax^T) \in V_{(x,Y)}P$. Therefore, $T_{(x,Y)}P = V_{(x,Y)}P + \mathscr{D}(x,Y)$. Now let $(a,B) \in V_{(x,Y)}P \cap \mathscr{D}(x,Y)$. Since $(a,B) \in \mathscr{D}(x,Y)$ we must have $(a,B) = (u, xu^T - ux^T)$ for some $u \in \mathbb{R}^m$. But, since $(a,B) \in V_{(x,Y)}P$ we must have $u = 0$ which implies that $B = 0$ so $V_{(x,Y)}P \cap \mathscr{D}(x,Y) = \{0\}$. Thus $T_{(x,Y)}P = V_{(x,Y)}P \oplus \mathscr{D}(x,Y)$.

Now we show that $TR_A(\mathscr{D}(x,Y)) = \mathscr{D}(R_A(x,Y))$ for each $(x,Y) \in P$ and for each $A \in G$. Note that

$$TR_A(x,Y,a,B) = (x, Y + A, a, B).$$

Thus

$$\begin{aligned} TR_A(\mathscr{D}(x,Y)) &= TR_A\big(\{(x,Y,a',B') \in TP \mid \\ &\qquad\qquad a' = u,\ B' = xu^T - ux^T \text{ for } u \in \mathbb{R}^m\}\big) \\ &= \{(x, Y + A, a', B') \in TP \mid \\ &\qquad\qquad a' = u,\ B' = xu^T - ux^T \text{ for } u \in \mathbb{R}^m\} \\ &= \mathscr{D}(R_A(x,Y)). \end{aligned}$$

This allows us to conclude that $\mathscr{D}$ is the horizontal distribution for a connection on $P(M,G)$. The connection form and the curvature form will be respectively denoted by ω and Ω. ■

Thus we see that the control system (2.1) may be regarded as a control system on a principal fibre bundle whose control inputs span the horizontal distribution of a connection on that bundle.

2.2. Controllability of (2.1). It is possible to examine the controllability of (2.1) directly. However, we will use the principal bundle structure to examine this. We first need to compute the connection one-form ω. Let ξ_{ij} be the element of $o(m)$ which has a 1 in the ij^{th} position and a -1 in the ji^{th} position. It is then a simple matter to see that

$$\big\{\xi_{ij} \mid \ i,j = 1,\dots,m,\ i < j\big\}$$

is a basis for $o(m)$. Now define differential one-forms α^{ij} for $i,j = 1,\dots,m$, $i < j$ by

$$\alpha^{ij} = \mathrm{d}y^{ij} + x^i \mathrm{d}x^j - x^j \mathrm{d}x^i.$$

2.2 LEMMA: *The connection form is given by $\omega = \alpha^{ij}\xi_{ij}$ (summation over $i < j$).*

Proof: We need to show that ω annihilates $\mathscr{D}$ and that ω applied to a vertical vector, $(0,B)$, yields the vector in $\mathfrak{g}$ whose infinitesimal generator gives $(0,B)$.

Let U be a horizontal vector at $(x,Y) \in P$. Thus

$$U = u^k \frac{\partial}{\partial x^k} + (u^l x^m - u^m x^l)\frac{\partial}{\partial y^{lm}}.$$

Now we compute

$$\begin{aligned} \alpha^{ij}(U) &= \alpha^{ij}\Big(u^k \frac{\partial}{\partial x^k} + (u^l x^m - u^m x^l)\frac{\partial}{\partial y^{lm}}\Big) \\ &= x^i \mathrm{d}x^j(u^k \frac{\partial}{\partial x^k}) - x^j \mathrm{d}x^i(u^k \frac{\partial}{\partial x^k}) + \mathrm{d}y^{ij}((u^l x^m - u^m x^l)\frac{\partial}{\partial y^{lm}}) \\ &= x^i u^j - x^j u^i + u^i x^j - u^j x^i = 0. \end{aligned}$$

Thus ω annihilates horizontal vectors.

Now let V be a vertical vector at $(x,Y) \in P$. Thus

$$V = B^{kl}\frac{\partial}{\partial y^{kl}}$$

for a skew-symmetric matrix B. Then

$$\begin{aligned} \alpha^{ij}(V) &= \alpha^{ij}\Big(B^{kl}\frac{\partial}{\partial y^{kl}}\Big) \\ &= \mathrm{d}y^{ij}(B^{kl}\frac{\partial}{\partial y^{kl}}) \\ &= B^{ij}. \end{aligned}$$

Therefore,

$$\omega(V) = B^{ij}\xi_{ij}, \quad \text{summation over } i < j.$$

But $B^{ij}\xi_{ij}$ is exactly the element of $\mathfrak{g}$ whose infinitesimal generator is V. Thus ω as given in the statement of the lemma is the connection one-form for $\mathscr{D}$. ■

Armed with an explicit formula for the connection one-form it is possible to prove the following result.

2.3 PROPOSITION: *Let $(x,Y) \in P$ and $\xi \in \mathfrak{g}$. Then $\xi = \Omega_{(x,Y)}(U,V)$ for horizontal vector fields U,V at (x,Y) if and only if $\xi = vu^T - uv^T$ for $u,v \in \mathbb{R}^m$.*

Proof: From Lemma 2.2 we have the connection form given by

$$\omega = \alpha^{ij}\xi_{ij}, \quad \text{summation over } i < j.$$

Thus we have

$$\begin{aligned} \boldsymbol{d}\omega &= \boldsymbol{d}\alpha^{ij}\xi_{ij}, \qquad \text{summation over } i < j \\ &= 2\mathrm{d}x^i \mathrm{d}x^j \xi_{ij}, \quad \text{summation over } i < j. \end{aligned}$$

Let U, V be horizontal vector fields given by

$$U(x,Y) = u^i(x,Y)\frac{\partial}{\partial x^i} + \left(u^i(x,Y)x^j - u^j(x,Y)x^i\right)\frac{\partial}{\partial y^{ij}}$$
$$V(x,Y) = v^i(x,Y)\frac{\partial}{\partial x^i} + \left(v^i(x,Y)x^j - v^j(x,Y)x^i\right)\frac{\partial}{\partial y^{ij}}$$

where $u, v\colon P \to \mathbb{R}^m$ are smooth maps. Then we may write

$$\begin{aligned}\Omega(U,V) &= \boldsymbol{d}\omega(U,V)\\ &= 2\mathrm{d}x^i\mathrm{d}x^j(U,V)\xi_{ij}, && \text{summation over } i<j\\ &= 2(\mathrm{d}x^i(U)\mathrm{d}x^j(V) - \mathrm{d}x^i(V)\mathrm{d}x^j(U))\xi_{ij}, && \text{summation over } i<j\\ &= 2(u^iv^j - u^jv^i)\xi_{ij}, && \text{summation over } i<j\\ &= vu^T - uv^T.\end{aligned}$$

Now if we fix $(x,Y) \in P$ we see that if $\xi = \Omega_{(x,Y)}(U,V)$ then $\xi = vu^T - uv^T$. A simple reversal of the above computations yields the converse. ■

If we define the bilinear map

$$\begin{aligned} f_m\colon \mathbb{R}^m \times \mathbb{R}^m &\to o(m)\\ (u,v) &\mapsto vu^T - uv^T.\end{aligned}$$

we know that by 1.7 the image of f_m generates a subspace of the Lie algebra of the holonomy group at any point. Furthermore, (2.1) is controllable if range(f_m) generates all of $\mathfrak{g} = o(m)$.

2.4 PROPOSITION: *The system (2.1) is controllable.*

Proof: Let $\{e_1, \dots, e_m\}$ be the standard basis for $\mathbb{R}^m$. Now note that $\xi_{ij} = f_m(e_j, e_i)$. Thus range(f_m) contains a basis for $o(m)$ and so generates $o(m)$. By our remarks made above this shows that (2.1) is controllable. ■

2.5 REMARKS:

1. Since the system (2.1) is a normal form for a class of control systems it is reasonable to ask if this class of systems possesses a group structure carried over from that which we have demonstrated in the normal form.
2. It is also interesting to consider when other classes of control distributions possess group structure. Thus, to examine these systems, it may be of advantage to use the group structure, especially the notion of fibre directions. □

2.3. Optimal control of (2.1). Brockett [1982] discusses optimal control of the system (2.1) in the sense that controls $u_1(t), \dots, u_m(t)$ are sought which steer the system from $(x_1, Y_1) \in P$ to $(x_2, Y_2) \in P$ in time $T = 1$ while minimising the cost function

$$d((x_1,Y_1)(x_2,Y_2)) = \int_0^1 \sqrt{\langle u(t), u(t)\rangle}\,\mathrm{d}t$$

We wish to cast these calculations in our principal fibre bundle formulation. Suppose that

$$U = u^i\frac{\partial}{\partial x^i} + (u^jx^k - u^kx^j)\frac{\partial}{\partial y^{ij}}$$

is a horizontal vector at $(x,Y) \in P$. Then is is easy to compute

$$T_{(x,Y)}\pi(U) = u^i\frac{\partial}{\partial x^i}$$

Thus we have the following vector bundle isomorphism over π.

$$\begin{aligned}\mathscr{D}\pi \triangleq T\pi \mid \mathscr{D}\colon \mathscr{D} &\to TB\\ (x,Y,u,xu^T - ux^T) &\mapsto (x,u)\end{aligned}$$

This immediately yields the following result.

2.6 LEMMA: *Let $\mathscr{C}((x_1,Y_1),(x_2,Y_2))$ be the set of horizontal curves which connect (x_1,Y_1) and (x_2,Y_2) in time $T = 1$. Let $c(t) \in \mathscr{C}((x_1,Y_1),(x_2,Y_2))$. Then the length of $c(t)$ defined by*

$$l(c(t)) = \int_0^1 \sqrt{\langle c'(t), c'(t)\rangle}\,\mathrm{d}t$$

is equal to the standard Euclidean length of the projected curve $\pi\circ c(t)$ on $\mathbb{R}^m$.

Thus the optimal control problem may be reworded as:

PROBLEM STATEMENT: *Among all curves $c(t) \in \mathscr{C}((x_1,Y_1),(x_2,Y_2))$ find the one whose base curve $\pi\circ c(t)$ has the minimal Euclidean length.*

Now we restrict ourselves to the case where $x_1 = x_2$. Thus we are considering the case where Y_1 and Y_2 lie in the fibre over the same point. Without loss of generality we may suppose that $x_1 = x_2 = 0$, and $Y_1 = 0$. We will let Y_2 be denoted by $\tilde{Y}$. To study this problem a key rôle will obviously be played by base curves which are loops beginning at $x = 0$ at $t = 0$ and ending at $x = 0$ at $t = 1$. To that end let C_0 be the set of all such curves and recall that concatenation of these curves determines a group structure on C_0. For each $c(t) \in C_0$ let $c_l(t)$ be the horizontal lift of $c(t)$ through $(0,0) \in P$. The map

$$\begin{aligned}*\colon \Sigma &\to C_0\\ \Phi(0,0) &\mapsto c(t)\end{aligned}\qquad \{g \in G \mid \ R_g(0,0) = c_l(1)\}$$

is an isomorphism.

3. The principal bundle structure in the chained form of Murray and Sastry

Another nonholonomic control system which exhibits a principal fibre bundle structure is given by Murray and Sastry [1991]. This control system has two inputs and has the form

$$\begin{aligned}\dot\xi^1 &= v^1\\ \dot\xi^2 &= v^2\\ \dot\xi^3 &= \xi^2 v^1\\ \dot\xi^4 &= \xi^3 v^1\\ &\vdots\\ \dot\xi^n &= \xi^{n-1} v^1\end{aligned} \tag{3.1}$$

3.1 PROPOSITION: *The change of variables*

$$\begin{aligned}x^1 &= \xi^1\\ x^2 &= \xi^2\\ x^3 &= -\xi^3 + \xi^1\xi^2\\ x^4 &= \xi^4 - \xi^1\xi^3 + \frac{1}{2}(\xi^1)^2\xi^2\\ &\vdots\\ x^n &= (-1)^n\xi^n + \sum_{i=2}^{n-1}(-1)^i\frac{1}{(n-i)!}(\xi^1)^{n-i}\xi^i\end{aligned} \tag{3.2}$$

is a global diffeomorphism of $\mathbb{R}^n$ *and the vector field given by (3.1) transforms to*

$$\begin{aligned}\dot x^1 &= u^1\\ \dot x^2 &= u^2\\ \dot x^3 &= x^1u^2\\ \dot x^4 &= \frac{1}{2}(x^1)^2u^2\\ &\vdots\\ \dot x^n &= \frac{1}{(n-2)!}(x^1)^{n-2}u^2\end{aligned} \tag{3.3}$$

under the coordinate change (3.2).

Proof: A calculation. ∎

The form (3.1) is called the ***chained form***, and the form (3.3) is called the ***power form***. For our purposes it is convenient to use the power form, so for the rest of the section we consider the control system (3.3).

3.1. The power form as a principal bundle. The control distribution is given by

$$\mathscr{D}(x) = \left\{v \in T_x\mathbb{R}^n \mid \ v^i = \frac{(x^1)^{i-2}}{(i-2)!}v, \quad i = 3,\dots,n\right\} \tag{3.4}$$

We wish to show that this is the horizontal distribution on a principal fibre bundle whose total space is $P = \mathbb{R}^n$. The group will be $G = \mathbb{R}^{n-2}$ and the group action is given by

$$((a^1,\dots,a^{n-2}),(x^1,\dots,x^n)) \mapsto (x^1,x^2,x^3+a^1,\dots,x^n+a^{n-2}))$$

The orbit space, M, is diffeomorphic to $\mathbb{R}^{n-2}$. The canonical projection from P to M may be taken to be

$$\pi(x^1,x^2,\dots,x^n) = (x^1,x^2)$$

It is easy to see that for $\xi \in \mathfrak{g} \simeq \mathbb{R}^{n-2}$, the infinitesimal generator ξ_P is given by

$$\xi_P(x) = \sum_{i=3}^{n}\xi^{i-2}\frac{\partial}{\partial x^i}.$$

Thus the vertical subbundle is given by

$$V_xP = \left\langle \frac{\partial}{\partial x^3},\dots,\frac{\partial}{\partial x^n}\right\rangle.$$

3.2 PROPOSITION: *The distribution given by (3.4) is a connection on the principal fibre bundle* $P(M,G)$.

Proof: First we show that $T_xP = V_xP \oplus \mathscr{D}(x)$. For $v \in T_xP$ we may write

$$(v^1,\dots,v^n) = \left(v^1,v^2,x^1v^2,\dots,\frac{(x^1)^{n-2}}{(n-2)!}v^2\right) - \left(0,0,v^3-x^1v^2,\dots,v^n-\frac{(x^1)^{n-2}}{(n-2)!}v^2\right)$$

The first vector in the decomposition of v lies in $\mathscr{D}(x)$ and the second vector lies in V_xP. This shows that $T_xP = V_xP + \mathscr{D}(x)$. Now let $v \in V_xP \cap \mathscr{D}(x)$. Since $v \in V_x$ we have $v^1 = v^2 = 0$. It is then easy to see from (3.4) that this must mean that $v = 0$. Thus $T_xP = V_xP \oplus \mathscr{D}(x)$.

Now we show that $TR_a(\mathscr{D}(x)) = \mathscr{D}(R_a(x))$ for each $x \in P$ and $a \in G$. Note that

$$TR_a(x,v) = ((x^1,x^2,x^3+a^1,\dots,x^n+a^{n-2}),(v^1,\dots,v^n)).$$

Thus

$$\begin{aligned}TR_a(\mathscr{D}(x)) &= TR_a\left(\left\{(x,v) \in TP \mid \ v^i = \frac{(x^1)^{i-2}}{(i-2)!}v^2, \quad i = 3,\dots,n\right\}\right)\\ &= \left\{((x^1,x^2,x^3+a^1,\dots,x^n+a^{n-2}),(v^1,\dots,v^n)) \in TP \mid \right.\\ &\qquad \left. v^i = \frac{(x^1)^{i-2}}{(i-2)!}v^2, \quad i = 3,\dots,n\right\}\\ &= \mathscr{D}(R_a(x))\end{aligned}$$

This completes the proof. ∎

3.2. Controllability of (3.3). We examine controllability of the power form using the principal bundle structure. Note that the distribution $\mathscr{D}$ may be written as

$$\mathscr{D} = \ker\left\{\alpha^1, \dots, \alpha^{n-2}\right\}$$

for

$$\alpha^i = \frac{(x^1)^i}{i!}\mathrm{d}x^2 - \mathrm{d}x^{i+2}, \qquad i = 1, \dots, n-2$$

We may use this to determine the connection form for the connection $\mathscr{D}$.

3.3 LEMMA: *Let $\{e_1, \dots, e_{n-2}\}$ be the standard basis for $\mathbb{R}^{n-1}$. The $\mathbb{R}^{n-2}$-valued differential one-form*

$$\omega = \alpha^1 e_1 + \cdots + \alpha^{n-2} e_{n-2}$$

is the connection form for the connection $\mathscr{D}$ on $P(M, G)$.

Proof: We must show that ω annihilates $\mathscr{D}$ and that, when applied to a vertical vector v, ω gives the vector in $\mathfrak{g}$ whose infinitesimal generator gives v.

Let u be a horizontal vector at $x \in P$. Then

$$u = \left(u^1, u^2, x^1u^2, \dots, \frac{(x^1)^{n-1}}{(n-2)!}u^2\right)$$

Then

$$\omega(u) = (x^1 - x^1)u^2 e_1 + \cdots + \left(\frac{(x^1)^{n-1}}{(n-2)!} - \frac{(x^1)^{n-1}}{(n-2)!}\right)u^2 e_{n-1} = 0$$

so ω annihilates horizontal vectors.

Now let v be a vertical vector at $x \in P$. Thus $v = (0, 0, \xi^1, \dots, \xi^{n-2})$. It is then clear that

$$\omega(v) = (\xi^1, \dots, \xi^{n-2}) \triangleq \xi$$

and the vector $\xi \in \mathfrak{g}$ has the property that $\xi_P(x) = v$. Thus ω is the connection form for the connection $\mathscr{D}$. ■

It is now a simple matter to explicitly compute the curvature form for the connection $\mathscr{D}$.

3.4 LEMMA: *Let $u, v \in T_xP$. If Ω denotes the curvature form for the connection $\mathscr{D}$, then*

$$\Omega(u, v) = \sum_{i=1}^{n-2} \frac{(x^1)^{i-1}}{(i-1)!}(u^1v^2 - u^2v^1)e_i$$

Proof: From 3.3 we know that the connection form is given by

$$\omega = \sum_{i=1}^{n-2}\left(\frac{(x^1)^i}{i!}\mathrm{d}x^2 - \mathrm{d}x^{i+2}\right)e_i.$$

Thus

$$\boldsymbol{d}\omega = \sum_{i=1}^{n-2} \frac{(x^1)^{i-1}}{(i-1)!}\mathrm{d}x^1 \wedge \mathrm{d}x^2 e_i.$$

Now if u, v are vectors at $x \in P$ we have

$$\begin{aligned}
\Omega(u, v) &= \boldsymbol{d}\omega(u_h, v_h) \\
&= \sum_{i=1}^{n-2} \frac{(x^1)^{i-1}}{(i-1)!}(\mathrm{d}x^1(u_h)\mathrm{d}x^2(v_h) - \mathrm{d}x^1(v_h)\mathrm{d}x^2(u_h))e_i \\
&= \sum_{i=1}^{n-2} \frac{(x^1)^{i-1}}{(i-1)!}(u_h^1 v_h^2 - u_h^2 v_h^1)e_i \\
&= \sum_{i=1}^{n-2} \frac{(x^1)^{i-1}}{(i-1)!}(u^1v^2 - u^2v^1)e_i
\end{aligned}$$

It is now possible to show that (3.3) is controllable.

3.5 PROPOSITION: *The system (3.3) is controllable.*

Proof: By 1.7 it is sufficient to find $n-2$ points $x_1, \dots, x_{n-2} \in P$ and $n-2$ pairs $((u_1^1, u_1^2), (v_1^1, v_1^2)), \dots, ((u_{n-2}^1, u_{n-2}^2), (v_{n-2}^1, v_{n-2}^2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $x_1 \sim x_2 \sim \cdots \sim x_{n-2}$ and such that

$$\left\{\Omega(x_1)((u_1^1, u_1^2), (v_1^1, v_1^2)), \dots, \Omega(x_{n-2})((u_{n-2}^1, u_{n-2}^2), (v_{n-2}^1, v_{n-2}^2))\right\} \tag{3.5}$$

is a basis for $\mathbb{R}^m$. Let $x_1, \dots x_{n-2} \in P$ be such that $x_1^j = \cdots = x_{n-2}^j = 0, \quad j = 2, \dots, n$. Clearly $x_1 \sim x_2 \sim \cdots \sim x_{n-2}$ since one can pick the control u^1 to steer to any value of x^1 desired and leave all the other coordinates fixed by setting the control $u^2 = 0$. Now choose $((u_1^1, u_1^2), (v_1^1, v_1^2)), \dots, ((u_{n-2}^1, u_{n-2}^2), (v_{n-2}^1, v_{n-2}^2))$ so that

$$u_1^1v_1^2 - u_1^2v_1^1 = \cdots = u_{n-2}^1v_{n-2}^2 - u_{n-2}^2v_{n-2}^1 = 1$$

This will be true, for example, if

$$((u_1^1, u_1^2)(v_1^1, v_1^2)) = \cdots = ((u_{n-2}^1, u_{n-2}^2)(v_{n-2}^1, v_{n-2}^2)) = ((1, 0), (0, 1))$$

We claim that we can choose $x_1^1, \dots, x_{n-2}^1$ so that (3.5) is a basis for $\mathbb{R}^m$. We will make the assumption that none of the x^1's are zero. Note that

$$\begin{aligned}
\Omega(x_i)((u_i^1, u_i^2), (v_i^1, v_i^2)) = e_1 + x_i^1 e_2 + \frac{(x_i^1)^2}{2!}e_3 + \\
\cdots + \frac{(x_i^1)^{n-3}}{(n-3)!}e_{n-2}, \qquad i = 1, \dots, n-2
\end{aligned}$$

Thus (3.5) is a basis for $\mathbb{R}^m$ if and only if the matrix

$$\begin{bmatrix}
1 & x_1^1 & \frac{(x_1^1)^2}{2} & \cdots & \frac{(x_1^1)^{n-3}}{(n-3)!} \\
1 & x_2^1 & \frac{(x_1^2)^2}{2} & \cdots & \frac{(x_1^2)^{n-3}}{(n-3)!} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{n-2}^1 & \frac{(x_{n-2}^1)^2}{2} & \cdots & \frac{(x_{n-2}^1)^{n-3}}{(n-3)!}
\end{bmatrix}$$

has full rank. Note that we may write

$$\begin{bmatrix} 1 & x_1^1 & \frac{(x_1^1)^2}{2} & \cdots & \frac{(x_1^1)^{n-3}}{(n-3)!} \\ 1 & x_2^1 & \frac{(x_1^2)^2}{2} & \cdots & \frac{(x_1^2)^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-2}^1 & \frac{(x_{n-2}^1)^2}{2} & \cdots & \frac{(x_{n-2}^1)^{n-3}}{(n-3)!} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & x_1^1 & (x_1^1)^2 & \cdots & (x_1^1)^{n-3} \\ 1 & x_2^1 & (x_1^2)^2 & \cdots & (x_1^2)^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-2}^1 & (x_{n-2}^1)^2 & \cdots & (x_{n-2}^1)^{n-3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{(n-3)!} \end{bmatrix}$$

Thus the proposition will be proved if we can prove the following lemma.

3.6 LEMMA: *For $m > 0$ the real matrix*

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{m-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & a_m & a_m^2 & \cdots & a_m^{m-1} \end{bmatrix}$$

is invertible if and only if the points $a_1, \ldots, a_m \in \mathbb{R}$ are distinct and non-zero.

Proof: Necessity is clear. For sufficiency, suppose that the above matrix is not of full rank and that $a_1, \ldots, a_m$ are distinct and non-zero. Then there exists constants $c_0, c_1, \ldots, c_m \in \mathbb{R}$, not all zero, such that

$$c_0(1, \ldots, 1) + c_1(a_1, \ldots, a_m) + \cdots + c_m(a_1^{m-1}, \ldots, a_m^{m-1}) = 0$$

Therefore

$$\begin{aligned} c_0 + c_1 a_1 + \cdots + c_m a_1^{m-1} &= 0 \\ &\vdots \\ c_0 + c_1 a_m + \cdots + c_m a_m^{m-1} &= 0 \end{aligned}$$

This implies that the polynomial

$$f(t) = c_0 + c_1 t + \cdots + c_m t^{m-1}$$

has m distinct non-zero roots. However $f(t)$ has at most $m-1$ distinct roots so we arrive at a contradiction. ▼

Since the lemma is proved the proof of the proposition is also complete. ■

The following result gives a description of the points such that the curvature form at these points generates the holonomy algebra.

3.7 COROLLARY: *For $x_1, \ldots, x_{n-2} \in P$, the set*

$$\bigcup_{i=1}^{n-2} \text{range}(\Omega(x_i))$$

generates $\mathfrak{g} = \mathbb{R}^{n-2}$ if and only if $x_1^1, \ldots, x_{n-2}^1$ are distinct and non-zero.

Proof: Follows directly from the proof of Proposition 3.5. ■

3.3. The holonomy associated to a control input for (3.3). Now we determine the holonomy associated to a particular control input $(u^1(t), u^2(t))$ in (3.3). This may be phrased in principle bundle language as: We shall determine the horizontal lift of an arbitrary curve on M. We begin with simple closed curves (i.e., curves which do not intersect themselves). Let $c(t) = (x^1(t), x^2(t))$ be a simple closed curve on M (assumed to be parameterised on the interval $I = [0, 1]$), and let $(x_0^3, \ldots, x_0^n)$ lie in the fibre over $c(0)$. We wish to compute the values of $(x_1^3, \ldots, x_1^n) \in \pi^{-1}(c(1))$. Observe from (3.3) that the evolution of $(x^3, \ldots, x^n)$ is a function only of $x^1(t)$ and $u^2(t)$. Thus we may compute the evolution of each component $(x_1^3(t), \ldots, x_1^n(t))$ independently of the others. The following lemma determines how to perform that calculation.

3.8 LEMMA: *Consider the system*

$$\begin{aligned} \dot{\xi}^1 &= v^1 \\ \dot{\xi}^2 &= v^2 \\ \dot{\xi}^3 &= f^1(\xi^1, \xi^2)v^1 + f^2(\xi^1, \xi^2)v^2 \end{aligned}$$

Suppose that the control inputs $(v^1(t), v^2(t))$ describe a simple closed curve in the (ξ^1, ξ^2) plane. Let D denote the domain whose boundary $\partial\mathsf{D}$ is the curve described by the control inputs. Then

$$\xi_0^3 - \xi_1^3 = \pm \iint_{\mathsf{D}} \left(\frac{\partial f^2(\xi^1, \xi^2)}{\partial \xi^1} - \frac{\partial f^1(\xi^1, \xi^2)}{\partial \xi^2} \right) \mathrm{d}\xi^1 \mathrm{d}\xi^2$$

We take the plus sign if the curve $\partial\mathsf{D}$ is positively oriented, otherwise we take the minus sign.

Proof: By Green's theorem we have

$$\int_{\partial \mathsf{D}} \left(f^1(\xi^1, \xi^2)\, d\xi^1 + f^2(\xi^1, \xi^2) \mathrm{d}\xi^2 \right) = \iint_{\mathsf{D}} \left(\frac{\partial f^2(\xi^1, \xi^2)}{\partial \xi^1} - \frac{\partial f^1(\xi^1, \xi^2)}{\partial \xi^2} \right) \mathrm{d}\xi^1 \mathrm{d}\xi^2$$

The integral on the left is taken in a positively oriented (i.e., counterclockwise) sense. If we parameterise the boundary $\partial\mathsf{D}$ by the curve defined by the control inputs we may write

$$\int_{\partial \mathsf{D}} \left(f^1(\xi^1, \xi^2) \mathrm{d}\xi^1 + f^2(\xi^1, \xi^2) \mathrm{d}\xi^2 \right) = \int_0^1 \left(f^1(\xi^1(t), \xi^2(t)) v^1(t) + f^2(\xi^1(t), \xi^2(t)) v^2(t) \right) \mathrm{d}t$$

However, we also have

$$\begin{aligned} &\dot{\xi^3} = f^1(\xi^1,\xi^2)v^1 + f^2(\xi^1,\xi^2)v^2 \\ \implies \qquad &\xi^3_1 - \xi^3_0 = \int_0^1 \big(f^1(\xi^1(t),\xi^2(t))v^1(t) + f^2(\xi^1(t),\xi^2(t))v^2(t)\big)\,\mathrm{d}t \end{aligned}$$

This proves the lemma. ■

Recall that from (3.3) we have

$$\begin{aligned} \dot{x^1} &= u^1 \\ \dot{x^2} &= u^2 \\ \dot{x^j} &= \frac{(x^1)^{j-2}}{(j-2)!}u^2 \end{aligned}$$

for $j = 3, \ldots, n$. Therefore, by Lemma 3.8 we have

$$x^j_1 - x^j_0 = \iint_{\mathsf{D}} \frac{(x^1)^{j-3}}{(j-3)!}\mathrm{d}x^1\mathrm{d}x^2, \qquad j = 3, \ldots, n$$

Note that

$$\frac{(x^1)^{j-3}}{(j-3)!}\mathrm{d}x^1\mathrm{d}x^2$$

is the $(j-2)^{nd}$ component of the curvature form Ω. Let's compute the holonomy in the instance where D is a circle of radius R. The circle is shown in Figure 1. The region D is given by

$$\mathsf{D} = \big\{(x^1, x^2 \in \mathbb{R}^2 \mid \ \|(x^1,x^2) - (x^1_0 + R\cos\phi, x^2_0 + R\sin\phi)\| = R\big\}$$

As coordinates for D we will use

$$\begin{aligned} r &= \sqrt{(x^1 - x^1_0 - R\cos\phi)^2 + (x^2 - x^2_0 - R\sin\phi)^2} \\ \theta &= \arctan\left(\frac{x^2 - x^2_0 - R\sin\phi}{x^1 - x^1_0 - R\cos\phi}\right) \end{aligned}$$

with the inverse coordinate change being

$$\begin{aligned} x^1 &= x^1_0 + R\cos\phi + r\cos\theta \\ x^2 &= x^2_0 + R\sin\phi + r\sin\theta \end{aligned}$$

We then have

$$\begin{aligned} x^j_1 - x^j_0 &= \iint_{\mathsf{D}} \frac{(x^1)^{j-3}}{(j-3)!}\mathrm{d}x^1\mathrm{d}x^2 \\ &= \int_0^{2\pi}\int_0^R \frac{(r\cos\theta + R\cos\phi + x^1_0)^{j-3}}{(j-3)!}\,r\mathrm{d}r\mathrm{d}\theta \\ &= \frac{1}{(j-3)!}\int_0^R r\bigg(\int_0^{2\pi}(a + b\cos\theta)^{j-3}\,\mathrm{d}\theta\bigg)\,\mathrm{d}r \end{aligned}$$

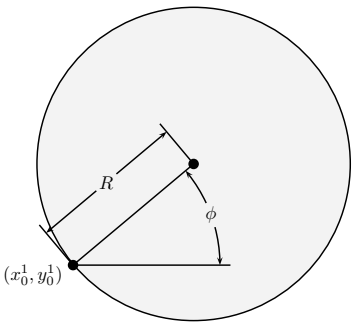

FIGURE 1. The domain D.

where

$$a = R\cos\phi + x^1_0$$

The integral with respect to θ may be obtained (by *Mathematica*) as

$$\int_0^{2\pi}(a + r\cos\theta)^{j-3}\,\mathrm{d}\theta = 2a^{j-3}\,{}_2F_1\big(\tfrac{1}{2} - \tfrac{j-3}{2}, -\tfrac{j-3}{2}, 1, \tfrac{r^2}{a^2}\big)$$

Here ${}_2F_1(\alpha,\beta;\delta;x)$ is the ***hypergeometric function*** which is a solution of the differential equation

$$x(1-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + [\delta - (\alpha+\beta+1)x]\frac{\mathrm{d}y}{\mathrm{d}x} - \alpha\beta y = 0$$

Thus we get

$$x^j_1 - x^j_0 = \frac{1}{(j-3)!}\int_0^R {}_2F_1\big(\tfrac{1}{2} - \tfrac{j-3}{2}, -\tfrac{j-3}{2}, 1, \tfrac{r^2}{a^2}\big)\,r\mathrm{d}r$$

References

Ambrose, W. and Singer, I. M. [1953] *A theorem on holonomy*, Transactions of the American Mathematical Society, **75**, 428–443.

Brockett, R. W. [1982] *Control theory and singular Riemannian geometry*, in *New Directions in Applied Mathematics*, P. J. Hilton and G. S. Young, editors, Springer-Verlag, New York-Heidelberg-Berlin.

Kobayashi, S. and Nomizu, K. [1963] *Foundations of Differential Geometry*, Volume I, number 15 in Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers, New York, ISBN 0-470-49647-9.

Murray, R. M. and Sastry, S. S. [1991] *Steering nonholonomic systems in chained form*, in *Proceedings of the 31st IEEE Conference on Decision and Control*, pages 1121–1126, Institute of Electrical and Electronics Engineers, Brighton, England.

San Martin, L. and Crouch, P. E. [1984] *Controllability on principal fibre bundles with compact structure group*, Systems & Control Letters, **5**(1), 35–40.