Anisotropic Fluid Solutions to Einstein's Equations in Spherical Symmetry

by

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Abstract

The main objective of this thesis is to develop and investigate a class of spherically symmetric solutions to Einstein's field equations for anisotropic matter distributions. This work is partially motivated by a recent theorem of Andreasson [1], on the upper limit of the tenuity ratio for objects possessing equations of state of the form $p_r + 2p_t = \Omega\rho$, where p_r, p_t and ρ are the radial pressure, tangential pressure and energy density, respectively and Ω is an unspecified constant characteristic parameter of the fluid. An immediate goal was to extend this theorem to the broader class of all linear barotropic equations of state. A solution generating algorithm which yields infinitely many analytical, physically reasonable, stable fluid distributions for general linear barotropes (with and without the cosmological constant Λ) is presented and analyzed. Other general aspects of spherical anisotropic matter distributions are discussed, including historically significant developments, wave propagation and stability.

Acknowledgments

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Chapter 1

Introduction

Prior to Maxwell's development of the theory of electrodynamics, the fundamental laws of physics were thought to be invariant under the group of Galilean transformations. The revelation that the speed of light in vacuum is a universal constant necessitated the development of the concept of Lorentz invariance and the special theory of relativity. Now, over a century has passed since the field equations of the general theory of relativity were fully formulated. In the following, we will explore a brief introduction to the mathematical framework of the theory. Concepts developed throughout this section will be employed later in the analysis. Much of the introductory notation section is included such that this document is self contained, and can be glossed over for readers familiar with the mathematical background of the theory of general relativity.

1.1 Introductory Notation

The locus of events in general relativity is called the space-time, and can be described by an equivalence class of pairs ($\mathcal{M}$, $\mathbf{g}$), where $\mathcal{M}$ is a C^∞ Hausdorff manifold, and $\mathbf{g}$ is a Lorentzian metric on the manifold $\mathcal{M}$. Let R^n denote the set of all n-tuples x^α, $\alpha = \{0, 1, \dots n\}$. A map ψ between two open subsets of R^n, say a and a', is C^r if the coordinates x'^α of the image point $\psi(p)$ in a' are r-times continuously differentiable functions of the coordinates x^α of p in a. A C^r n-dimensional manifold is a set $\mathcal{M}$ and a collection of charts, i.e. subsets of $\mathcal{M}$, say m_α and their one to one maps ψ to R^n such that the union of all sets m_α covers the set $\mathcal{M}$. In the intersection of any two subsets m_α and m_β of $\mathcal{M}$ the coordinates spanning m_α must be r-times continuously differentiable functions of the coordinates spanning m_β. Stating that the manifold $\mathcal{M}$ is Hausdorff implies that for $p, q \in \mathcal{M}$ $\exists$ disjoint open sets A and B, where $p \in A$ and $q \in B$. The Cartesian product of two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, is another manifold denoted $\mathcal{M}_1 \times \mathcal{M}_2$. For two points $p \in \mathcal{M}_1$ and $q \in \mathcal{M}_2$ with local coordinate neighbourhoods U and V respectively, the point $(p, q) \in \mathcal{M}_1 \times \mathcal{M}_2$ exists in $U \times V \subset \mathcal{M}_1 \times \mathcal{M}_2$ with coordinates (x^i, y^j), where x^i are the coordinates of $p \in U$ and y^i are the coordinates of $q \in V$. The use of smooth manifolds as a model for space-time seems natural given our conception of space and time as being continuous quantities$-$ a presumption which has proven useful in describing gravitation on familiar length scales.

Let $T_q(\mathcal{M})$ or simply T_q denote the tangent space to $\mathcal{M}$ at the point q. The tangent space is an n-dimensional vector space representing the set of all directions at a point. One can think of an element of this set, say $\mathbf{U} \in T_q$, as an arrow at the point q pointing in the direction of a curve, say γ, passing through q. The metric tensor $\mathbf{g}$ at the point $q \in \mathcal{M}$ designates a magnitude to each vector $\mathbf{X} \in T_q$, $(|g(\mathbf{X},\mathbf{X})|)^{\frac{1}{2}}$, and defines the 'cos angle' between any two vectors $\mathbf{X}, \mathbf{Y} \in T_q$

$$\frac{g(\mathbf{X},\mathbf{Y})}{(|g(\mathbf{X},\mathbf{X}) \cdot g(\mathbf{Y},\mathbf{Y})|)}. \tag{1.1}$$

$\mathbf{g}$ is a symmetric tensor of rank (0,2), the components of which (denoted $g_{\alpha\beta}$) can be specified with respect to some basis.

A C^s curve, say $\gamma(t)$, in $\mathcal{M}$ is a map of an interval of R^1 into $\mathcal{M}$. The contravariant vector $(\partial/\partial t)_\gamma|_{t_0}$ tangent to $\gamma(t)$ at a point $\gamma(t_0)$ is an operator that maps a C^1 function f at the point $\gamma(t_0)$ in to the number $(\partial f/\partial t)_\gamma|_{t_0}$. For some local coordinates $(x^1 \dots x^n)$ in the neighbourhood of a point p, we can write

$$\left(\frac{\partial f}{\partial t}\right)|_{\gamma(t_0)} = \frac{dx^j}{dt}\frac{\partial f}{\partial x^j}|_{\gamma(t_0)}. \tag{1.2}$$

Thus the tangent vectors to the curve γ at some point constitute elements of a vector space spanned by the coordinate basis $(\partial/\partial x^1|_p, \dots \partial/\partial x^n|_p)$ and any element of T_p can be expressed by a linear combination of these basis elements. A vector $\mathbf{V} \in T_p$ can be written using any set $\mathbf{E}_a$ $(a \in \{1 \dots n\})$ of linearly independent basis vectors as

$$\mathbf{V} = V^a \mathbf{E}_a, \tag{1.3}$$

where the coefficients V^a are numbers representing the components of $\mathbf{V}$ with respect to the chosen basis of vectors at p and we note the use of summation convention where a repeated covariant and contravariant index is summed over.

We define a covariant vector (a 1-form) as an operator belonging to the dual space of T_p denoted T_p^*. The covariant vector is a linear map defined on T_p that takes a vector as an argument and produces a number. For $\alpha \in T_p^*$ and $u \in T_p$

$$\alpha(u) = \langle \alpha, u \rangle \in R \tag{1.4}$$

A set of basis vectors $\{\mathbf{E}_a\}$ at a point p can be used to construct a unique dual set of 1-forms $\{\mathbf{E}^a\}$ at p such that

$$\langle \mathbf{E}^a, \mathbf{E}_b \rangle = \delta^a_b \tag{1.5}$$

Any 1-form at a point can be expressed in terms of the dual basis covectors ($\omega = \omega_a \mathbf{E}^a$). The Cartesian product of m vector spaces and n covector spaces at a point p forms the space Π^m_n. A tensor $\mathbf{T}$ of type (n, m) at p is a multilinear function which maps m vectors and n 1-forms in to a number. The space of all such tensors is the tensor product space,

$$T^m_n(p) = T_p \otimes \cdots \otimes T_p \otimes T_p^* \otimes \cdots \otimes T_p^*, \tag{1.6}$$

where there are m factors of T_p, and n factors of T_p^*.

Given a coordinate basis $\{\partial/\partial x^\alpha\}$, one can define an important quantity called the line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{1.7}$$

which we immediately note is a scalar quantity:

$$g_{\alpha\beta} dx^\alpha dx^\beta = \bar{g}_{\gamma\delta} d\bar{x}^\gamma d\bar{x}^\delta \tag{1.8}$$

The line element is a representation of the length of an infinitesimal coordinate displacement between x^α and $x^\alpha + dx^\alpha$ in an arbitrary space-time and is a common way to represent the components of the metric tensor $g_{\alpha\beta}$ with respect to a specified basis.

To prove that the line element is a scalar we needed to know how the metric tensor and the contravariant vector dx^α behave under coordinate transformations from say x^α to $\bar{x}^\alpha$. In general, an object $T^{\alpha_1\ldots\alpha_m}_{\beta_1\ldots\beta_n}$ that under the transformation $x^\alpha \longmapsto \bar{x}^\alpha$ obeys

$$\bar{T}^{\alpha_1\ldots\alpha_m}_{\beta_1\ldots\beta_n} = \frac{\partial\bar{x}^{\alpha_1}}{\partial x^{\gamma_1}}\cdots\frac{\partial\bar{x}^{\alpha_m}}{\partial x^{\gamma_m}}\frac{\partial x^{\delta_1}}{\partial\bar{x}^{\beta_1}}\cdots\frac{\partial x^{\delta_n}}{\partial\bar{x}^{\beta_n}}T^{\gamma_1\ldots\gamma_m}_{\delta_1\ldots\delta_n} \tag{1.9}$$

represents the components of a tensor with respect to the coordinate basis. It is important to understand that the 'ordinary' partial derivative of a tensor with respect to coordinates does not transform as a tensor. For example the partial derivative of a rank 1 tensor transforms like

$$\frac{\partial\bar{v}^\alpha}{\partial\bar{x}^\gamma} = \frac{\partial x^\delta}{\partial\bar{x}^\gamma}\frac{\partial^2\bar{x}^\alpha}{\partial x^\delta\partial x^\beta}v^\beta + \frac{\partial\bar{x}^\alpha}{\partial x^\beta}\frac{\partial x^\delta}{\partial\bar{x}^\gamma}\frac{\partial v^\beta}{\partial x^\delta}. \tag{1.10}$$

We are motivated by this to introduce a method of differentiation which transforms as a tensor under general coordinate transformations. To do so briefly, we first introduce the Christoffel symbol $\Gamma^\alpha_{\beta\gamma}$ (see e.g. [2]) as the object which transforms under $\bar{x} \longmapsto x$ via

$$\bar{\Gamma}^\lambda_{\mu\nu} = \frac{\partial\bar{x}^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial\bar{x}^\mu}\frac{\partial x^\sigma}{\partial\bar{x}^\nu}\Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial\bar{x}^\nu}\frac{\partial x^\sigma}{\partial\bar{x}^\mu}\frac{\partial^2\bar{x}^\lambda}{\partial x^\rho\partial x^\sigma}. \tag{1.11}$$

The Christoffel symbol clearly does not represent the components of a tensor. One then constructs the covariant derivative of a vector as

$$\nabla_\lambda V^\mu \equiv \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa}V^\kappa = V^\mu_{;\lambda}. \tag{1.12}$$

It can be verified that the covariant derivative is tensorial under general coordinate transformations. The contravariant analogue is simply the covariant derivative raised via the inverse metric tensor. Note that the covariant and contravariant derivatives can be applied to general tensors of rank (m, n) and that the raising and lowering of indices commutes with the operation of covariant differentiation.

Some curve $x^\alpha(w)$ parametrized by the variable w is called a geodesic if the covariant derivative of the tangent vector to the curve (say $u \equiv d/dw$) with respect to itself is proportional to u,

$$\nabla_u u \equiv u^\alpha \nabla_\alpha u = f(w)u \tag{1.13}$$

An affine parameter ($w = \lambda$) is one for which

$$\nabla_u u = 0. \tag{1.14}$$

Thus along an affinely parameterized geodesic curve, the quantity

$$2L \equiv u^\alpha u_\alpha \tag{1.15}$$

is constant. A trajectory is called null if it has $2L = 0$. To further distinguish timelike versus spacelike trajectories using the sign of L, one needs to address the concept of signature.

Given $g_{\alpha\beta}(x^\gamma)$, it is always possible to transform $x^\alpha \longmapsto \bar{x}^\alpha$ such that at some point p the metric tensor $\bar{g}_{\alpha\beta}$ is diagonal with elements equal to ± 1. For x^α and $\bar{x}^\alpha \in \Re$, the number of diagonal components $= +1$ and the number of diagonal components $= -1$ is fixed $\forall$ p. The signature of the metric tensor is defined as the number of diagonal components $= +1$ minus the number of diagonal components $= -1$. Every 4-dimensional space of signature ± 2 is locally Minkowskian (see e.g. [2]).

For signature +2,

- $u^\alpha u_\alpha = -1$: “timelike”
- $u^\alpha u_\alpha = +1$: “spacelike”

For signature -2,

- $u^\alpha u_\alpha = +1$: “timelike”
- $u^\alpha u_\alpha = -1$: “spacelike”

In general, throughout the bulk of this text, a signature of +2 will be used unless otherwise stated.

We previously showed that the line element is a scalar. The value of this scalar in general is position dependent. Thus in general along a curve, say γ, which is parametrized by some quantity λ,

$$\frac{d}{d\lambda}(ds^2)|_\gamma = \frac{d}{d\lambda}(g_{\alpha\beta}dx^\alpha dx^\beta)|_\gamma \neq 0 \tag{1.16}$$

A curve x^α with tangent vector components v^α, along which the scalar ds^2 is fixed is called a Killing trajectory, the tangent to which is called a Killing vector. It can be shown that if the line element is fixed along a curve, the tangent to that curve necessarily satisfies the Killing equation

$$\nabla_\beta v_\alpha + \nabla_\alpha v_\beta = 0. \tag{1.17}$$

Each coordinate x^i that the metric tensor is independent of corresponds to a solution of Killing's equation. A congruence in an open region of a spacetime, say $\mathcal{O}$, is a family of non intersecting curves such that $\forall$ points $q \in \mathcal{O}$, there is a single curve passing through the point. For an affinely parametrized geodesic with tangent u passing through a congruence of Killing trajectories, there is a corresponding 'constant of the motion',

$$g_{\alpha\beta}u^{\alpha}v^{\beta} = C_0, \tag{1.18}$$

where C_0 is a constant which is conserved along the geodesic. Associated with each Killing vector is a unique constant of the motion.

The metric tensors $\mathbf{g}$ and $\hat{\mathbf{g}}$ are called conformal if

$$\hat{\mathbf{g}} = \omega^2 \mathbf{g} \tag{1.19}$$

for some real-valued, non-zero ω which is a suitably differentiable function of the coordinates. Clearly conformal transformations to the metric tensor preserve angles and ratios of magnitudes since

$$\frac{\mathbf{g}(A,B)}{\mathbf{g}(X,Y)} = \frac{\hat{\mathbf{g}}(A,B)}{\hat{\mathbf{g}}(X,Y)}. \tag{1.20}$$

An important consequence is that conformally related spacetimes have the same null cone structure.

Since a thorough discussion of important quantities such as the Riemann, Ricci and Weyl tensors can be found in any introductory textbook, deep discussion of these will be omitted from this section. It will suffice to mention that a four dimensional sig ± 2 space for which Riemann vanishes is called flat (Minkowski) and that the components of Riemann can be decomposed (for dimension $n \geq 4$) in terms of those of Ricci ($R_{\alpha\beta}$) and Weyl ($C_{\alpha\beta\gamma\delta}$) in the following way:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = & \frac{1}{n-2}\left(g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta} + g_{\beta\delta}R_{\alpha\gamma}\right) \\ & - \frac{R}{(n-1)(n-2)}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) + C_{\alpha\beta\gamma\delta} \end{aligned} \tag{1.21}$$

Above R is the Ricci scalar $R = R_{\mu\nu}g^{\mu\nu}$. A pseudo-Riemannian manifold $(\mathcal{M}, \mathbf{g})$ is called conformally flat if $\forall$ points $q \in \mathcal{M}$, $\exists$ a neighbourhood $\mathcal{U}$ of q and a smooth function f defined on $\mathcal{U}$ such that the curvature of $e^{2f}\mathbf{g}$ vanishes on $\mathcal{U}$. The necessary and sufficient condition for conformal flatness in an N-dimensional space-time (for $N \geq 4$) is the vanishing of the Weyl tensor:

$$C_{\alpha\beta\gamma\delta} = 0 \tag{1.22}$$

An important feature of the Weyl tensor is that it remains invariant under conformal transformations to the metric tensor (i.e. under transformations to the metric which involve scalar multiplication by a position dependent conformal factor).

A fundamental tensor on which the theory of general relativity is based is referred to as the Einstein tensor, and can be constructed via Ricci and the metric tensor:

$$G^{\alpha}_{\beta} \equiv R^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}R \tag{1.23}$$

An important differential identity due to Bianchi, which holds $\forall$ $g_{\alpha\beta}(x^{\gamma})$ states that

$$\nabla_{\epsilon}R_{\alpha\beta\gamma\delta} + \nabla_{\gamma}R_{\alpha\beta\delta\epsilon} + \nabla_{\delta}R_{\alpha\beta\epsilon\gamma} = 0. \tag{1.24}$$

A profound consequence of the Bianchi identity is that the Einstein tensor is divergence free.

$$\nabla_\alpha G^\alpha_\beta = 0 \tag{1.25}$$

One of the beautiful characteristics of the general theory of relativity is that it describes the dynamics of bodies in the presence of local energy and matter distributions using a general (non-Euclidean) geometric framework. An appealing feature is that the tensorial nature of Einstein's equations indicates that they must have the same form regardless of what set of coordinates is chosen.

The theory of general relativity describes distributions of matter and energy via a symmetric rank 2 tensor called the energy-momentum tensor denoted by $T^{\alpha\beta}$. On an open set $\mathcal{U}$, the energy momentum tensor vanishes if and only if the 'matter fields' described by $T^{\alpha\beta}$ are zero on $\mathcal{U}$. The local conservation of energy and momentum specifies that $T^{\alpha\beta}$ is divergence free.

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{1.26}$$

Courtesy of Lovelock [3], it has been shown that the only field equations (for $n = 4$) based on the metric tensor and its first two partial derivatives which are divergence free are the Einstein field equations,

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1.27}$$

where Λ is an important unspecified constant (the "cosmological constant"). Notice that Bianchi's identity and Lovelock's theorem render the local conservation of energy and momentum a necessary feature of the theory.

The usual procedure in 'solving' the field equations is to specify a stress-energy-momentum tensor $T_{\alpha\beta}$ and subsequently determine the metric tensor (thus yielding the geometry). In some cases, however, specification of the components of the metric tensor with respect to a basis can ensure that an analytical solution is attainable. In general, solutions attained through assumptions about the metric will not yield physically reasonable results. Providing a relationship between the components of the energy-momentum tensor with respect to a basis (e.g. an equation of state) allows us to investigate systems which we hypothesize to be physically realistic, yet attaining a solution without recourse to numerical integration is unlikely. Note that given a metric tensor, the physical interpretation of $T_{\alpha\beta}$ is not necessarily unique.

Einstein's field equations constitute a set of coupled, non-linear partial differential equations. The symmetric nature of the metric tensor implies that in an n-dimensional spacetime there are in general $n(n+1)/2$ independent field equations. Further constraint is due to divergence freedom of the Einstein tensor. Obtaining exact solutions, even in the most idealized scenarios presents a significant challenge.

1.2 Schwarzschild Exterior Solution

In this section, we examine the Schwarzschild exterior solution published by the German polymath Karl Schwarzschild in 1915 [4] during his service on the Russian front in World War I. The Schwarzschild exterior solution is the metric describing the vacuum exterior to a spherically symmetric massive object.

Note that every spherically symmetric spacetime can be written in the form

$$ds^2 = ds_\Sigma^2 + R^2 d\Omega^2, \tag{1.28}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of a 2-sphere, and $R = R(x^1, x^2)$ with x^1 and x^2 the coordinates on Σ (which has signature $= 0$). Since every two dimensional space admits orthogonal coordinates, ds_Σ^2 can be diagonalized and we can choose, pathologically, the general spherically symmetric metric of the form (see e.g. [5]),

$$ds^2 = -e^{2\Phi(r,t)}dt^2 + \left(1 - \frac{2m(r,t)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.29}$$

Using the line element one can immediately solve the vacuum field equations without Λ,

$$G_{\alpha\beta} = 0. \tag{1.30}$$

It is worth noting at this point that the analysis herein relies heavily on tensor algebra software, primarily GRTensorIII [6]. Using such software, calculating relevant tensors such as Einstein is a simple task once the metric is specified. Furthermore, I will note here the use of geometrical units [7]. Throughout the thesis, I will in general work with units where the ratio of 'mass' to distance is unitless. In the ordinary SI unit system the energy momentum tensor in the field equations has a factor G/c^4.

In examining the field equations it becomes clear that the vacuum condition ($G_{\alpha\beta} = 0$) requires that the metric function $m(r,t) \equiv m$ is constant with respect to both t and r, and that

$$\Phi(r,t) = \frac{1}{2}\left[\ln(r - 2m) - \ln(r)\right] + F(t) \tag{1.31}$$

After some straightforward algebra, one can obtain the familiar form of the Schwarzschild solution,

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.32}$$

An important point is that the Schwarzschild solution is 'static'; the components of the metric are independent of the coordinate t and therefore the space admits at least one hypersurface orthogonal timelike Killing vector [8].

Birkhoff's theorem [2] states that every spherically symmetric solution to Einstein's vacuum field equations without Λ is static and asymptotically flat. The Schwarzschild solution is the unique spacetime exterior to a spherical, non-rotating massive body. Recall the Newtonian analogue (coined the "superb theorem" by Chandrasekhar [9]) which states that the gravitational field at a point external to a spherically symmetric mass distribution is the same as though the mass were all concentrated directly at the center of the distribution. The Birkhoff theorem also states that the metric tensor for a spherical cavity inside a spherically symmetric mass distribution is necessarily Minkowski. Again we find analogy in Newton's finding that the gravitational field interior to a spherically symmetric mass shell vanishes. An interesting consequence of the theorem is that a fixed mass star which is experiencing radial spherical pulsations cannot produce gravitational waves (the only effect of the oscillations will be to change the location of the stellar surface) since the external field is necessarily the unique static, asymptotically flat Schwarzschild solution.

The dynamics of particles in a gravitational field are obtained through analysis of the geodesics of the metric describing a particular spacetime. Free-falling massive particles travel along timelike geodesics whereas massless particles, such as photons, travel along null geodesics [10]. For the Schwarzschild solution, the timelike geodesic equations reduce to the Newtonian equations of motion in the limit of large $\frac{r}{m}$.

There is much more to be said about the Schwarzschild exterior solution. For example, the coordinate singularity which occurs at $r = 2m$ can be removed in so-called 'maximal extensions' of the solution (see [11] and citations therein). Carter-Penrose diagrams [10] present an invaluable tool in visualizing the geometry and causal structure of the Schwarzschild solution. We must keep in mind that Λ is permitted to exist in the field equations and that the spherically symmetric solution for $\Lambda \neq 0$ in which the energy momentum tensor vanishes has a significantly different character.

1.3 Hydrostatic Equilibrium

Consider the standard Newtonian treatment of hydrostatic equilibrium. For a spherically symmetric configuration of matter, let r represent the radial coordinate, measured outward from the origin of the distribution. Spherical symmetry and staticity impose the constraint that the physical parameters such as the pressure p, mass m and density ρ are strictly functions of r. The mass internal to some radius r is related to the density ρ via

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr. \tag{1.33}$$

$$dm(r) = 4\pi r^2 \rho(r) dr \tag{1.34}$$

Let $r = R$ be the radius at which the fluid terminates, i.e. $p(R) = 0$. At this radius, the total mass of the object is defined to be $m(R) = M$.

An infinitesimal cylinder some distance r from the center, having height dr along the r axis, with unit cross-sectional area will have a difference in pressure between the two faces perpendicular to r. This difference, dp represents a buoyant force acting on the cylinder. Note that this infinitesimal cylinder has a mass given by $\rho(r)dr$. The gravitational force of attraction between $m(r)$ and $\rho(r)dr$ is the same as that for a mass $m(r)$ located at the origin and $\rho(r)dr$ located at r.

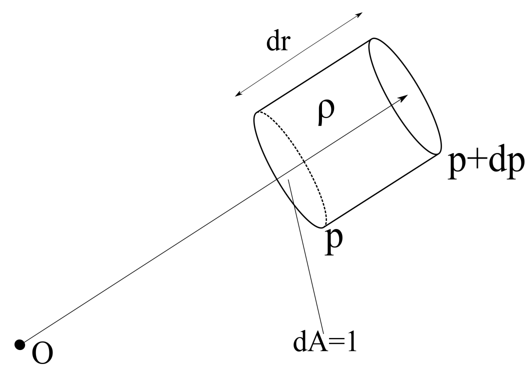

Figure 1.1: The standard Newtonian equation for hydrostatic equilibrium can be derived by considering an infinitesimal cylinder along the radial axis of length dr with mass $\rho\ dr$ which has a pressure gradient across the faces. Note that the pressure is decreasing outward, and so $dp < 0$.

Hydrostatic equilibrium implies that the sum of all forces acting on a given small volume element of a fluid is equal to zero. In this case, the force dp is balanced with the gravitational attraction experienced by the cylinder, which via Newton's theory yields the relationship

$$\frac{dp}{dr} = -\frac{m\rho}{r^2}. \tag{1.35}$$

The familiar fundamental expression in hydrostatic equilibrium can be developed as

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi\rho. \tag{1.36}$$

Now, instead, consider the case in which the pressure distribution is not isotropic. We consider the fluid element at r with length along the radial axis dr and angular width $d\phi$ which has a force due to some radial pressure on its radial faces and a separate force due to some transverse pressure on its transverse faces.

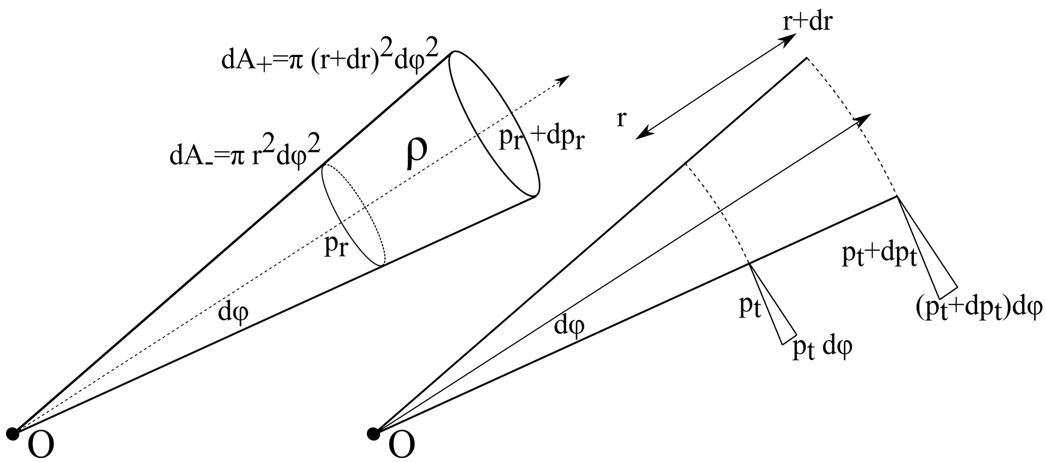

Figure 1.2: Consider Newtonian hydrostatic equilibrium for a fluid in which the pressure on the transverse faces of a fluid element differs from the pressures on the faces intersected by the radial line.

The inner radial face has area $dA_- = \pi r^2 d\phi^2$ and the outer radial face has area $dA_+ = \pi(r+dr)^2 d\phi^2$. The radial pressure $p_r(r)$ at r results in a force $\vec{F}_{p_r-} = p_r dA_- \hat{r}$ on the inner face and a force $\vec{F}_{p_r+} = -(p_r + dp_r)dA_+\hat{r}$ on the outer face. There exists a gravitational force on the mass ρdV toward the interior mass $m(r)$. To lowest order, the volume of the fluid element dV is $\pi r^2 dr d\phi^2$. The gravitational force is $\vec{F}_g = \frac{-G\rho m(r)dV}{r^2}\hat{r}$. Notice that the faces intersected by the radial line are only approximately circular. In fact, they are spherical caps. To compute the contribution of the transverse pressure to the hydrostatic equilibrium condition, one must recognize that the force due to p_t acting inwards on the transverse faces has a non-zero component in the radial direction. The transverse pressure results in a force in the outward radial direction $\vec{F}_{p_t} = p_t d\phi dA_{sides}$, where $dA_{sides} = 2\pi r dr d\phi$ to lowest order.

Hydrostatic equilibrium implies that the sum of all the forces acting on a small fluid element is equal to zero. We thus obtain the Newtonian hydrostatic equilibrium equation for a sphere of anisotropic fluid as

$$\frac{dp_r}{dr} + \frac{2}{r}\left(p_r - p_t\right) + \frac{m\rho}{r^2} = 0. \tag{1.37}$$

1.3.1 The Energy Momentum Tensor

For a general matter distribution in the context of general relativity, specification of the energy momentum tensor is a necessary step toward understanding the geometry and dynamics of a fluid. If u^α represents the velocity field of some arbitrary continuous medium, with flow-lines being parametrized by some proper time variable λ,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{1.38}$$

$$u^\alpha = \frac{dx^\alpha}{d\lambda}. \tag{1.39}$$

Consider first a form of distributed matter called 'dust'. At every point of a distribution of dust, it is possible to erect a locally inertial frame in which the particles in a given volume are at rest. For N non-interacting particles with rest mass m in a box with volume V_0 at rest in the locally inertial frame S', the number density of particles in the box is $n_0 = N/V_0$. Since each particle in the box is at rest, the total energy density is $\rho_0 = n_0 m$. In another frame S in which the box and it's contents are moving with velocity $\vec{v}$, the volume of the box appears smaller ($V = V_0\sqrt{1-v^2}$) since the box length is Lorentz contracted in the direction of motion. Then in S, the number density is $n = N/V = n_0/\sqrt{1-v^2}$. The fluid four velocity in S can be written as

$$u^\alpha = u^t(1, \vec{v}). \tag{1.40}$$

Therefore in S, the number density can be written as

$$n = n_0 u^t. \tag{1.41}$$

Each particle in S has energy defined by the t-component of the momentum four-vector $p^t = mu^t$ [2], so the total energy density can be written as

$$\rho = np^t = n_0 u^t m u^t = \rho_0 u^t u^t. \tag{1.42}$$

The quantity ρ_0 is a frame independent scalar defined by measurement of the local energy density in the locally inertial frame which is at rest with respect to the distribution of dust. The energy density as measured by a locally inertial frame moving relative to the dust rest frame can be interpreted as the $t{-}t$ component of a symmetric rank-two tensor. The energy momentum tensor for dust is

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu. \tag{1.43}$$

1.3.2 Perfect Fluids

In general $T_{00}(q)$, by construction, describes the energy density measured at a point q, which in the 'co-moving' coordinate system (i.e. the system in which $u^\alpha = \delta^\alpha_0$) can be written $T_{00}(q) \underset{*}{=} \rho$. Notice that in these coordinates, then $T_{\alpha\beta}u^\alpha u^\beta \underset{*}{=} T_{00}$. The resulting scalar expression is true in all coordinate systems:

$$T_{\alpha\beta}u^{\alpha}u^{\beta} = \rho \tag{1.44}$$

One can imagine a slightly more complicated fluid composed of individual particles which exhibit, for example, completely random thermal motions even in a frame where the fluid as a whole in a given neighborhood is at rest. Consider such a fluid in which the constituent particles are non-interacting, called a "perfect fluid". The components of the energy momentum tensor, T^i_0, $i = \{1, 2, 3\}$, comprise a vector describing energy flux through a surface element orthogonal to the direction of flow. In a perfect fluid, the only transport of energy is due to mass flow and thus, in co-moving coordinates $T^i_0 \stackrel{*}{=} T^i_{\alpha}u^{\alpha} \stackrel{*}{=} 0$ (where $\stackrel{*}{=}$ denotes equivalence only in a particular coordinate system). Then $T^{\beta}_{\alpha}u^{\alpha} \stackrel{*}{=} Cu^{\beta}$ where C is some unspecified coefficient. It is a straightforward exercise to show that $C = \rho$ and obtain the tensor equation

$$T^{\alpha}_{\beta}u^{\beta} = \rho u^{\alpha}. \tag{1.45}$$

Consider a particle at some point q in our space-time. The set of all four vectors v^{α} orthogonal to the velocity field at q constitutes a 3-volume co-moving with the particle that was at q. Perfect fluids are characterized by isotropic pressure distributions. Then, in this co-moving volume element, a pressure p exerted on a surface element with area σ results in a force $f = \sigma p$ normal to σ in some direction n^i, $i = \{1, 2, 3\}$. The components T^i_j of the energy momentum tensor form the standard stress tensor. Note that

$$T^i_j \sigma n^j = \sigma p n^i. \tag{1.46}$$

Thus each arbitrary vector n^i in the co-moving subspace orthogonal to the velocity field is an eigenvector of T^i_j, with an eigenvalue p.

$$T^i_j = p\delta^i_j \tag{1.47}$$

$\forall$ points q in the manifold $\mathcal{M}$ it is possible to choose a subspace of $T_q(\mathcal{M})$ orthogonal to the velocity field, and erect an orthogonal basis in which the previous expression is valid.

A set of four independent vector fields spanning $T_q(\mathcal{M})$ is a tetrad. Choose an orthonormal tetrad e^α_i, $i = \{0,1,2,3\}$ such that $e^\alpha_0 = u^\alpha$ and e^α_k, $k = \{1,2,3\}$, represent eigenvectors of the stress tensor. Now

$$T_{ij} = e^\alpha_i e^\beta_j T_{\alpha\beta} \tag{1.48}$$

and inversely

$$T_{\alpha\beta} = e^i_\alpha e^j_\beta T_{ij}. \tag{1.49}$$

We are now in a position to write out the stress energy tensor of a perfect fluid in the pleasing form

$$T^{\alpha\beta} = (p+\rho)\, u^\alpha u^\beta + p g^{\alpha\beta} \tag{1.50}$$

where p and ρ are scalars which we call the pressure and energy density as measured by a locally inertial co-moving observer.

Using the general static spherically symmetric metric, applying the $\nabla_\alpha T^\alpha_r = 0$ conservation equation, and solving the $r-r$ Einstein equation for $\Phi'(r)$, one obtains an important relationship referred to as the Tolman-Oppenheimer-Volkov (TOV) equation, discussed as early as 1925 in the doctoral thesis of Georges Lemaitre [12].

$$\frac{d\Phi}{dr} = \frac{dp}{dr}\left(\frac{1}{\rho + p}\right) = -\frac{m + 4\pi p r^3}{r(r - 2m)} \tag{1.51}$$

In order to integrate the system and explicitly obtain relevant functions, additional information in the form of, e.g., a heuristic assumption or an equation of state is required. It can be shown that specification of a single 'generating function' (e.g. the metric function Φ) can yield infinitely many spherically symmetric perfect fluid solutions [5]. In examining static spherically symmetric perfect fluid distributions, the problem of solving Einstein's field equations reduces to the problem of finding a consistent and physically reasonable solution to the TOV equation.

From the TOV equation, one may assert that the pressure has a local maximum at the origin. If $\exists$ a smooth barotropic function $p(\rho)$ then

$$\frac{dp}{dr} = \frac{dp}{d\rho}\frac{d\rho}{dr}. \tag{1.52}$$

The function $p(\rho)$ is a barotropic equation of state relating the radial pressure and the energy density. Later in the analysis, we will assume a general linear barotropic equation of state as a method of relating the principal stresses of the energy momentum tensor in more general fluids.

1.4 The Junction Conditions Between Spacetimes

In order to understand the boundary between a stellar object and the region exterior to it, it is necessary to formulate a robust theory of spacetime junction conditions. This is particularly important as the coordinates used in the exterior and interior regions are not necessarily continuous. Suppose we want to relate the coordinates in two regions $\mathcal{M}^-$ and $\mathcal{M}^+$, referred to as the interior and exterior spacetimes respectively. Let the boundary between these surfaces be described by a timelike 3-surface denoted Σ. Let the coordinates intrinsic to Σ be denoted ζ^i with ${}^{\pm}x^\alpha(\zeta^i)$ representing the coordinates in $\mathcal{M}^\pm$ at the hypersurface Σ. The three metric (first fundamental form) induced on Σ is

$$g_{ij} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \zeta^i}\frac{\partial x^\beta}{\partial \zeta^j}. \tag{1.53}$$

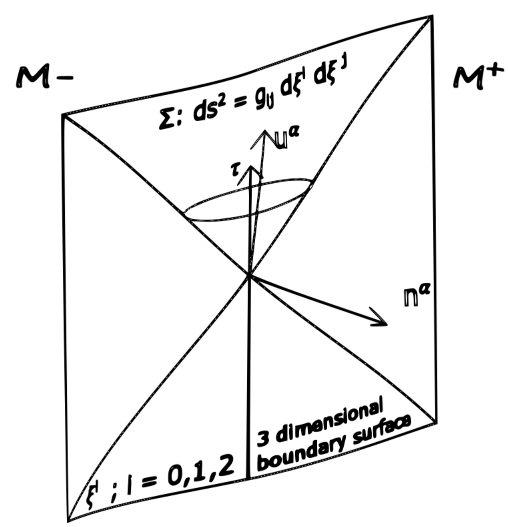

Figure 1.3: The junction between two 4-D space times $\mathcal{M}^\pm$ through some 3-D hypersurface of discontinuity Σ imposes the Darmois-Israel conditions on the first and second fundamental forms.

With the notation that

$$[A] \equiv A|^{+}_{\Sigma} - A|^{-}_{\Sigma} \tag{1.54}$$

we state the first junction condition, the continuity of the first fundamental form as

$$[g_{ij}] = 0. \tag{1.55}$$

Thus, the metric induced on Σ by both $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$ must be equivalent.

The extrinsic curvature, referred to as the second fundamental form, is defined as

$$K_{ij} = \nabla_{\beta} n_{\alpha} \frac{\partial x^{\alpha}}{\partial \zeta^{i}} \frac{\partial x^{\beta}}{\partial \zeta^{j}} \tag{1.56}$$

where n_{α} is a unit spacelike vector normal to Σ pointing from the interior to the exterior spacetime. The second junction condition is the continuity of the second fundamental form.

$$[K_{ij}] = 0 \tag{1.57}$$

Together (1.55) and (1.57) form what are known as the Darmois-Israel junction conditions [13]. Any timelike surface Σ which satisfies the junction conditions is called a boundary surface.

1.4.1 Junction Between Spherically Symmetric Static Spacetime and a Vacuum Exterior

As a brief example of a junction between two spacetimes, consider the matching, by way of a timelike boundary hypersurface Σ, of an interior static spherically symmetric fluid distribution with the exterior Schwarzschild solution. For the interior metric we have

$$ds_-^2 = -e^{2\Phi(\mathrm{r})}dT^2 + \left(1 - \frac{2m(\mathrm{r})}{\mathrm{r}}\right)^{-1} d\mathrm{r}^2 + \mathrm{r}^2 \mathrm{d}\Omega^2. \tag{1.58}$$

Note that the metric intrinsic to any three dimensional spherically symmetric timelike surface can be written as

$$ds_\Sigma^2 = -d\tau^2 + R^2(\tau)d\Omega^2, \tag{1.59}$$

where τ is the proper time on Σ. The exterior metric is given by Schwarzschild (notice that we are using separate coordinates r, r, t and T)

$$ds_+^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.60}$$

Without loss of generality, the coordinates θ and ϕ have been chosen to be continuous through Σ. At Σ, $\mathrm{r} = \mathrm{r}_\Sigma$ and $r = r_\Sigma$.

$$[g_{\theta\theta}] = [g_{\phi\phi}] = 0 \implies r_\Sigma = \mathrm{r}_\Sigma = R \tag{1.61}$$

Examining timelike trajectories on the boundary, one obtains the condition

$$d\tau^2 = (1 - \frac{2m}{r_\Sigma})dt^2 = e^{2\Phi(\mathrm{r}_\Sigma)}dT^2 \tag{1.62}$$

In the Schwarzschild region (note that $(x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi)$,

$$u^\alpha = \left(\dot{t}, 0, 0, 0\right) \tag{1.63}$$

and thus (since $u^\alpha n_\alpha = 0$ and $n^\alpha n_\alpha = 1$), we find

$$n_\alpha = \left(0, \frac{1}{\sqrt{1-\frac{2M}{r_\Sigma}}}, 0, 0\right) \tag{1.64}$$

Similarly, in the interior region we find

$$u^\alpha = \left(\dot{T}, 0, 0, 0\right), \tag{1.65}$$

$$n_\alpha = \left(0, \frac{1}{\sqrt{1-\frac{2m(\mathrm{r}_\Sigma)}{\mathrm{r}_\Sigma}}}, 0, 0\right). \tag{1.66}$$

Since the spacelike normal n_α only has r or r components, the second fundamental form becomes

$$K_{ij} = -n_\gamma \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial \zeta^i} \frac{\partial x^\beta}{\partial \zeta^j}. \tag{1.67}$$

It now becomes an algorithmic procedure to determine the implications of the continuity of the second fundamental form. With the Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$ calculated using any appropriate software, the condition $[K_{ij}] = 0$ implies that $m(r_\Sigma) = m$ and

$$\Phi'|_\Sigma = \frac{m}{r_\Sigma(r_\Sigma - 2m)} \tag{1.68}$$

which, upon comparison with the TOV equation, implies that the pressure at the boundary surface goes to zero.

$$p(r_\Sigma) = 0 \tag{1.69}$$

Note that the TOV equation can be derived from the r-component of the divergence of the stress energy momentum tensor. It follows that, even if the principal stresses of T^α_β are not equivalent (e.g. an 'anisotropic fluid'), the condition that the radial pressure p vanishes at the boundary surface Σ still holds. In this case we do not require that the (diagonalized) components of the stress energy momentum tensor other than T^r_r vanish at $r = R$. In general the energy density is discontinuous across the boundary.

1.5 On the Physical Acceptability of Fluid Solutions to the Field Equations

A study published in 1998 by Delgaty and Lake [14] addresses the problem of whether a given solution to the field equations can be considered physically reasonable. Out of 127 static, spherically symmetric, perfect fluid (SSSPF) candidate solutions, only 9 passed the authors' criteria for physical acceptability. It is important to understand that one can always supply a metric tensor and solve for the stress energy tensor, however, this method most often yields unphysical fluid distributions.

Thus, in order to characterize the solutions which make physical sense, the authors propose a robust list of criteria which the solutions must satisfy:

1. For a solution to represent a perfect fluid, the pressure must be isotropic, which constrains the elements of the Einstein tensor (i.e. $G^r_r = G^\theta_\theta$). Note that the assumption of spherical symmetry necessitates $G^\theta_\theta = G^\phi_\phi$.

   In spherical symmetry the constraint $G^r_r = G^\theta_\theta$ restricts us to scenarios in which the principal stresses of the energy momentum tensor are equivalent. This condition will be abandoned later in this analysis when we consider anisotropic fluids.

2. The solutions must be regular at the origin. In the case of the static spherically symmetric metric, this implies the origin is conformally flat, and thus the only non-zero invariants there are those of Ricci. The Ricci invariants can be expressed at the origin in terms of the metric functions, and used to express the pressure and energy density there.

   For a SSSPF distribution there are two independent Ricci invariants.

$$R = 8\pi(\rho - 3p) \tag{1.70}$$

$$r_1 = 3(2\pi(\rho + p))^2 \tag{1.71}$$

   and only one independent Weyl invariant

$$w_1 = 48\pi \left(\frac{m}{r^2} - \frac{4\pi\rho}{3} \right)^2 . \tag{1.72}$$

   Regularity of the Ricci invariants at the origin imposes the condition that ρ and p must be finite at $r = 0$.

3. Positive energy density and pressure at the origin.

   Despite some recent theoretical investigations of exotic objects possessing regions of negative pressure (e.g. gravastars [15][16]) we restrict our study to positive definite pressure and energy densities.

4. There must be some positive finite radius $r \equiv R$ at which the pressure reduces to zero, and the solution is matched by way of a boundary surface on to a Schwarzschild exterior.

This condition ensures that the solution is not cosmological, i.e. that the fluid distribution is confined within some region Σ. This condition can also be extended to junction an arbitrary fluid on to Schwarzschild de Sitter (SdS). Carrying out the same procedure as in the last section except with an SdS exterior, one obtains the same result that $p(R) = 0$, which fixes the radius of the fluid distribution.

5. The pressure and energy density are required to be monotonically decreasing in the outward radial direction to the boundary.

   Since we have a positive maximal pressure at the origin, and at some finite radius the pressure decreases to zero, the monotonicity condition on p is superfluous. Consider some $r < R$ at which point $p'(r) = 0$. The TOV equation indicates that this requires $p < 0$ which is ruled out by previous criteria.

6. A last condition is that the quantity $dp/d\rho$, referred to as the sound speed, is less than unity.

   Every SSSPF spacetime admits a barotropic equation of state $p(\rho)$ regardless of whether this form can be obtained explicitly. Consider a perturbed spacetime with a sound wave propagating through it. Now, take the pressure to be some function of energy density but also of some 'perturbed fluid variables', say ψ_i. Then, we may write

$$\delta p = \frac{\partial p}{\partial \rho}\delta p + \sum_i \frac{\partial p}{\partial \psi_i}\delta\psi_i. \tag{1.73}$$

   If variations in the pressure do not result in variations of the perturbed fluid variables (e.g. the entropy per particle remains fixed) the process can be considered isentropic, and one obtains the isentropic sound speed $dp/d\rho$.

There is more to be said regarding whether the stress-energy tensor can be considered physically reasonable [8, 15, 17]. Assume that at every point the stress energy tensor can be represented using a set of orthonormal basis vectors.

$$T^{\alpha\beta} = \rho \hat{e}_0^\alpha \hat{e}_0^\beta + p_1 \hat{e}_1^\alpha \hat{e}_1^\beta + p_2 \hat{e}_2^\alpha \hat{e}_2^\beta + p_3 \hat{e}_3^\alpha \hat{e}_3^\beta \tag{1.74}$$

$$g_{\alpha\beta} \hat{e}_\mu^\alpha \hat{e}_\nu^\beta = \eta_{\mu\nu} \tag{1.75}$$

The energy density ρ and the principal stresses represent eigenvalues of the stress energy tensor corresponding to the basis eigenvectors $\hat{e}_\mu^\alpha$. For example, an anisotropic fluid admits the decomposition

$$T^{\alpha\beta} = (\rho + p_t) \hat{e}_0^\alpha \hat{e}_0^\beta + (p_r - p_t) \hat{e}_1^\alpha \hat{e}_1^\beta + p_t g^{\alpha\beta}. \tag{1.76}$$

Furthermore, we can represent arbitrary future directed timelike vectors in terms of the basis,

$$u^\alpha = \gamma(\hat{e}_0^\alpha + a\hat{e}_1^\alpha + b\hat{e}_2^\alpha + c\hat{e}_3^\alpha) \tag{1.77}$$

where a, b and c are unspecified functions of the coordinates which are subject to the condition

$$a^2 + b^2 + c^2 < 1, \tag{1.78}$$

and $\gamma \equiv (1 - a^2 - b^2 - c^2)^{-1/2}$.

It is reasonable to suspect that the energy density as measured by an arbitrary observer must be positive [10]. Thus we may state for an arbitrary observer with

4-velocity u^α,

$$T_{\alpha\beta}u^\alpha u^\beta \geq 0. \tag{1.79}$$

For a completely arbitrary fluid, at the point q, this implies the 'weak energy conditions' (WEC),

$$\rho \geq 0, \quad \text{and} \quad \rho + p_i > 0. \tag{1.80}$$

One can replace u^α with an arbitrary future directed null vector and obtain the 'null energy condition' (NEC), which it can be shown is implied by the WEC. The 'dominant energy condition' (DEC) is obtained by also imposing the condition that the local energy flow vector $T^{\alpha\beta}u_\alpha$ is non-spacelike. This implies that in any orthonormal basis, the energy density 'dominates' the other components of $T^{\alpha\beta}$, i.e.

$$T^{00} \geq \left|T^{\alpha\beta}\right|. \tag{1.81}$$

Thus the DEC is simply the WEC with the added condition that individually, the principal stresses should not exceed the energy density.

1.6 Constraining the Tenuity of Fluid Spheres

A question of considerable importance in the theoretical investigation of potential stellar models concerns the parameter referred to herein as the tenuity. The tenuity α is defined as the ratio of the total mass of a stellar body $M \equiv m(r_\Sigma)$ to its radius $r_\Sigma \equiv R$.

$$\alpha \equiv \frac{2M}{R} \tag{1.82}$$

Spherical symmetry implies the existence of canonical coordinates in which the stress energy tensor is diagonalizable. Consider that the principal stresses of the stress energy tensor are not equal. In this case we write the rank (1,1) tensor as

$$T^{\mu}_{\nu} = \mathrm{diag}[-\rho(r), p_r(r), p_t(r), p_t(r)] \tag{1.83}$$

which, without specification of the functions ρ, p_r and p_t is completely general. We can subsequently write the field equations (where here, for simplicity Λ is neglected).

$$8\pi\rho = \frac{2m'}{r^2} \tag{1.84}$$

$$8\pi p_r = -\frac{2}{r^3}\left(\Phi' r(2m - r) + m\right) \tag{1.85}$$

$$8\pi p_t = -\frac{1}{r^3}\left(r^2(\Phi'^2 + \Phi'')(2m - r) + (\Phi' r + 1)(m' r - m) - \Phi' r^2\right) \tag{1.86}$$

The quantity $2m(r)/r$ plays a fundamental role in understanding the geometry of a spherically symmetric static spacetime. For example, bounding the tenuity has immediate observational implications as it directly limits the maximum red shift of particles in the field of a self-gravitating object. It is a well known result due to pioneering work of Schwarzschild [2] and, decades later, Buchdahl [18], that a spherically symmetric perfect fluid with a monotone decreasing energy density profile must have $\alpha \leq 8/9$ (the value found by Schwarzschild in his analysis of the constant density incompressible fluid).

The Birkhoff theorem states that the gravitational field exterior to a spherical matter distribution is described via Schwarzschild regardless whether the matter distribution is 'endowed with radial motion'[9].

The coordinate singularity at the Schwarzschild limit ($r = 2M \equiv R_s$) challenges our interpretation of the spacelike and timelike character of the coordinates t and r. If we consider a matter distribution responsible for the gravitational field, which is confined to some $R < R_s$, then the character of the coordinates dt and dr must change across the Schwarzschild limit. However the theorem due to Buchdahl [18] states that for a sphere of fluid in hydrostatic equilibrium with some isotropic pressure and a monotonically decreasing energy density profile this problem is of no concern since

$$R \geq \frac{9}{8} R_s. \tag{1.87}$$

As pointed out by several authors, e.g. [1], the assumptions made by Buchdahl in generating this inequality are very restrictive and do not hold, for example, for a simple soap bubble which has a film possessing some surface tension. The programme due to Buchdahl neglects solutions for which ρ is kept general, for example the important case of thin matter shells. Furthermore, the Schwarzschild interior which maximizes $2M/R$ only saturates the bound when the radial pressure goes to infinity resulting in an unphysical violation of the dominant energy condition. Thus the inequality due to Buchdahl is not sharp in the sense that if we also impose that the DEC is satisfied, it may be the case that for the very restrictive class of solutions considered, $2M/R$ is bounded by some value even less than $8/9$.

Thus, an important question arises regarding the tenuity of fluid distributions which are not so restrictive. For example, one may consider whether it is possible to constrain the tenuity for a more general fluid in hydrostatic equilbrium. A step in this direction is due to a recent theorem by Andreasson [1] which concerns *all* static spherically symmetric solutions to the field equations for which the energy density $\rho \geq 0$, the radial pressure $p_r \geq 0$ and the tangential pressure $p_t \geq 0$ satisfy the equation of state $p_r + 2p_t \leq \Omega\rho$, for constant $\Omega > 0$ stating that

$$\alpha \leq \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2}. \tag{1.88}$$

Consider an anisotropic fluid with the slightly more general equation of state

$$p_r + \chi p_t = \Omega\rho. \tag{1.89}$$

The field equations in curvature coordinates provide relationships between the physical variables and the metric functions as

$$p_r = \frac{-1}{4\pi r^3}\left(2\Phi' m r - \Phi' r^2 + m\right), \tag{1.90}$$

$$p_t = \frac{-1}{8\pi r^3}\left(2\Phi'^2 r^2 m - \Phi'^2 r^3 + \Phi' m' r^2 + 2\Phi'' r^2 m - \Phi'' r^3 + \Phi' m r - \Phi' r^2 + m' r - m\right), \tag{1.91}$$

$$\rho = \frac{m'}{4\pi r^2}. \tag{1.92}$$

Substituting in to the equation of state and simplifying yields the relation

$$\begin{aligned} m' =& \frac{1}{r(\chi + \chi r\Phi' + 2\Omega)}(r^2(\chi r\Phi'' + \Phi'(\chi + \chi r\Phi' + 2)) \\ &- m(-\chi + r(2\chi r\Phi'' + \Phi'(\chi + 2\chi r\Phi' + 4)) + 2)) \end{aligned} \tag{1.93}$$

Note that we have

$$\Phi' = \frac{m + 4\pi r^3 p_r}{r(r-2m)} \tag{1.94}$$

and thus

$$\Phi'' = \frac{1}{r^2(r-2m)^2}\Big[r^2\left(4\pi r^2\left(2p_r m' + r p_r' + p_r\right) + m'\right) \\ - 2m\left(4\pi r^3\left(r p_r' + 2p_r\right) + r\right) + 2m^2\Big] \tag{1.95}$$

Subbing Φ' and Φ'' in to the expression for $m'(r)$ and examining the surface $r = R$, we can solve for the radial pressure gradient and obtain the result

$$p_r'(R) = \frac{m'(R)(2R\Omega - M\chi - 4M\Omega)}{4\pi\chi R^3(R-2M)}. \tag{1.96}$$

Regardless whether the pressure distribution inside the fluid sphere we are modelling is monotonically decreasing outward, by definition the surface $r = R$ is where the radial pressure vanishes, so since we assert the non-negativity of pressure it follows that $p_r'(R) < 0$. Thus the assertion that $m'(R) > 0$, in conjunction with the above result implies a lower bound on the tenuity.

$$2R\Omega - M\chi - 4M\Omega < 0 \implies \frac{4\Omega}{4\Omega + \chi} < \frac{2M}{R} \tag{1.97}$$

With the parameter $\chi = 2$, using the Andreasson result, the tenuity must lie in the range

$$\frac{2\Omega}{1+2\Omega} < \frac{2M}{R} \leq \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2}. \tag{1.98}$$

Andreasson's method for deriving the upper bound on the tenuity requires sophisticated techniques from Lebesgue integration theory and is advantageous in that it provides a unique solution which saturates the boundary.

In chapter 3 we employ a different, arguably more intuitive method to constrain tenuity and obtain more generalized bounds which encompass both Buchdahl's and

Andreasson's inequalities but with less restrictive initial assumptions.

1.7 Lagrangian Analysis for Geodesics in Spherically Symmetric Static Geometries

Along a geodesic curve C affinely parametrized by λ with tangent $u^\alpha = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ the scalar quantity $2L = u^\alpha u_\alpha$ (the Lagrangian $L = L(x^\alpha, \dot{x}^\alpha, \lambda)$) is fixed. The line element

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{1}{1 - \frac{2m(r)}{r}}dr^2 + r^2 d\Omega^2, \tag{1.99}$$

can be used to construct the Lagrangian

$$2L = -e^{2\Phi(r)}\dot{t}^2 + \frac{1}{1 - \frac{2m(r)}{r}}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin(\theta)^2\dot{\phi}^2. \tag{1.100}$$

The curve C passing through two points, say p_1 and p_2, minimizes the functional

$$S(C) = \int_{p_1}^{p_2} L d\lambda \tag{1.101}$$

and is thus geodesic, if and only if along C the Lagrangian obeys the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha}. \tag{1.102}$$

It is instructive to examine these equations to find quantities which may be conserved along C (so-called "constants of the motion") which simplify our analysis. Consideration of θ yields the expression

$$\frac{d}{d\lambda}\left(r^2\dot{\theta}\right) = r^2 \sin(\theta)\cos(\theta) \tag{1.103}$$

which has the solution $\theta = \pi/2$ implying that all orbits are planar. Since the Lagrangian is independent of ϕ, associated with $\dot{\phi}$ is a constant of the motion l which we refer to as the angular momentum.

$$\frac{d}{d\lambda}\left(r^2\dot{\phi}\right) = 0 \implies l \equiv r^2\dot{\phi} \tag{1.104}$$

Similarly, since the metric is static, we find a constant of the motion γ associated with $\dot{t}$, which we refer to as the energy.

$$\frac{d}{d\lambda}\left(-e^{2\Phi(r)}\dot{t}\right) = 0 \implies \gamma \equiv e^{2\Phi(r)}\dot{t} \tag{1.105}$$

The Lagrangian can be rewritten in terms of l and γ

$$2L = -\frac{\gamma^2}{e^{2\Phi(r)}} + \frac{\dot{r}^2}{1 - \frac{2m(r)}{r}} + \frac{l^2}{r^2}. \tag{1.106}$$

It is always possible to normalize u^α such that $2L$ is ± 1 or 0. Thus to investigate timelike trajectories, consider $2L = -1$. Furthermore, we can examine circular orbits by setting $\dot{r} = \ddot{r} = 0$, or radial geodesics by setting $l = 0$. Thus Lagrangian analysis presents a significant tool for examining orbits in the context of general relativity.

For a timelike trajectory with $\dot{r} = \ddot{r} = 0$ one can verify that the geodesic equation

$$\ddot{r} + \Gamma^r_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \tag{1.107}$$

implies that the constants of the motion l and γ are constrained via

$$\gamma = \frac{e^{\Phi(r)}}{\sqrt{1 - r\Phi'(r)}} \tag{1.108}$$

and

$$l = \frac{r\sqrt{r\Phi'(r)}}{\sqrt{1 - r\Phi'(r)}}. \tag{1.109}$$

It can be shown (see e.g. [19]) that for these timelike circular orbits to be stable under small radial perturbations requires that

$$3\Phi' + r\Phi'' > 2r\Phi'^2. \tag{1.110}$$

Chapter 2

Motivation for the Study of Anisotropic Fluids

In this section, the goal is to motivate the study of stress energy momentum tensors in spherical symmetry which deviate from the perfect fluid prescription. From a purely mathematical perspective, an anisotropic stress energy tensor is interesting because it presents a new challenge in that the principal stresses of the energy-momentum tensor are no longer assumed equal. Physically, however, we wish to investigate where an anisotropic fluid prescription more accurately models a given situation, or how the introduction of local anisotropy affects a physical system. As an example, it was explicitly established in [1] that the tenuity of a static fluid model is altered by the relaxation of the pressure isotropy constraint.

A secondary goal of this section of the thesis is to examine some of the present archive of general relativistic fluid research. In particular we will examine historically significant works due to authors such as Tolman, Lemaitre and Einstein to give some supposed insight in to those authors' motivations for studying the subject. Obviously a major goal of studying relativistic fluids is finding direct application in the modelling of celestial bodies.

The difficulty of finding analytic solutions to the field equations of general relativity is well known. The non-linearity of the equations provides further difficulty in that the construction of new solutions via superposition of previously obtained ones is not permitted. Even for the simple case of spherically symmetric static perfect fluids, the first two decades after the establishment of relativity saw only two well known analytic solutions [20]. At the time of Tolman's highly regarded 1939 publication on static perfect fluids, Einstein's cosmological solution with constant pressure and density, and Schwarzschild's interior solution for a constant density fluid with monotonically decreasing pressure profile were the only fluid solutions which had entered the mainstream of physics. In the decade prior to this, however, Lemaitre had obtained explicit solutions with Λ for radial pressure-free distributions of matter "each concentric layer of which supports its own weight by purely transverse stresses" [20, 21]. Tolman was aware of the work due to Lemaitre and regarded it as interesting, but relegated discussion of it to a brief footnote outside the main body of his publication.

At the time (and somewhat still at present) the inclusion of the cosmological constant in fluid research was largely neglected except in works due to physicists such as Lemaitre and de Sitter.

The 1939 work due to Tolman treated perfect fluids with Λ in a method which yielded the previously mentioned solutions as well as "a number of others which may have a measure of physical interest" [20]. Tolman uses the field equations assuming a static spherically symmetric perfect fluid and remarks that it is advantageous to specify relations between the metric functions to simplify the field equations rather than providing an equation of state relating the pressure and energy density. Tolman then demonstrates the power of his method by specifying simple relationships between metric functions which yield the Einstein universe, the Schwarzschild de Sitter solution, the Schwarzschild interior solution as well as five other solutions.

Solution IV of Tolman's paper is of particular interest as it represents the first compressible fluid solution which has a monotonically decreasing radial pressure profile. With the metric given as

$$ds^2 = -e^{2\Phi}dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.1}$$

the solution imposes the constraint that the term $e^{2\Phi}\Phi'/r$ is constant implying that

$$e^{2\Phi} = B^2(1 + \frac{r^2}{A^2}), \tag{2.2}$$

where A and B are constants. Tolman's solution IV has three free parameters which alter the structure of the solutions. In particular, the relative size of these parameters determine the central pressure p_c and density ρ_c and the radius r_b at which $p = 0$ as well as the energy density at this boundary radius.

As previously mentioned, earlier in that decade the Belgian Jesuit Georges Lemaitre was also making considerable contributions to the study of relativistic fluids. In a 1933 paper, translated in 1997 by Malcolm MacCallum [22], the general TOV equation for a spherically symmetric system with transverse stresses was developed and used to analyze quasi-static uniform energy density spheres, generalizing the Schwarzschild incompressible fluid and the Einstein universe. Lemaitre's work in this vein resulted in other interesting solutions including his "progressive loading of the universe" model [22].

Another 1939 publication of interest is due to the father of relativity theory himself [23]. Einstein was unsatisfied with the notion of a singular region produced by a self gravitating mass. His comments in this 1939 work criticize the Schwarzschild incompressible fluid as an unphysical distribution since "elastic waves would have to travel with infinite velocity". As such, Einstein was uninterested that for a given energy density the incompressible fluid model predicts a coordinate singularity in the metric for distributions with a sufficiently large radius. Einstein introduces a model which he describes as resembling an idealized spherical star cluster, in which a random distribution of non-interacting particles orbit on circular geodesics around a central origin. The purpose of the paper was to demonstrate that for physically realistic matter distributions, the Schwarzschild horizon can not exist. It is a revival of the ingenious ideas in Einstein's paper due to Lake [24] which is of great relevance in motivating studies of non-perfect fluid distributions.

Consider a spherically symmetric static space-time, the metric of which is generated by a distribution of non-interacting particles with rest mass m and 4-momentum p_α which orbit on stable circular timelike geodesics about the origin. For a local number density of particles n, the energy momentum tensor at some spatial point q is given by [23, 24]

$$T^\alpha_\beta|_q \equiv \frac{n}{m} < p^\alpha p_\beta >, \tag{2.3}$$

where the angled brackets represent an average at q, taken over all the trajectories passing through q. The total angular momentum ($L^2 \equiv p_\theta^2 + (\sin\theta)^{-2} p_\phi^2$) is constant along each trajectory through q. A further assumption that *all* trajectories through

q have the same angular momentum yields the important results that

$$< p_\theta >=< p_\phi >=< p_\theta p_\phi >= 0 \tag{2.4}$$

and

$$< p_\theta^2 >= \frac{< p_\phi^2 >}{\sin^2 \theta_q} = \frac{L^2}{2}. \tag{2.5}$$

Thus for a continuous distribution comprised of non-interacting particles on orbits with a variety of radii r, the angular momentum distribution $L = L(r)$ and number density $n = n(r)$ yield the following non-zero components of the energy momentum tensor:

$$T_t^t = -mn \left(1 + \left(\frac{L}{mr}\right)^2\right), \tag{2.6}$$

$$T_\theta^\theta = T_\phi^\phi = \frac{nL^2}{2mr^2}. \tag{2.7}$$

Notice how this energy momentum distribution is manifestly 'imperfect', in that it is held together purely by the hoop stresses generated from the $\theta - \theta$, $\phi - \phi$ components of the energy momentum tensor. Since $T_r^r = 0$ for this distribution, one immediately has from the field equations that

$$M(r) = \frac{r^2 \Phi'}{1 + 2r\Phi'}, \tag{2.8}$$

where $M(r)$ and $\Phi(r)$ are those functions which appear in the metric in curvature coordinates as seen previously. Defining $\mathcal{L} \equiv \frac{L}{mr}$, the remaining field equations yield that

$$r\Phi' = \frac{\mathcal{L}^2}{1 + \mathcal{L}^2}. \tag{2.9}$$

Thus for a given angular momentum distribution, the mass distribution $M(r)$ is obtained immediately.

$$M(r) = \frac{r\mathcal{L}^2}{1 + 3\mathcal{L}^2} \tag{2.10}$$

Lastly, the trace of the energy momentum tensor provides additional constraint. Namely $T^t_t + T^\theta_\theta + T^\phi_\phi = -mn$ yields that

$$4\pi mn = \frac{\mathcal{L}(2r\mathcal{L}' + \mathcal{L}(1 + 3\mathcal{L}^2))}{r^2(1 + \mathcal{L}^2)(1 + 3\mathcal{L}^2)^2}. \tag{2.11}$$

The general relativistic potential $\Phi(r)$ can be obtained (up to quadrature) directly from observations of redshift (rotation curves) from distant galaxies [19]. Thus in this case, where $T^r_r = 0$, one immediately infers the angular momentum distribution and thus obtains the so-called number-distribution nm. It is a fact that any observed rotation curve whatsoever could be attributed to a particular angular momentum distribution of the non-interacting particles used in constructing this model.

An important article published in 1974 by Bowers and Liang [25] is acclaimed as one of the first investigations of an exact relativistic hydrostatic equilibrium solution for a locally anisotropic fluid. Bowers and Liang derive the generalized TOV expression and venture to examine how surface redshift changes for a constant density fluid due to the relaxation of local isotropy. Spherically symmetric static anisotropic distributions in relativity constitute a system of 3 equations in 5 unknowns. Thus an additional constraint is required to solve the constant density fluid system. The TOV equation for the constant density fluid takes the form

$$\frac{dp_r}{dr} = \frac{2}{r}\left(p_t - p_r\right) - \frac{4\pi r}{1 - 8\pi r^2 \rho_0/3}\left(p_r^2 + \frac{\rho_0}{3}\left(4p_r + \rho_0\right)\right). \tag{2.12}$$

The additional constraint used by Bowers and Liang is essentially cooked up to simplify the above expression. Consider the additional equation of state

$$p_t - p_r \equiv \frac{Cr^2}{1 - 8\pi r^2 \rho_0/3}\left(p_r^2 + \frac{\rho_0}{3}\left(4p_r + \rho_0\right)\right), \tag{2.13}$$

which results in a solvable first order differential equation for the radial pressure. Defining $Q \equiv \frac{1}{2} - \frac{3C}{4\pi}$ we arrive at the Bowers-Liang solution

$$p_r = \rho_0\left(\frac{(1 - 2m/r)^Q - (1 - 2M/R)^Q}{3(1 - 2M/R)^Q - (1 - 2m/r)^Q}\right). \tag{2.14}$$

The central pressure is

$$p_c = \rho_0\left(\frac{1 - (1 - 2M/R)^Q}{3(1 - 2M/R)^Q - 1}\right). \tag{2.15}$$

Notice that for a finite central pressure,

$$\frac{2M}{R} < 1 - 3^{-1/Q}. \tag{2.16}$$

The case $C = 0$ corresponds to the isotropic distribution, and the inequality above reproduces the Buchdahl result ($2M/R < 8/9$). Physically reasonable models have $M/R > 0$, which constrains $Q \geq 0$, i.e. $C \geq -\frac{2\pi}{3}$. Thus for the solution cooked up by Bowers and Liang p_r is monotone decreasing. For those solutions with $C > 0$, i.e. $p_r < p_t$, the tenuity can become arbitrarily close to unity.

In the 1980s, a Brazilian mathematician Patricio Letelier [26] investigated the following sum of two non-interacting perfect fluids in spherical symmetry,

$$T^{\mu\nu}(u_1, u_2) = t^{\mu\nu}(u_1) + t^{\mu\nu}(u_2), \tag{2.17}$$

where $t^{\mu\nu}(u_i) = (\rho_i + p_i)u_i^\mu u_i^\nu - p_i g^{\mu\nu}$, and since the fluids are non-interacting, each obeys a respective conservation law, $t^{\mu\nu}(u_i)_{;\mu} = 0$. Letelier demonstrated that by making simple transformations to the timelike flow vectors, the energy momentum tensor for a fluid composed of two non-interacting perfect fluids can be expressed in the ordinary form of an anisotropic fluid. In the appendix this work is extended to demonstrate that the sum of N non-interacting perfect fluids yields an anisotropic fluid. Thus a general understanding of anisotropic energy momentum tensors can find application in the analysis of combinations of non-interacting ideal gases.

That one can conceive of physically reasonable spherical matter distributions which are not yet adequately studied in general relativity is motivation in and of itself for studying anisotropic matter. Fluids which have unequal principal stresses represent the most general spherical masses and it has been hypothesized for half a century that highly dense matter (e.g. neutron stars) may be governed by anisotropic equations of state [27, 28]. In the following sections of the thesis we will investigate and discuss further some of those interesting features which distinguish anisotropic matter.

Chapter 3

Constraining the Tenuity of Fluid Spheres

3.1 Massive Anisotropic Spheres in General Relativity

In this section the goal is to investigate and generalize to the anisotropic case a paper published by Hermann Bondi in 1964 [29], in which the tenuity of a spherical isotropic fluid is constrained without assumption of equation of state.

The analysis by Bondi begins with the Einstein field equations for a spherically symmetric static spacetime comprised of a locally isotropic fluid distribution. We immediately impose that $T^r_r \neq T^\theta_\theta$ and have the field equations below.

$$8\pi\rho = \frac{2m'}{r^2} \tag{3.1}$$

$$8\pi p_r = \frac{-2}{r^3}\left(\Phi' r(2m-r)+m\right) \tag{3.2}$$

$$8\pi p_t = \frac{-1}{r^3}\left(r^2(\Phi'^2+\Phi'')(2m-r)+(\Phi' r+1)(m' r-m)-\Phi' r^2\right) \tag{3.3}$$

We remark here that we consider only those cases for which ρ, p_r, and p_t are everywhere non-negative finite. Furthermore, we are interested only in non-cosmological solutions and so impose the existence of a finite $r = R$ such that $p_r(R) = 0$. Following the method of Bondi, we introduce the following substitutions:

$$x = \frac{2m}{r} \tag{3.4}$$

$$y = 8\pi r^2 p_r \tag{3.5}$$

and introduce a new third variable to account for the local anisotropy which we call the anisotropy function Δ,

$$\Delta = \frac{p_t}{p_r} - 1. \tag{3.6}$$

Note that for isotropic fluids $\Delta = 0$ and we should re-obtain the results in [29]. Also note that since we require strictly non-negative radial and tangential pressures, $\Delta > -1$. Direct substitution in to the $r-r$ component of the field equations yields that

$$\Phi' r = \frac{x+y}{2(1-x)}. \tag{3.7}$$

Substitution of the above result in to the $\theta - \theta$ component yields the fundamental equation of hydrostatic equilibrium in an unfamiliar form

$$\frac{2}{r}\frac{dr}{dx} = \frac{1-x}{H(x,y,\Delta)}\left(\frac{dy}{dx} - \alpha(x,y)\right), \tag{3.8}$$

where we have defined the functions $\alpha(x, y)$ and $H(x, y, \Delta)$ as

$$\alpha = -\frac{x+y}{2(1-x)}, \tag{3.9}$$

$$H = y(1-x)(1+\Delta) - \frac{1}{4}(x+y)^2 . \tag{3.10}$$

Note that if the function $y(x)$ is given along with some equation of state, the above relation determines $x(r)$ and thus one immediately obtains the radial pressure function. In fact, all physical parameters are obtained by the specification of an equation of state and the function $y(x)$. Consider the energy density:

$$8\pi r^2 \rho = 2m' = (xr)' = x + x'r \tag{3.11}$$

We may write (as in [29]) that

$$8\pi r^2 \rho = x \left(\frac{dy/dx - \beta}{dy/dx - \alpha} \right) \tag{3.12}$$

where

$$\beta = -\frac{y}{x} \left(\frac{4 - 5x - y}{2(1-x)} + 2\Delta \right) \tag{3.13}$$

Note that the quantity β above reduces to the corresponding quantity defined in Bondi when $\Delta = 0$.

The expressions defined above in terms of the parametric quantities x, y and Δ allow us to dissect the x-y plane in to various regions of interest. The curves used to divide the plane were first studied by Bondi for the isotropic case. We will demonstrate the effect that non-zero Δ has on such curves of interest and attempt to constrain the tenuity of anisotropic matter distributions without explicitly stating an equation of state.

First, consider the family of parabolas (P) defined by the integral curves of the differential equation

$$\frac{dy}{dx} = \alpha. \tag{3.14}$$

Such curves correspond to thin mass shells, specific members of P being determined by specifying the constant C in the solution

$$y = x - 2 \pm \sqrt{C\,(1-x)}. \tag{3.15}$$

Note that the family P is unaffected by the introduction of local anisotropy. Each element of P (except $C = 0$ which corresponds to the line $y = x - 2$) is a parabola with vertex at $(x = 1, y = -1)$. We only consider the half-family $C > 0$, since those members corresponding to $C < 0$ lie outside the domain of x that is of interest $(0 \leq x < 1)$. Along the parabolas in P, the coordinate r is constant whereas the energy density is infinite.

Along the family of curves (S) in the $x - y$ plane for which

$$\frac{dy}{dx} = \beta, \tag{3.16}$$

the energy density ρ is equal to zero. β depends directly on the anisotropy function Δ, and so using β to constrain the tenuity requires specification of the function Δ, which it should be clear is a function of x and y.

Perhaps the most fundamental function used in Bondi's analysis is H. As in the isotropic case, the curve $H = 0$ divides the x-y plane in to a "core" and an "envelope".

For $\Delta = 0$, $H = 0$ is a hyperbola, however varying the anisotropy function results in many interesting shapes which we will see restrict the maximum attainable tenuity. Note that the condition $\Delta = 0$ at $x = y = 0$ restricts all $H = 0$ curves to intersect the origin.

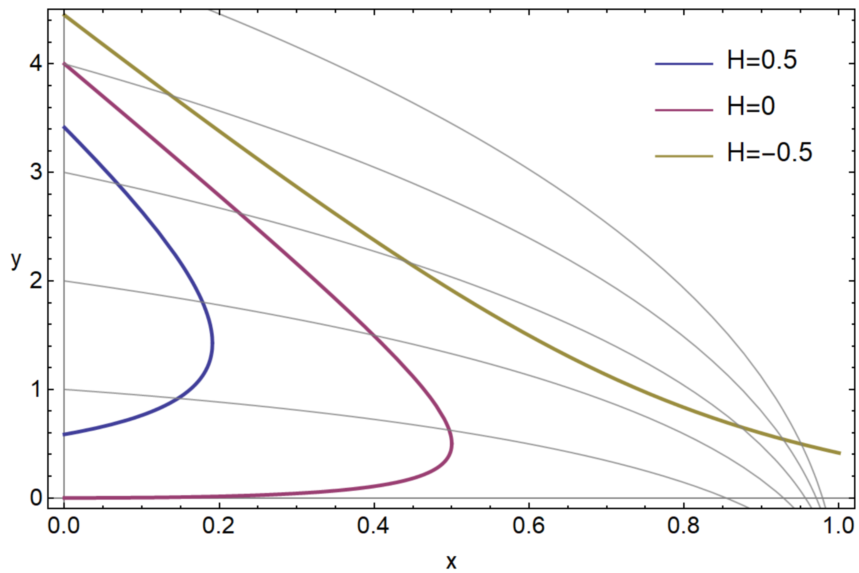

Figure 3.1: The curves $H = 0$ divide the x-y plane in to a core ($H > 0$) and an envelope ($H < 0$) in both the isotropic and anisotropic cases. A structure curve representing an equilibrium solution must tend toward integral curves of the family P for non-decreasing constant C in the core. The family P is shown in gray for $C = \{9, 16, 25, 36, 49\}$, with larger values of C intersecting the y-axis at larger values.

Let a particular equilibrium solution be denoted $y(x)$. We call $y(x)$ a "structure curve" as in [29] and note that such a curve will have the following general properties: (i) $y(0) = 0$, (ii) $y(x)$ is in the quadrant $(x, y) \geq 0$, (iii) $y(x)$ returns to zero at some $x \in (0, 1)$. Also note that if our equilibrium solution crosses an $H = 0$ curve, it must do so with a gradient $dy/dx = \alpha$ in order to avoid divergent r.

Note that the variable r is increasing outwards until at most $r = 2m$ corresponding to $x = 1$. Thus, for $x < 1$ a structure curve $y(x)$ must have $dr/dx > 0$. In the core,

$H > 0$, we therefore require that along $y(x)$, $dy/dx > \alpha$. Equivalently, we may state that in the core, a structure curve representing an equilibrium fluid sphere must tend towards integral curves of the family P for non-decreasing values of the constant C. The opposite is true of the envelope, wherein our structure curve must return to $y = 0$ intersecting integral curves of the family P for non-increasing constant C. We will see that to maximize the x at which $y(x)$ returns to zero (hereafter called x_R) the curve must intersect $H = 0$ at the highest possible value of the constant C, and descend to the x-axis along the member of P corresponding to that constant.

As in the isotropic case, at any point in the $x - y$ plane, the density is determined by gradient of $y(x)$. Also note that

$$\alpha - \beta = \frac{2H}{x(1-x)}. \tag{3.17}$$

Since we are concerned with $x < 1$, inside the core ($H > 0$), $\alpha > \beta$. Similarly, in the envelope ($H < 0$), $\alpha < \beta$. On the x-axis, $y = 0$ and thus $\beta = 0$ and $H < 0$. Thus supposing the density ρ is non-negative amounts to the assertion that $x > 0$, unless $\rho = 0$ everywhere. A lemma apparently due to Bondi (which holds in the presence of local anisotropy) is that for $x < 1$, a fluid sphere of non-negative energy density does not admit regions of negative pressure. Consider that there is some region of negative pressure (i.e. $y < 0$) and that we have imposed the constraint that there exists a boundary surface at which the pressure vanishes. Thus there exists a surface for which $y = 0$, and the above conditions hold. Moving inwards from the $y = 0$

surface one finds

$$8\pi r^2\rho = \frac{x\,(dy/dx)}{dy/dx + \left(\frac{x}{2(1-x)}\right)} = \frac{(1-x)\,dy}{x\,(-dr/r)} \geq 0 \tag{3.18}$$

We therefore find the requirement that dy be non-negative, which proves the lemma, since we are moving inwards from a surface of zero y, contrary to the assumption that the region internal to this surface had negative y.

To summarize our results thus far, we have the following constraints on the general equilibrium solution $y(x)$. The curve must lie in the $(x, y) \geq 0$ quadrant. There must exist a radius R at which $p(R) = 0$ which implies $\exists\ x_R$ such that $y(x_R) = 0$. In the region $H > 0$, $y(x)$ must tend towards members of P corresponding to non-decreasing values of the constant C since we require $dy/dx > \alpha$. The opposite is true for $H < 0$. Lastly, we require that the solution $y(x)$ passes through the curve $H = 0$ tangent to a member of P. The introduction of anisotropy Δ serves to shift the curve $H = 0$ toward or away from the y-axis depending on whether $\Delta < 0$ or $\Delta > 0$ respectively. As the anisotropy increases from zero, the curve $H = 0$ intersects the family (P) for larger values of the constant C.

The largest possible value of x_R is obtained from the equilibrium solution $y(x)$ which starts from the origin, and intersects $H = 0$ along the curve belonging to the family (P) with the largest value of the constant C. For $y(x)$ to intercept the x-axis at a maximum value, it must descend through the curves $H < 0$ along the integral curve through which it intersected $H = 0$. Clearly the highest x_R will be obtained by $y(x)$ intersecting $H = 0$ at $x = 0$. The value of y at this intersection point (y_{int}) directly depends on the anisotropy Δ.

In the case where $\Delta = 0$, we obtain the Bondi result that $y_{\text{int}} = 4$ and the maximum value of x_R is given by the x-intercept of the integral curve of $dy/dx = \alpha$ with $C = 36$ which has the explicit value:

$$x_{R,\text{isotropic}} = 12\sqrt{2} - 16 \approx 0.971 \tag{3.19}$$

In this scenario the equilibrium fluid solution consists of a core and an envelope, the envelope being represented by an integral curve from the family P and thus a thin mass shell. Within the core, the solution $y(x)$ which maximizes the constant C at which $H = 0$ is intersected is irrelevant, provided that $dy/dx > \alpha$.

The introduction of anisotropy suggests that the value of x_R for a self-gravitating locally anisotropic distribution can be arbitrarily close to zero or unity depending on how close Δ is to -1 or positive infinity. For $\Delta < 0$, the $H = 0$ curve is shifted toward the y-axis and intersects the family P for smaller values of C. When $\Delta > 0$, the curve $H = 0$ can intersect the family P for arbitrarily high values of C. To see this, consider a non-zero Δ model which intersects $H = 0$ tangent to a member of the family P. The coordinates of this point are found by solving the system

$$y = x - 2 + \sqrt{C(1-x)}, \tag{3.20}$$

$$0 = y(1-x)(1+\Delta) - \frac{1}{4}(x+y)^2. \tag{3.21}$$

If a structure curve is to intersect $H = 0$ along some P-curve corresponding to a value C, at the point of intersection Δ must take the value

$$\Delta = \frac{C + 12 - 8x - 8\sqrt{C(1-x)}}{4(x - 2 + \sqrt{c(1-x)})}. \tag{3.22}$$

Recall that the larger the value of C for which this intersection occurs, the larger the value of x_R. Note that in the limit of $C >> 1$, the expression above for Δ goes as $\sqrt{C}$. Therefore, provided that $\Delta \to \infty$, x_R can become arbitrarily close to unity.

Since the value y_{int} at which $H = 0$ is dependent on Δ (which in general is a function), in order to find the maximum x_R for a particular equilibrium solution, we must specify an equation of state, and solve the equation $H = 0$. For example, consider the equation of state which we examine in more detail later in the thesis,

$$p_r + \chi p_t = \Omega \rho. \tag{3.23}$$

Given this equation of state, we find

$$\Delta = \frac{\Omega \rho}{\chi p_r} - \frac{1}{\chi} - 1. \tag{3.24}$$

Substituting the parametric functions for ρ and p_r results in a differential equation for $H = 0$, which has the solution

$$h_0(x) = \frac{-1}{\chi}\left(2 - 2x + \chi x \pm 2\sqrt{(1-x)(1 + x(\chi + \chi\Omega - 1))}\right). \tag{3.25}$$

The resulting $H = 0$ curve in the x-y plane does not have the simple shape that the constant Δ or $\Delta = 0$ curves had. As a result, maximizing x_R is not as simple as finding the member of P with the largest value of C that intersects $H = 0$ at y_{int}. The non-trivial x-intercept of $H = 0$ occurs at

$$x_{\text{int}} = \frac{4\Omega}{\chi + 4\Omega}. \tag{3.26}$$

Note that our structure curve, if it is to return to $y = 0$ must cross $H = 0$, and must do so with a gradient $dy/dx = \alpha$. Thus the x_{int} above is a minimum value for $2M/R$ for those equilibrium solutions with the equation of state $p_r + \chi p_t = \Omega\rho$, in agreement with that lower bound found in the introduction.

3.2 The Buchdahl Limit as Part of a More General Procedure

In this section, we again begin by parameterizing the field equations in terms of the variables $x \equiv 2m/r$ and $y \equiv 8\pi r^2 p_r$. We note that this section is motivated by the Stalker-Karageorgis work [30]. It can be shown that the field equations then reduce to

$$8\pi r^2 \rho = 2\dot{x} + x, \tag{3.27}$$

$$8\pi r^2 p_r = y, \tag{3.28}$$

$$8\pi r^2 p_t = \frac{x+y}{2(1-x)}\dot{x} + \dot{y} + \frac{(x+y)^2}{4(1-x)}, \tag{3.29}$$

where $\dot{} \equiv d/d\beta$, with $\beta = 2\ln(r)$. Note that this β is not the same as the function defined in the previous section.

We are interested in the inequality which generalizes that due to Andreasson for fluids with equations of state of the form $p_r + \chi p_t = \Omega\rho$ as are considered in more detail later in this work. Thus consider

$$p_r + \chi p_t \leq \Omega\rho. \tag{3.30}$$

Substituting the above expressions in terms of variables x and y and rearraging yields the inequality

$$\dot{x}(\chi(x+y) - 4\Omega(1-x)) + 2\dot{y}\chi(1-x) \leq 2(\Omega x - y)(1-x) - \chi\frac{(x+y)^2}{2}. \tag{3.31}$$

Let

$$z(x,y) \equiv -2(\Omega x - y)(1-x) + \chi\frac{(x+y)^2}{2}. \tag{3.32}$$

Define the two parameter function w_{pq} which generalizes that in [30],

$$w_{pq}(x,y) \equiv \frac{1}{1-x}(q(1-x) + p(1+y))^2. \tag{3.33}$$

With

$$q = 4\Omega + \chi, \tag{3.34}$$

$$p = \chi, \tag{3.35}$$

we have

$$w(x,y) = \frac{1}{1-x}((4\Omega+\chi)(1-x) + \chi(1+y))^2. \tag{3.36}$$

Next, examine the rate of change of this function with respect to β.

$$\dot{w}(x,y) = \left(\frac{2\chi + 4\Omega - (\chi+4\Omega)x + \chi y}{(1-x)^2}\right)(((4\Omega+\chi)x + \chi y - 4\Omega)\dot{x} + 2\chi(1-x)\dot{y}) \tag{3.37}$$

Notice how the function $\dot{w}(x,y)$ relates to our previously derived inequality. The result obtained is that

$$\dot{w}(x,y) \le -z(x,y)\left(\frac{2\chi + 4\Omega - (\chi + 4\Omega)x + \chi y}{(1-x)^2}\right) \tag{3.38}$$

Define the quantity $A(x,y)$ as

$$A(x,y) \equiv \left(\frac{2\chi + 4\Omega - (\chi + 4\Omega)x + \chi y}{(1-x)^2}\right) \tag{3.39}$$

Therefore,

$$\dot{w}(x,y) \le -z(x,y)A(x,y) \tag{3.40}$$

Now, it is clear that $A(x,y)$ is non-negative since

$$2\chi + 4\Omega - (\chi + 4\Omega)x + \chi y = (\chi + 4\Omega)(1-x) + \chi(1+y) \tag{3.41}$$

and thus $x \le 1$, $y \ge 0$ implies $A(x,y) \ge 0$. Therefore we arrive at the important conclusion that when $z > 0$, the function $w(x,y)$ is decreasing. Thus

$$w(x,y) \le \max_{0\le x\le 1, z\le 0\le y} w(x,y) = w(0,0) = (2\chi + 4\Omega)^2. \tag{3.42}$$

Therefore the maximum value attained for x is the x-intersection of the parametric curve $w(x,y) = (2\chi + 4\Omega)^2$. We obtain the conclusion that given $p_r + \chi p_t \le \Omega\rho$, there is an upper bound on $x = 2m/r$, namely

$$\frac{2M}{R} \le \frac{8\Omega(\chi + 2\Omega)}{(\chi + 4\Omega)^2}. \tag{3.43}$$

The cases where we demand the strong energy condition is satisfied are equivalent to the constraint that $\chi = 2$, $\Omega = 1$, in which case we obtain the familiar Buchdahl limit

$$x \leq \frac{8}{9}. \tag{3.44}$$

Thus the Buchdahl limit can be obtained through more general considerations of the field equations using less restrictive assumptions. Namely we make no restrictions on monotonicity of the energy density or pressures.

Chapter 4

Calculating the Speed of Sound for Spherical Matter Distributions

In this section, our goal is to apply the theoretical framework developed in an important article due to Carter [31] to calculate the speed of sound in perfect fluid and anisotropic fluid distributions. A full appreciation of this framework, which we call the "Carter-Quintana formalism", requires advanced techniques in differential geometry and so the details have been left to the appendix; even there we only provide a loose overview of the theory.

Thus in the following we accept, as given, the theory of elastic fluids presented in seminal works such as that due to Carter and Quintana [31]. Another source worthy of investigation is a manuscript due to Karlovini and Samuelsson [32] which essentially reformulates the Carter-Quintana theory with minor modifications and includes important discussions on sound-wave propagation. Furthermore, the Karlovini-Samuelsson manuscript provides an extensive bibliography on research regarding the subject of relativistic elastic fluids. A useful reference for this section is also due to

Carter [33], and can be seen as a basis for the investigations in this portion of the thesis. As in [33], we address the issue of sound wave propagation in elastic media using a classical method which Carter attributes to the French mathematician Jacques Hadamard.

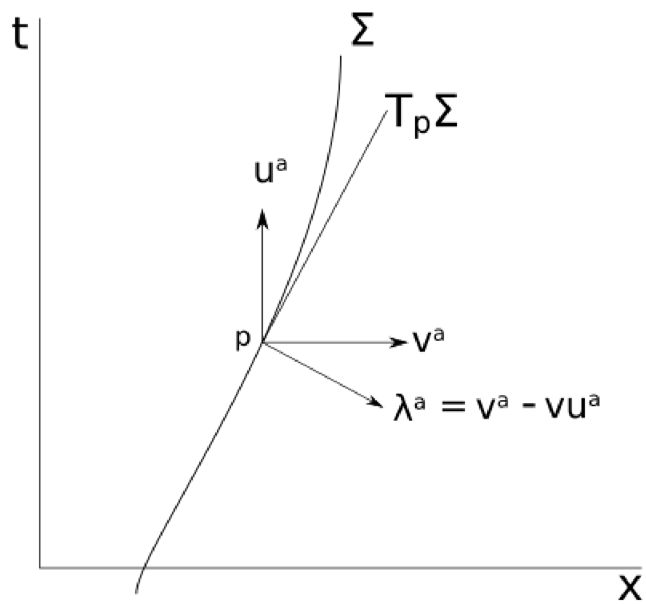

Figure 4.1: Consider the propagation of wave front Σ having spatial direction v^a and speed v with respect to the fluid flow vector u^a. The normal to Σ is in the direction of the vector $\lambda^a = v^a - vu^a$ which is defined such that contraction with the flow vector yields the speed v.

At this point, we take it as understood that the stress energy tensor for a general fluid can be decomposed with respect to its overall flow vector u^a and some tensor p^{ab} defined on the subspace orthogonal to u^a as

$$T^{ab} = \rho u^a u^b + p^{ab}. \tag{4.1}$$

The tensor p^{ab} is defined as (see [31])

$$p^{ab} = -2\frac{\partial \rho}{\partial \gamma_{ab}} - \rho\gamma^{ab}, \tag{4.2}$$

where the metric tensor on that subspace orthogonal to u^a is

$$\gamma_{ab} = u_a u_b + g_{ab}. \tag{4.3}$$

As in the Carter-Quintana formalism, in any given comoving coordinate system, specification of ρ and p^{ab} as functions of the six independent components of γ_{ab} in the same comoving coordinate system constitutes an equation of state. It can be shown via the conservation law $T^{ab}{}_{;b} = 0$ that ρ and p^{ab} are subject to the constraints that

$$\dot{\rho} = -\rho(u^c)_{;c} - p^{cd}u_{c;d}, \tag{4.4}$$

$$(p^{ab})_{;c}u^c = 2u^{(a}p^{b)c}\dot{u}_c + 2p^{c(a}u^{b)}_{;c} - p^{ab}(u^c)_{;c} - E^{abcd}u_{c;d} \tag{4.5}$$

where E^{abcd} is the bisymmetric relativistic elasticity tensor (see [31], [32] or [33]) defined by

$$E^{abcd} = -2\frac{\partial(p^{ab})}{\partial(\gamma_{cd})} - p^{ab}\gamma^{cd}. \tag{4.6}$$

We consider a wave front Σ moving in the spatial direction v^a with speed v respect to the flow u^a at point p. The vector $\lambda^a = v^a - vu^a$ is normal to Σ and defined such that contraction with the flow vector yields the propagation speed v. The wave front is timelike, and thus has an everywhere spacelike normal.

$$\lambda^a\lambda_a \geq 0 \implies v^2 \leq 1 \tag{4.7}$$

We directly follow [33] by employing the Hadamard method to arrive at a characteristic wave equation for variously polarized sound waves travelling along the principal directions of our fluid spheres.

Across Σ it is assumed that the flow vector u^a and the metric and its first derivative are continuous. The acceleration $\dot{u}^a$ is permitted to have a discontinuity. Given these assumptions, γ_{ab} and therefore the functions of state ρ, p^{ab} and E^{abcd} are continuous. However, the first derivatives of the functions of state are permitted to be discontinuous. As in our discussion of spacetime junctions, let square brackets denote jump discontinuities in a given quantity. We then let

$$[\dot{u}^a] \equiv \alpha \iota^a, \tag{4.8}$$

where ι^a is called the polarization vector and is normalized such that $\iota^a \iota_a = 1$.

We use the fact that the discontinuity in the gradient of a continuous function across a hypersurface is normal to the hypersurface (see e.g. [10]). Therefore $\exists$ σ, κ^a and π^{ab} on Σ such that

$$[\rho_{,c}] = \sigma \lambda_c \tag{4.9}$$

$$[(u^a)_{;c}] = \kappa^a \lambda_c \tag{4.10}$$

$$[(p^{ab})_{;c}] = \pi^{ab} \lambda_c. \tag{4.11}$$

It is a straightforward algebraic procedure to use the equations of motion specified above to determine σ, κ^a and π^{ab} in terms of the functions of state and α, v, λ^a and ι^a. Proceeding in this manner leads to the characteristic equation

$$\left(v^2(\rho\gamma^{ac} + p^{ac}) - (E^{abcd} - \gamma^{ac}p^{bd})v_b v_d\right)\iota_c = 0. \tag{4.12}$$

In order to proceed, firstly the energy momentum tensor must specified. Subsequently, to determine sound speed, we must specify which direction and polarization we wish to consider.

4.1 Speed of Sound in a Spherically Symmetric Perfect Fluid

For a perfect fluid the stress energy tensor is given as

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}, \tag{4.13}$$

and thus $p^{ab} = p\gamma^{ab}$. Inserting this in to the characteristic equation results in the following eigensystem problem:

$$\left(-2v^2 \frac{\partial \rho}{\partial \gamma_{ac}} + \left(2\frac{\partial p}{\partial \gamma_{cd}}\gamma^{ab} + p(\gamma^{ab}\gamma^{cd} - \gamma^{ac}\gamma^{bd})\right) v_b v_d\right) \iota_c = 0 \tag{4.14}$$

We then proceed by specifying the spatial direction in which the wave propagates. Consider the case in which the propagation vector is specified as $v^a = e^{-\lambda}\delta^a_r$ where

$$e^{2\lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1}. \tag{4.15}$$

Then the waves propagating in the radial direction satisfy the expression

$$\left(-2v^2 \frac{\partial \rho}{\partial \gamma_{ac}} + e^{2\lambda}\left(2\frac{\partial p}{\partial \gamma_{cr}}\gamma^{ar} + p(\gamma^{ar}\gamma^{cr} - \gamma^{ac}\gamma^{rr})\right)\right) \iota_c = 0. \tag{4.16}$$

We then have a three dimensional eigenvector problem where the eigenvalues v^2 represent the speed in the radial direction. The three equations correspond to the choice for the index a. Choosing $a = r$ and supposing there exists some relation $\rho(p)$ yields the familiar result that

$$v^2 = \frac{\partial p}{\partial r}\left(\frac{\partial \rho}{\partial r}\right)^{-1} \tag{4.17}$$

for radially polarized sound waves travelling in the radial spatial direction. The cases $a = \theta$ and $a = \phi$ are degenerate and yield the eigenvalue

$$v^2 = -p\left(r\frac{\partial \rho}{\partial r}\right)^{-1} \tag{4.18}$$

corresponding to the case of θ and ϕ polarized waves travelling in the radial direction.

4.2 Speed of Sound in a Spherically Symmetric Anisotropic Fluid

In the orthonormal basis $\{u^a, k^a, l^a, j^a\}$, the general energy momentum tensor for a spherically symmetric static matter distribution admits the decomposition

$$T^{ab} = \rho u^a u^b + p_r k^a k^b + p_t(l^a l^b + j^a j^b), \tag{4.19}$$

where u^a is the flow tangent, k^a is a normalized radial spacelike vector and l^a and j^a are the eigenvectors of T^{ab} corresponding to the degenerate eigenvalue p_t. Using this decomposition, the pressure tensor p^{ab} takes the form

$$p^{ab} = p_r k^a k^b + p_t(l^a l^b + j^a j^b). \tag{4.20}$$

For the purposes of this subsection, we are interested only in those cases for which the propagation direction of a sound wave coincides with the spatial eigenvectors of the energy momentum tensor. Thus we are interested in the cases $v_b = \{k_b, l_b, j_b\}$, where $k_b = e^{\lambda}\delta^r_b$, $l_b = r\delta^\theta_b$, $j_b = r\sin\theta\delta^\phi_b$. Furthermore, we will only calculate the speed of sound for the two independent transverse acoustic waves corresponding to radial directed, radially polarized or angularly directed, angularly polarized sound waves.

For the case $v_b = k_b$, the characteristic equation reduces to

$$\begin{aligned} 0 =& \Bigg\{ v^2 \left((\rho + p_r)e^{-2\lambda}\delta^a_r\delta^c_r + \frac{(\rho + p_t)}{r^2}(\delta^a_\theta\delta^c_\theta + \frac{1}{\sin^2\theta}\delta^a_\phi\delta^c_\phi) \right) \\ &+ 2e^{2\lambda}\frac{\partial}{\partial\gamma_{cr}}(p_r e^{-2\lambda})\delta^a_r + p_r\left(2e^{-2\lambda}\delta^a_r\delta^c_r + \frac{1}{r^2}(\delta^a_\theta\delta^c_\theta + \frac{1}{\sin^2\theta}\delta^a_\phi\delta^c_\phi)\right)\Bigg\}\iota_c. \end{aligned} \tag{4.21}$$

We then consider either that the polarization vector is also radially directed, the magnitude of which is irrelevant. Sound speeds are labeled hereafter via the notation $(v^2)_{pq}$, where p represents the propagation direction and q represents the polarization. We find the speed of sound for radially-directed, radially-polarized sound waves in the anisotropic fluid as

$$(v^2)_{rr} = \frac{-p_r'}{\lambda'(\rho + p_r)}. \tag{4.22}$$

The transverse waves for the cases $v_b = l_b$ and $v_b = j_b$ are degenerate. For $v_b = l_b$ the characteristic equation reduces to

$$\begin{aligned} 0 =& \Bigg\{ v^2 \left((\rho + p_r)e^{-2\lambda}\delta^a_r\delta^c_r + \frac{(\rho + p_t)}{r^2}(\delta^a_\theta\delta^c_\theta + \frac{1}{\sin^2\theta}\delta^a_\phi\delta^c_\phi) \right) \\ &+ 2r^2\frac{\partial}{\partial\gamma_{c\theta}}\left(\frac{p_t}{r^2}\right)\delta^a_\theta + p_t\left(\frac{1}{r^2}(2\delta^a_\theta\delta^c_\theta + \frac{1}{\sin^2\theta}\delta^a_\phi\delta^c_\phi) + e^{-2\lambda}\delta^a_r\delta^c_r\right)\Bigg\}\iota_c, \end{aligned} \tag{4.23}$$

and we find that the speed of sound for θ-directed waves with polarization along the

θ direction is

$$(v^2)_{\theta\theta} = \frac{-rp_t'}{(\rho + p_t)}. \tag{4.24}$$

In the following section of the thesis, we algorithmically generate an infinite class of anisotropic fluid solutions. The results presented here will be used to verify or reject the physicality of various members of that family of solutions using the criteria that sound propagation speeds must be subluminal.

Chapter 5

Solving Einstein's Equations Given an Equation of State

Having motivated the study of anisotropic matter, our goal now is to discuss the primary objective of this thesis: to develop and investigate an infinite class of spherically symmetric static spacetimes composed of anisotropic fluid.

To this end, we begin our analysis by considering a spherically symmetric matter distribution in static equilibrium which by assumption has a stress energy tensor that is locally anisotropic. Spherical symmetry implies that the existence of canonical coordinates in which the energy momentum tensor is diagonalizable,

$$T_{\mu}^{\nu} = \mathrm{diag}(-\rho(r), p_r(r), p_\theta(r), p_\phi(r)). \tag{5.1}$$

Recall that $T^{\alpha\beta}$ gives the flux of the α^{th} component of the momentum 4-vector across a surface of constant x^β, and thus by symmetry $p_\theta = p_\phi \equiv p_t$. We will hereafter refer to ρ, p_r, and p_t as the energy density, radial pressure and tangential pressure respectively, which as yet are completely unspecified and general. p_r and p_t are also referred to as principal stresses of the energy momentum tensor. In contrast to perfect fluids, the principal stresses of T^μ_ν throughout an anisotropic distribution are not assumed equal.

We seek physically realistic solutions to the Einstein field equations with Λ,

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{5.2}$$

One immediately obtains the relevant Einstein field equations from the metric and its source which we have taken to be an anisotropic fluid described in the canonical frame by the stress-energy tensor above. Note that in the comoving coordinate system, we can express the fluid four velocity vector as

$$u^\alpha = e^{-\Phi(r)}\delta^\alpha_t. \tag{5.3}$$

The field equations are

$$8\pi\rho = \frac{2m'}{r^2} - \Lambda, \tag{5.4}$$

$$8\pi p_r = -\frac{2}{r^3}\left[\Phi' r(2m - r) + m\right] + \Lambda, \tag{5.5}$$

$$8\pi p_t = -\frac{1}{r^3}\left[r^2(\Phi'^2 + \Phi'')(2m - r) + (\Phi' r + 1)(m' r - m) - \Phi' r^2\right] + \Lambda. \tag{5.6}$$

Straightforward calculation subsequently yields the effective gravitational mass as a function of the energy density,

$$m = 4\pi\int_0^r \rho(x)x^2 dx + \Lambda\frac{r^3}{6} \tag{5.7}$$

the source equation for Φ,

$$\Phi' = \frac{m + 4\pi r^3 p_r - \Lambda r^3/2}{r(r-2m)} \tag{5.8}$$

and the generalized TOV equation,

$$p_t = \frac{r}{2}\left(p_r' + (\rho + p_r)\Phi'\right) + p_r. \tag{5.9}$$

At this point, it is constructive to pause, and restate that all static spherically symmetric metrics are anisotropic fluid 'solutions' to Einstein's equations. It is important to emphasize that we seek solutions which are physically sensible. To this end, we impose several criteria which the relevant physical parameters are required to satisfy:

1. Regularity (meaning no singularities), and in particular at the origin.

2. Positive but monotonically decreasing energy density, radial pressure and tangential pressure.

3. A finite $r = R$ such that $p_r(R) = 0$, defining a boundary surface for the distribution.

4. All standard energy conditions are satisfied, except the dominant energy condition which may be violated if $p_t > \rho$, a condition which may be too restrictive (see [34]).

5. We consider that our fluid sphere is composed of material with the general linear barotropic equation of state $p_r + \chi p_t = \Omega\rho$.

We impose these preliminary conditions in order to restrict our considerations to physically reasonable situations. For example, there is at present no evidence suggesting that measurements of negative energy density or negative pressure are physical. It should be remarked that in condition 2, we restrict our solutions to have monotonically decreasing energy density profiles. This restriction is not entirely necessary, and solutions without monotone energy density profiles are not, in general, unphysical. In condition 5, we stipulate that the fluid is described by a linear barotropic equation of state with constant parameters χ and Ω (> 0) determining the nature of the fluid.

The DEC implies that $\rho \geq p_r$ and $\rho \geq p_t$. Note that if the latter is to hold at the boundary $r = R$, where $p_r = 0$, the parameters Ω and χ must satisfy $\Omega \leq \chi$. The motivation for the DEC is that the flow of matter remain non-spacelike. That the tangential pressure p_t is not associated with such flows may render the condition $p_t \leq \rho$ unnecessary. If we allow $p_t > \rho$ but demand $p_r \leq \rho$, we require the less restrictive constraint $\Omega \leq 1 + \chi$.

We seek non-cosmological solutions and therefore require that the radial pressure p_r terminates at some finite $r = R$, where the fluid can be matched, by way of a boundary surface, on to a Schwarzschild (or Schwarzschild de Sitter) vacuum of mass $m(R) \equiv M$.

Rewriting Einstein's equations in terms of the physical quantities ρ, p_r and p_t allows us to demonstrate the utility of the linear barotropic equation of state. Substitution of the physical quantities ρ, p_r and p_t in to the condition 5 yields a differential equation of first order in the derivatives of the metric function m and second order in the derivatives of the metric function Φ. Given an explicit $m(r)$, obtaining Φ requires the solution of a complicated Riccati type differential equation. However, given $\Phi(r)$, the differential equation for m is a simple first order linear ODE. Consequently, we develop an algorithmic solution generating technique which ultimately yields infinitely many physically reasonable anisotropic fluid solutions.

Upon substituting ρ, p_r and p_t in the equation of state, we obtain a differential equation for m of the form

$$m' = b - ma, \tag{5.10}$$

where a and b are functions of r, namely

$$a = \frac{2 + 4r\Phi' + \chi(r\Phi' + 2r^2(\Phi'^2 + \Phi'') - 1)}{r\,(\chi + \chi r\Phi' + 2\Omega)}, \tag{5.11}$$

$$b = \frac{r((2+\chi)\Phi' + r(\chi(\Phi'^2 + \Phi'') + \Lambda(1 + \chi + \Omega)))}{\chi + \chi r\Phi' + 2\Omega}. \tag{5.12}$$

The first order linear ODE has the solution for m as

$$m = \frac{\int b e^{\int a dr} dr + C}{e^{\int a dr}}. \tag{5.13}$$

An explicit solution is obtained through specification of a function Φ for which m can be obtained without recourse to numerical integration [5]. Although supplying a metric tensor most often yields unphysical fluid distributions, one can supply an infinite number of Φ functions which satisfy our previously discussed criteria for physical acceptability.

It must be noted that the solution generating function Φ can not be chosen completely arbitrarily. Φ must be sufficiently smooth and such that the invariants constructed from the Riemann tensor are finite at the origin. Furthermore, for the isolated spherical distributions of interest in this work, the function Φ is subject to junction conditions at the fluid boundary.

Spherically symmetric spacetimes have four independent scalar polynomial invariants (three Ricci invariants and one Weyl invariant) [35]. The Ricci invariants may be expressed as simple polynomials in ρ, p_r and p_t, and thus regularity of the energy density and pressures guarantees regularity of the Ricci invariants. The Weyl invariant, however, grows like

$$\frac{1}{r^6}\left(\frac{rp_r' + 2(p_r - p_t)}{\rho - 3p_r + 4p_t}\right) \tag{5.14}$$

as $r \to 0$ and thus we require $p_r'(0) = 0$ and $p_r(0) = p_t(0)$ for regularity at the origin.

Notice that

$$p_r = \frac{2r(r-2m)\Phi' - 2m + \Lambda r^3}{8\pi r^3} \geq 0 \tag{5.15}$$

implies that $\Phi'(0) = 0$ for regularity. Since $r > 2m$ for $r > 0$, for $p_r > 0$, $\Lambda \leq 0$ solutions must have Φ monotonically increasing with a regular minimum at $r = 0$. For $\Lambda > 0$, if $m \geq \Lambda r^3/2$, Φ must be monotone increasing.

5.1 An Infinite Number of Φ Functions

Tolman's method for discovering new perfect fluid solutions was to specify an ansatz about the metric which would render the field equations integrable [20]. The ansatz leading to the solution IV from his 1939 work was equivalent to the supposition that the metric function Φ obeyed the differential equation $(e^{2\Phi}\Phi'/r)' = 0$ and could thus be written in the form

$$\Phi = \frac{n}{2}\log\left(1 + \frac{r^2}{\alpha}\right) \tag{5.16}$$

(as in [5]) where n and α are constant parameters such that $n \geq 1$ and $\alpha > 0$. The specified Φ has a regular minimum at $r = 0$ and is monotone increasing.

Substituting this form of the metric function Φ in to the solution of the linear ODE above yields a closed form analytic solution for the effective gravitational mass $m(r)$ in terms of the variable parameters n, α, χ, Ω and Λ, thereby providing all the information required to investigate the relevant physical quantities of the system.

For the sake of brevity, we define the following functions:

$$f(r) = \frac{r^3 \alpha^{n-3} (r^2 + \alpha)^{2-n}}{2(1 + \chi + 3\Omega)} \left(1 + \frac{r^2(\chi + n\chi + 2\Omega)}{\alpha(\chi + 2\Omega)}\right)^{-1-\frac{(\chi+2\Omega)^2 + n(\chi - 3\chi\Omega - 4\Omega(\Omega-1))}{(\chi+2\Omega)(\chi+n\chi+2\Omega)}}, \tag{5.17}$$

$$\begin{aligned} h_a(r) = F_1\Big(&\frac{\chi + 3\Omega + 1}{\chi + 2\Omega}; a - n, \frac{n\,(\chi(3\Omega - 1) + 4(\Omega - 1)\Omega) - (\chi + 2\Omega)^2}{(\chi + 2\Omega)(n\chi + \chi + 2\Omega)}; \\ &\frac{2\chi + 5\Omega + 1}{\chi + 2\Omega}; -\frac{r^2}{\alpha}, -\frac{r^2(n\chi + \chi + 2\Omega)}{\alpha(\chi + 2\Omega)}\Big), \end{aligned} \tag{5.18}$$

where F_1 denotes the Appell hypergeometric function of two variables [36].

We thus obtain a closed form analytic expression for the effective gravitational mass as

$$m(r) = f(r)\left[\alpha\Lambda(1 + \chi + \Omega)h_1(r) + n(2 + n\chi)h_2(r) + n\chi(2 - n)h_3(r)\right]. \tag{5.19}$$

Once $m(r)$ is obtained, it is a simple computational task to obtain the energy density and the radial and tangential pressures. Given the complexity of these functions (which involve r-derivatives of m), they will not be written in this main body of text. It is, however, of much interest to study the nature of these solutions and determine their physicality.

5.2 Analyzing a Specific Equation of State

While the solution for $m(r)$ above is quite general, involving the cosmological constant, for the sake of analysis it is instructive to examine a scenario with a more tractable parameter space. Consider the variable equation of state discussed recently in the literature by Andreasson [1],

$$p_r + 2p_t = \Omega\rho. \tag{5.20}$$

Let us also assume that $\Lambda = 0$. In this case, the functions a and b described above have the form

$$a = \frac{nr\left(5\alpha + (2n+1)r^2\right)}{\left(\alpha + r^2\right)\left(\alpha(1+\Omega) + r^2(1+n+\Omega)\right)}, \tag{5.21}$$

$$b = \frac{nr^2\left(3\alpha + (n+1)r^2\right)}{\left(\alpha + r^2\right)\left(\alpha(1+\Omega) + r^2(1+n+\Omega)\right)}. \tag{5.22}$$

For brevity we make the following definitions,

$$g(r) = \frac{nr}{1+\Omega}\left(1 + \frac{r^2}{\alpha}\right)^{2-n}\left(1 + \frac{r^2(1+n+\Omega)}{\alpha(1+\Omega)}\right)^{\frac{n(1+2\Omega)}{2(1+n+\Omega)}-2} \tag{5.23}$$

$$k_a(r) = F_1\left(\frac{1}{2}, a-n, -\frac{n(1-2\Omega)+2(1+\Omega)}{2(1+n+\Omega)}, \frac{3}{2}, -\frac{r^2}{\alpha}, -\frac{r^2(1+n+\Omega)}{\alpha(1+\Omega)}\right). \tag{5.24}$$

We thus obtain the mass function $m(r)$ as

$$m(r) = g(r)\left[(1+n)k_1(r) + (1-2n)k_2(r) + (n-2)k_3(r)\right]. \tag{5.25}$$

Having obtained a closed form solution for m renders the system solved. All relevant physical quantities are now easily computed from the field equations. Though the solutions appear relatively complicated, involving the hypergeometric Appell functions, the curves they give rise to are physically reasonable, and have a large degree of variability based on the choice of parameters n, α and Ω.

The energy density follows from the field equations as

$$\begin{aligned}\rho(r) = \frac{n}{4\pi\left(r^2+\alpha\right)\left[\alpha(1+\Omega)+r^2(1+n+\Omega)\right]}\Big\{ &\big((n+1)r^2+3\alpha\big) - \\ \frac{g(r)}{r}\big((2n+1)r^2+5\alpha\big)\left[(1+n)k_1(r)+(1-2n)k_2(r)+(n-2)k_3(r)\right]&\Big\}.\end{aligned} \tag{5.26}$$

Note that the energy density obeys

$$4\pi r^2\rho = b - ma, \tag{5.27}$$

and thus the central density is given by evaluating the function $b/4\pi r^2$ at $r = 0$, namely

$$\rho_c = \frac{3n}{4\pi\alpha(1+\Omega)}. \tag{5.28}$$

From the $r - r$ field equation, one obtains a closed form expression of the radial pressure as

$$\begin{aligned}p_r(r) = \frac{n}{4\pi(r^2+\alpha)}\Bigg\{1 - \frac{1}{r^2}\Bigg[&\alpha^{\frac{n(2n+1)}{2(1+n+\Omega)}}\left(r^2+\alpha\right)^{2-n}\left((2n+1)r^2+\alpha\right)\Bigg] \\ &\times\left(\frac{1+\Omega}{\alpha(1+\Omega)+r^2(1+n+\Omega)}\right)^{\frac{2(1+\Omega)+n(1-2\Omega)}{2(1+n+\Omega)}} \\ &\times\left[(1+n)k_1(r)+(1-2n)k_2(r)+(n-2)k_3(r)\right]\Bigg\}.\end{aligned} \tag{5.29}$$

The central pressure p_c is obtained by taking the limit as $r \to 0$,

$$p_c = \frac{n\Omega}{4\pi\alpha(1+\Omega)}. \tag{5.30}$$

Comparing the expressions for the central energy density and pressure, we find that for this class of solutions,

$$p_c = \frac{\Omega}{3}\rho_c, \tag{5.31}$$

as expected from the equation of state, and the condition that at $r = 0$, $p_r = p_t$. Thus specification of the parameter Ω determines the ratio of central pressure to energy density.

Having neatly defined p_c and ρ_c allows us to simplify our parameter space by specifying that we are interested in solutions with central pressure $p_c = 1$. Thus we restrict the parameter α to obey the relationship $\alpha = \frac{n\Omega}{4\pi(1+\Omega)}$.

From the $\theta - \theta$ field equation, the tangential pressure is easily calculated. The form of this function is not included in this body of the thesis, but will be attached in the appendix. We verify that the limit of p_t as $r \to 0$ matches the value for p_r as required for regularity of the Weyl invariant.

Finally, to give the reader a chance to visualize this subset of solutions generated by the algorithm presented above, we display plots of the effective gravitational mass m, the energy density ρ, the radial and tangential pressures p_r and p_t, as well as the ratios m/r and p_r/p_t. We remark that the solutions obtained via this method satisfy all of our conditions for physical acceptability when $\Omega < 2$. Plots for the physical quantities associated with the $\Omega = 2$ and $\Omega = 3$ equations of state are included to demonstrate that, although the dominant energy condition is violated, m, ρ, p_r and

p_t are all well behaved. It is the superluminal sound propagation speed inside the fluid which discounts such solutions. The limiting case, $\Omega = 2$, satisfies the dominant energy condition only in the limit that $n \to \infty$.

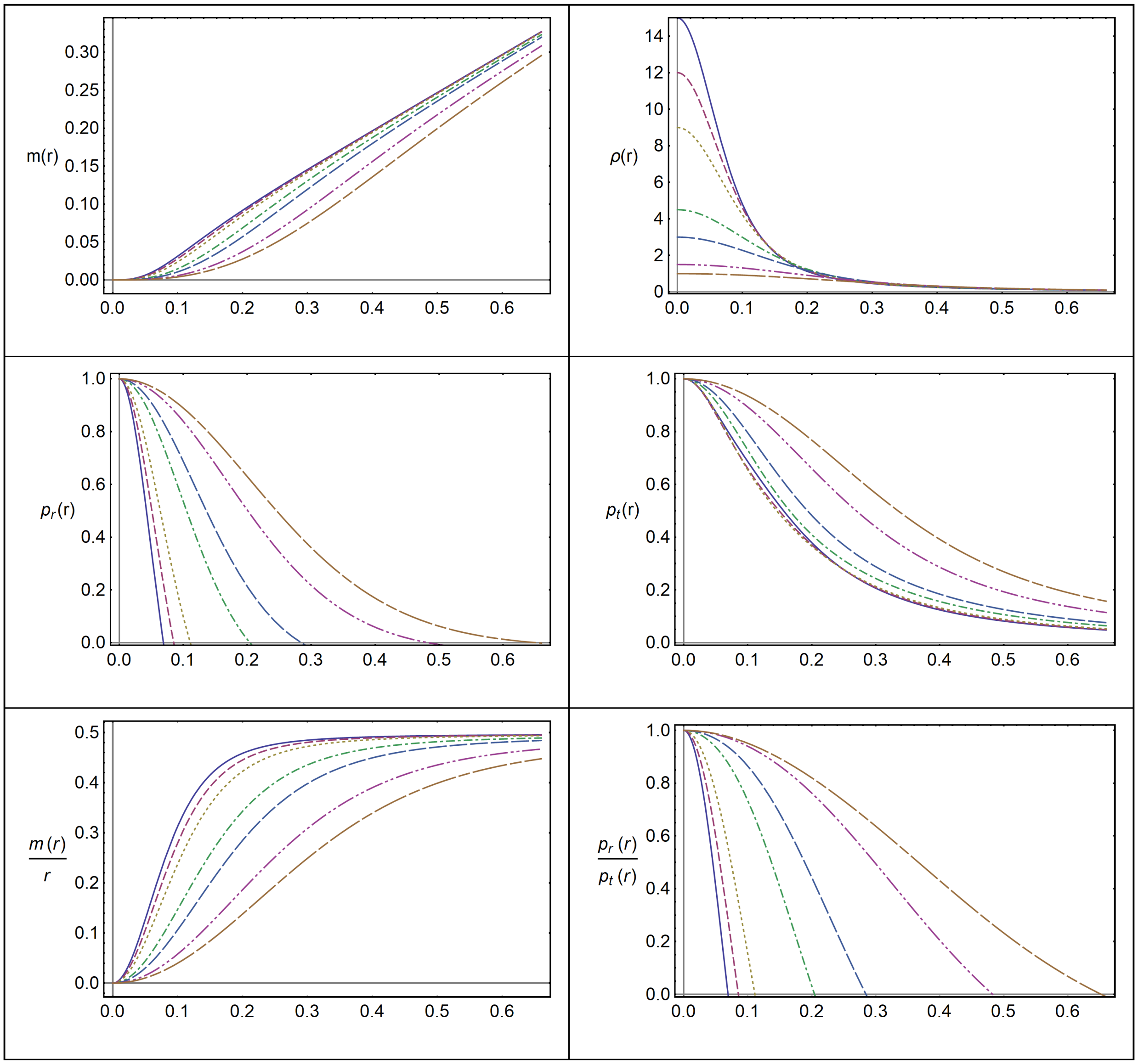

Figure 5.1: The effective gravitational mass, energy density, radial pressure, tangential pressure, mass/radius and radial/tangential pressure plots generated by the algorithm presented above, for the equation of state $p_r + 2p_t = \Omega\rho$, for $\Omega = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, 1, 2, 3$ with $n = 10$. The boundary radii, which can be ascertained from the radial pressure plots, increases with increasing Ω.

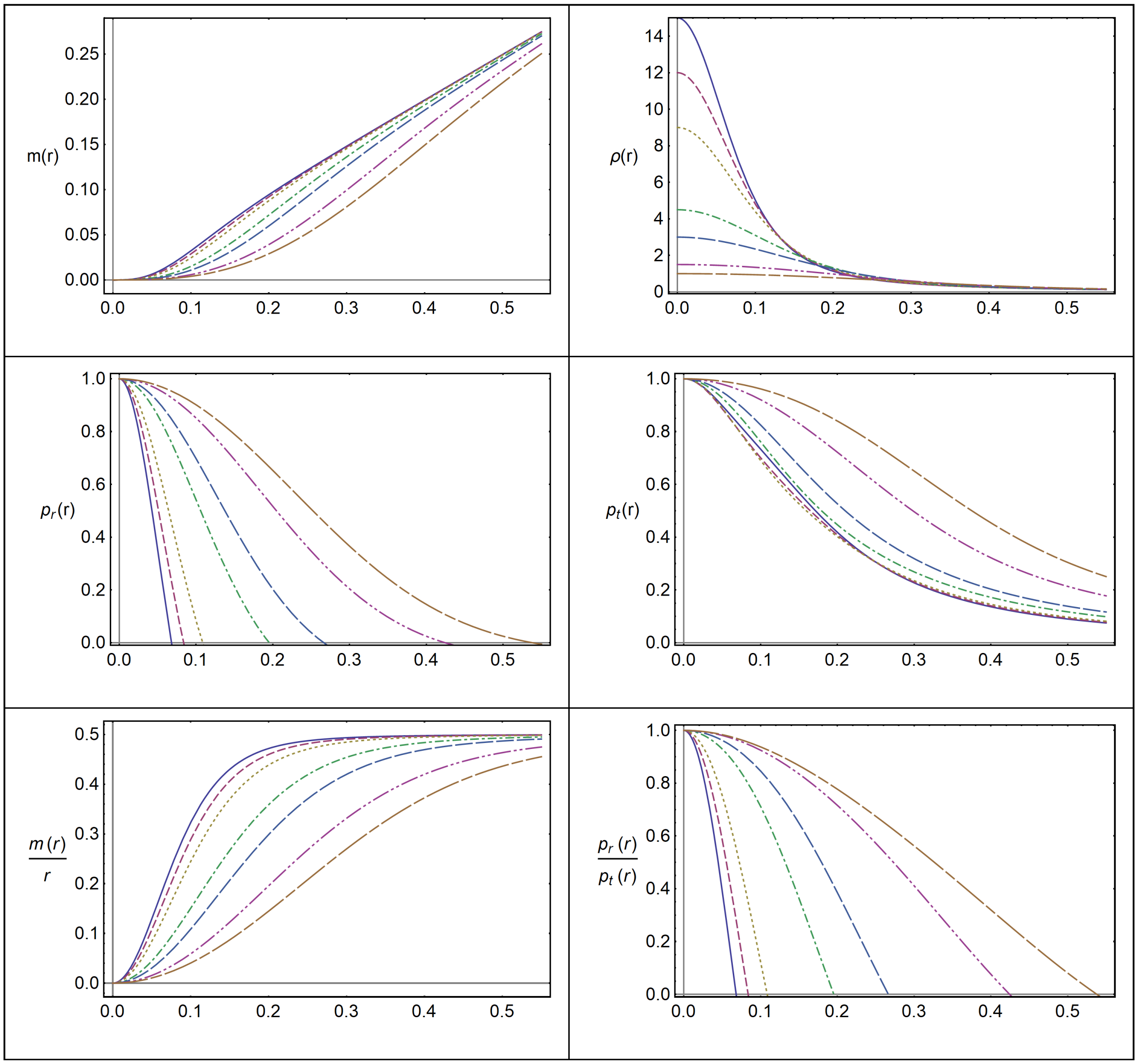

Figure 5.2: The effective gravitational mass, energy density, radial pressure, tangential pressure, mass/radius and radial/tangential pressure plots generated by the algorithm presented above, for the equation of state $p_r + 2p_t = \Omega\rho$, for $\Omega = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, 1, 2, 3$ with $n = 100$. The boundary radii, which can be ascertained from the radial pressure plots, increases with increasing Ω.

Figures 5.1 and 5.2 display plots of the physical quantities associated with this family of spacetimes, for various equations of state corresponding to different values of Ω. In 5.1, the metric parameter $n = 10$, whereas in 5.2, $n = 100$. For fixed Ω, notice that increasing n effectively serves to shift the boundary radius inwards. Also, note that the plots of the physical functions are shown over a domain of r extending beyond the value R to give the reader a more complete picture of the shape and behaviour of the curves. In reality, we require that at $r = R$ the solutions junction to a vacuum exterior.

The tenuity, $2M/R$, which has been of significant interest throughout this thesis is plotted in figure 5.3. We demonstrate the effect of the equation of state on tenuity for this class of solutions, varying the parameter n. To evaluate $2M/R$, a numerical root finder was applied to the radial pressure functions p_r to determine the value of R. The function $2m(r)/r$ is then evaluated at the values of R corresponding to the various pairs $\{n, \Omega\}$.

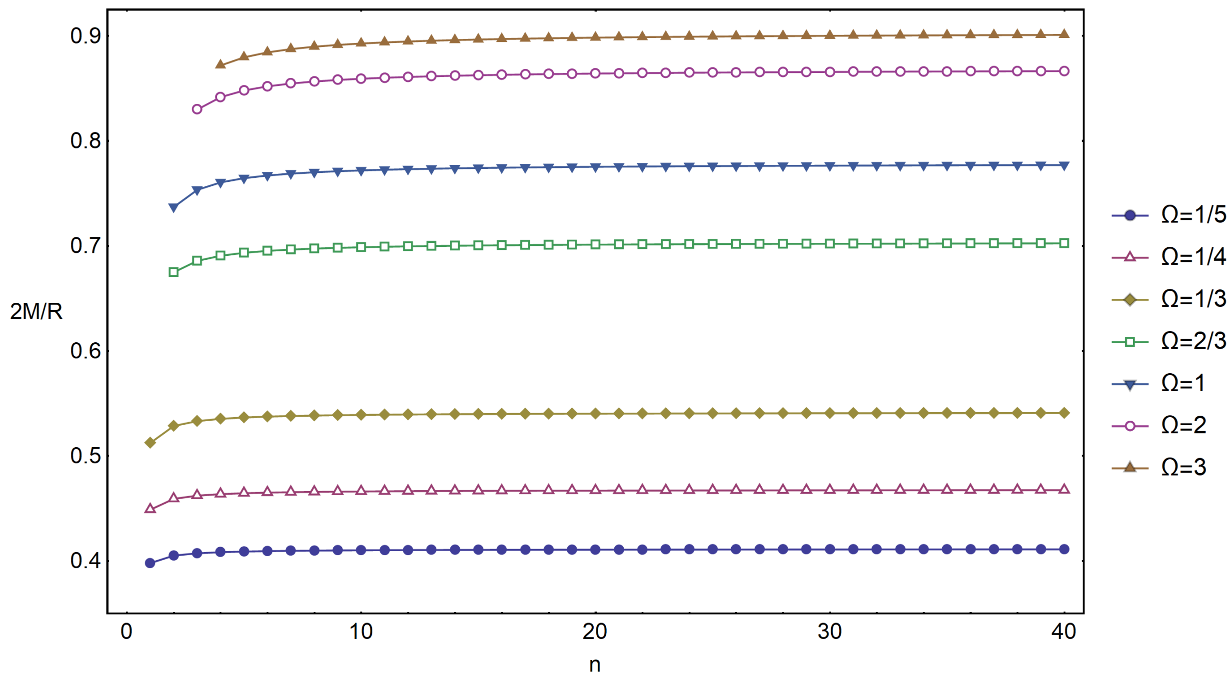

Figure 5.3: The tenuity $2M/R$ for $p_c = 1$ solutions as a function of the parameter n for varying equations of state corresponding to the values of Ω shown in the legend. It is easily verified that the tenuity for our solutions lie within the range predicted via earlier work in the thesis.

It has been demonstrated earlier in the thesis that for those linear barotropic equations of state $p_r + \chi p_t = \Omega\rho$, the tenuity is bounded above and below by

$$\frac{4\Omega}{4\Omega+\chi} < \frac{2M}{R} \leq \frac{8\Omega(\chi+2\Omega)}{(\chi+4\Omega)^2}, \tag{5.32}$$

and thus for the equation of state corresponding to $\chi = 2$, we verify that for the solutions presented in this section the tenuity bound

$$\frac{2\Omega}{1+2\Omega} < \frac{2M}{R} \leq \frac{4\Omega(1+\Omega)}{(1+2\Omega)^2} \tag{5.33}$$

is satisfied for all n.

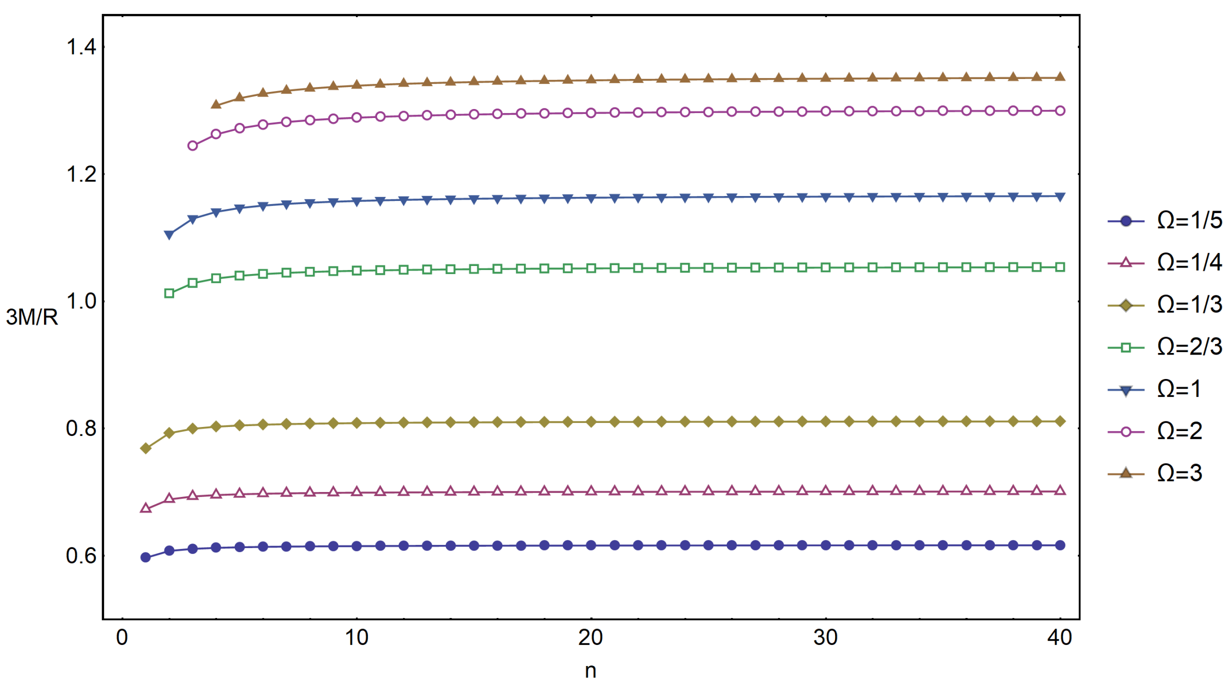

Figure 5.4: For considerations of photon trapping, it is the quantity $3M/R$ that is of interest. If $3M/R > 1$, there will be trapped photon orbits.

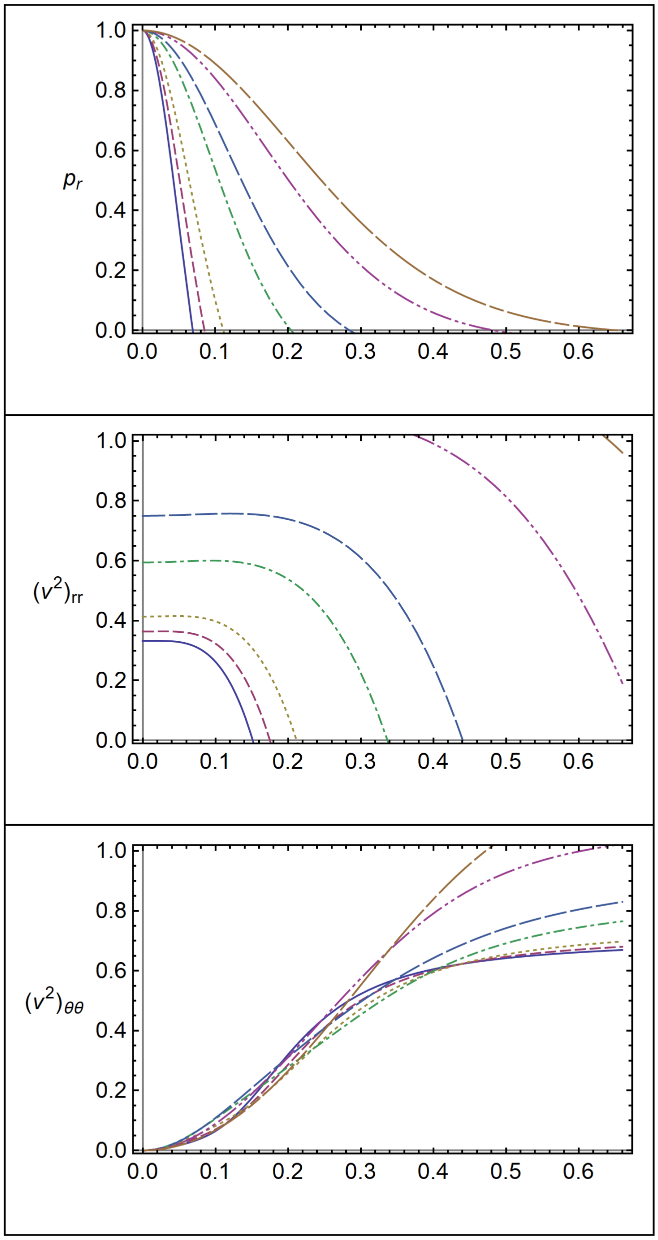

Figure 5.5: The sound speeds corresponding to radially or angularly directed longitudinal sound waves are plotted as functions of r, with the parameter $n = 10$ and $\Omega = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, 1, 2, 3$. For $\Omega < 2$, the sound waves are subluminal and well behaved throughout the fluid. The radial pressure plot is shown so that the reader may easily ascertain the boundary radius R of the corresponding fluids (which increases for increasing Ω).

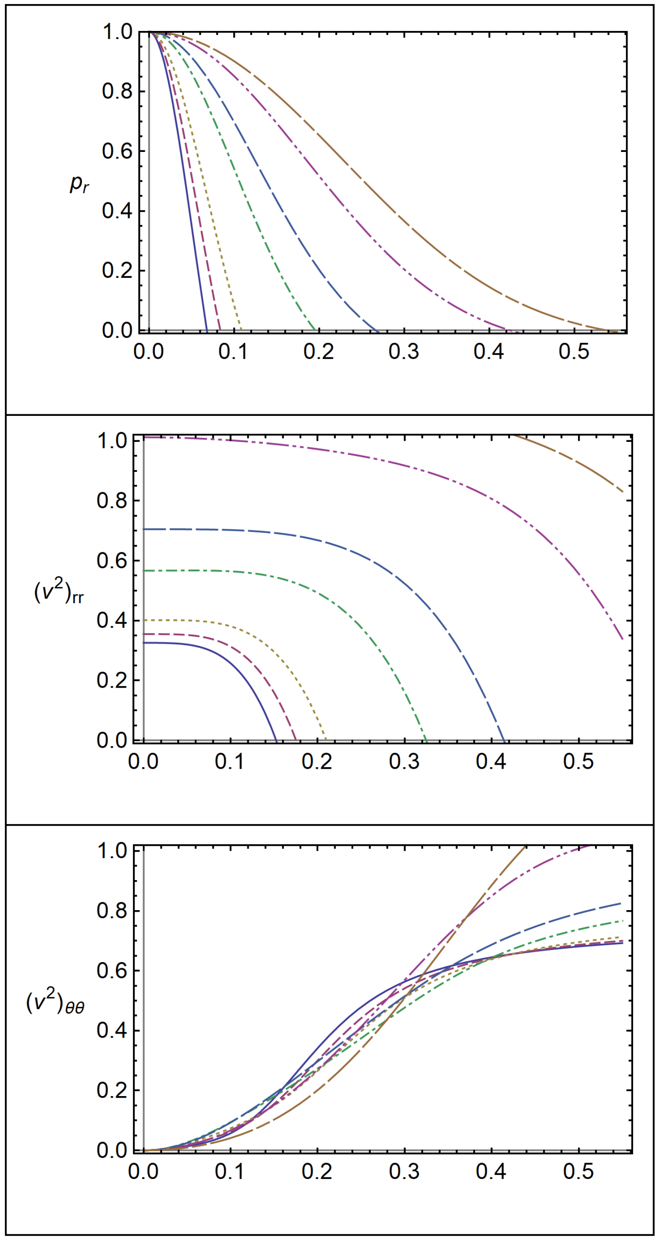

Figure 5.6: The sound speeds corresponding to radially or angularly directed longitudinal sound waves are plotted as functions of r, with the parameter $n = 100$ and $\Omega = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, 1, 2, 3$. For $\Omega < 2$, the sound waves are subluminal and well behaved throughout the fluid. The radial pressure plot is shown so that the reader may easily ascertain the boundary radius R of the corresponding fluids (which increases for increasing Ω).

In this section we presented a technique whereby one may algorithmically generate infinitely many previously unknown, physically reasonable anisotropic fluid solutions. This solution generating technique may find application in the modeling of celestial bodies, including neutron stars in order to place constraints on their equations of state. Of course, that the fluid in question is accurately described by such a simple linear barotropic model is unlikely. However, the basic schematic for algorithmically generating solutions using the seed function Φ has proved useful for other equations of state, too. That this method produces analytic, closed-form expressions for the physical functions that are easily dealt with using basic computer algebra software provides a valuable tool in investigating the properties of anisotropic matter.

Chapter 6

Addressing the Problem of Stability of Anisotropic Fluid Solutions

Having obtained a solution to the field equations for a spherical mass of fluid in hydrostatic equilibrium, our work is not necessarily complete. Though we have obtained a class of equilibrium solutions which satisfied all of our previously mentioned criteria for physical sensibility, any realistic stellar body must be stable against perturbations and it is important to determine the conditions under which the solutions obtained in the preceding section represent a stable matter configuration.

In this section we analyze whether spherically symmetric elastic fluid solutions of the type considered earlier this work are stable against radial perturbations which break the condition of staticity. The basis for our investigation is a 1964 paper by Chandrasekhar [37] which examines stability criteria for several idealized perfect fluid models. Anisotropic fluids possess an added degree of freedom, and thus require an additional constraint.

In calculations we will find it useful to use the non-static metric

$$ds^2 = -e^{2\Phi(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + r^2 d\Omega^2 \tag{6.1}$$

where

$$e^{2\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}. \tag{6.2}$$

Our goal is to examine the existence of stable anisotropic matter configurations corresponding to spherical masses which have a tenuity in excess of Buchdahl's bound. Recall that an elastic sphere admits the decomposition of the stress energy tensor in terms of the fluid flow vector u^α and the spacelike vector k^α

$$T^{\alpha\beta} = (\rho + p_t)u^\alpha u^\beta + p_t g^{\alpha\beta} + (p_r - p_t)k^\alpha k^\beta. \tag{6.3}$$

Suppose that a fluid element of our equilibrium solution is disturbed out of static equilibrium via a radial perturbation $v(r,t)$ such that $v(r,t) = \dot{\xi} = dr/dt$. The effect of the radial perturbation is to induce time dependent radial motions while preserving the spherical symmetry. At a fixed point we decompose the physical scalar fields p_r, p_t, ρ, and the metric functions λ (i.e. the mass function) and Φ as

$$f(r,t) = f_0(r) + \delta f(r,t) \tag{6.4}$$

where δf represents the small time dependent change in the quantity f due to the perturbation and the equilibrium solutions are denoted by a subscript naught.

The normalized fluid four velocity corresponding to a perturbed element is given by

$$u^\alpha = \left[\left(e^{2\Phi} - e^{2\lambda}\dot{\xi}^2\right)^{-1/2}, \left(e^{2\Phi} - e^{2\lambda}\dot{\xi}^2\right)^{-1/2} \dot{\xi}, 0, 0 \right]. \tag{6.5}$$

The normalized radial spacelike vector k^α follows from the orthogonality condition $k^\alpha u_\alpha = 0$,

$$k^\alpha = \left[e^{\lambda-\Phi} \left(e^{2\Phi} - e^{2\lambda}\dot{\xi}^2\right)^{-1/2} \dot{\xi},\ e^{\Phi-\lambda} \left(e^{2\Phi} - e^{2\lambda}\dot{\xi}^2\right)^{-1/2}, 0, 0 \right] \tag{6.6}$$

The field equations are expanded in powers of the perturbation quantity $\xi(r,t)$ and the Eulerian perturbations $\delta\Phi$, $\delta\lambda$, δp_r, δp_t and $\delta\rho$. To first order we re-obtain the familiar equations governing hydrostatic equilibrium used throughout previous sections of the thesis,

$$8\pi r^2 e^{2\lambda_0}\rho_0 = 2r\lambda_0' + e^{2\lambda_0} - 1, \tag{6.7}$$

$$8\pi r^2 e^{2\lambda_0} p_{r0} = 2r\Phi_0' - e^{2\lambda_0} + 1, \tag{6.8}$$

$$8\pi r e^{2\lambda_0} p_{t0} = (\Phi_0' - \lambda_0')(1 + r\Phi_0') + r\Phi_0''. \tag{6.9}$$

The unperturbed TOV equation is

$$\frac{2}{r}(p_{r0} - p_{t0}) + p_{r0}' + (\rho_0 + p_{r0})\Phi_0' = 0. \tag{6.10}$$

The first order perturbed field equations are

$$4\pi r^2 \delta\rho = \frac{\partial}{\partial r}\left(re^{-2\lambda_0}\delta\lambda\right), \tag{6.11}$$

$$4\pi r^2 e^{2\lambda_0}\delta p_r = r\delta\Phi' - \delta\lambda(1 + 2r\Phi_0'), \tag{6.12}$$

$$
\begin{aligned}
8\pi r\delta p_t =& e^{-2\lambda_0}\bigg(2\delta\lambda((\lambda_0' - \Phi_0')(1 + r\Phi_0') - r\Phi_0') - (1 + r\Phi_0')\delta\lambda' \\
&+ (1 - r\lambda_0' + 2r\Phi_0')\delta\Phi' + r\delta\Phi''\bigg) - e^{-2\Phi_0}r\delta\ddot{\lambda},
\end{aligned}
\tag{6.13}
$$

$$
\delta\dot{\lambda} = -4\pi r\dot{\xi}(\rho_0 + p_{r0})e^{2\lambda_0}. \tag{6.14}
$$

The last equation above, derived from the expression $G^{rt} = 8\pi T^{rt}$, is what encouraged us to introduce the quantity ξ which Chandrasekhar refers to as the "Lagrangian displacement" since we now find

$$
\delta\lambda = -4\pi r e^{2\lambda_0}\xi(\rho_0 + p_{r0}), \tag{6.15}
$$

or equivalently (via the hydrostatic equilibrium equations)

$$
\delta\lambda = -\xi(\lambda_0' + \Phi_0'). \tag{6.16}
$$

The first order perturbed TOV equation is

$$
\delta p_r' + (\delta\rho + \delta p_r)\Phi_0' + (p_{r0} + \rho_0)\left(\delta\Phi' + e^{2(\lambda-\Phi)}\ddot{\xi}\right) - \frac{2\delta p_t}{r} = 0. \tag{6.17}
$$

As in Chandrasekhar's work, the energy density perturbation $\delta\rho$ is obtained directly in terms of the Lagrangian displacement and equilibrium quantities. However, the pressure perturbations δp_r and δp_t can not be decoupled from $\delta\Phi$ without an additional assumption. Using the barotropic equation of state, we have a direct relation between δp_r and δp_t, so in order to proceed, we must only give some additional constraint which yields δp_r or δp_t as a function of the equilibrium quantities and the Lagrangian displacement. In order to proceed with the analysis, we will consider two cases. In the first case, we will suppose that δp_t is negligible compared to δp_r. In the second case, we will consider that the perturbation effects both the radial and tangential pressures equally, i.e. $\delta p_t = \delta p_r$.

We now introduce the assumption that all perturbations are separable with time dependence $e^{i\sigma t}$, where σ is a characteristic frequency. Let $\delta f = \delta f(r)$ now represent an amplitude of the perturbed quantity with time dependence as above. By reducing our system of equations we find an anisotropic analogue of the "pulsation equation" from Chandrasekhar's work corresponding to the specific class of linear barotropic equations of state considered throughout the thesis. The resulting differential equation, with the boundary conditions $\xi(r = 0) = 0$ and $\delta p_r(r = R) = 0$, constitutes a self-adjoint characteristic value problem for the eigenvalues σ^2 with eigenfunctions ξ.

$$L[\xi(r)] = \sigma^2 \xi(r) \tag{6.18}$$

The resulting linear operator L for either of the cases described above is not shown in this body of text due to its length. However, L is not exceedingly complex and is furthermore specified explicitly in terms of the equilibrium metric functions m_0 and Φ_0. It turns out that for the solutions generated via our algorithm L is a regular Sturm Liouville operator over our domain of interest, and thus all the useful results of the theory of Sturm-Liouville differential equations are applicable. Namely, if the distinct eigenvalues of the linear operator L are ordered such that $\sigma_0^2 < \sigma_1^2 < \cdots < \infty$, each σ_i^2 corresponds to the eigenfunction ξ_i with i zeros in $(0, R)$. In order to determine the stability of a given equilibrium solution it suffices to determine the sign of the smallest magnitude eigenvalue.

Using the algorithm from the preceding section, the equilibrium mass $m(r)$ and the radial pressure $p_r(r)$ functions are obtained explicitly in an analytic form for a given equation of state, defined by the choice of Ω and χ. We thus have a well defined Sturm-Liouville problem with Dirichlet boundary conditions, the boundary radius R being determined as previously by using a numerical root finder on the function p_r (this value of course being dependent on our choice of parameters). We fix the equation of state and the parameter n and use a numerical eigensystem package to determine the smallest magnitude eigenvalue. It is through this procedure that we examine the stability of specific solutions from our infinite class.

Importantly, we wish to investigate the stability of those static fluids which break the Buchdahl boundary. For example, consider the fluids with $p_r + 2p_t = \Omega\rho$. In this case, those solutions with $\Omega > 1$ permit a tenuity in excess of the upper bound found by Buchdahl. By testing various values of Ω we find in both the $\delta p_t = 0$ and $\delta p_r = \delta p_t$ cases that solutions with $\Omega > 1$ are not stable.

Table 6.1: A list of the stability tests conducted for various equations of state under the assumption that the perturbation to the tangential pressure is negligible.

χ	Ω	Smallest integer n such that $\exists$ finite boundary R	ρ_c	$2M/R > 8/9$	$p_r \leq \rho$	$p_t \leq \rho$	Stable?
1/4	1	9	5/4	Yes	Yes	No	No
1/2	1/4	2	6	No	Yes	Yes	No
1/2	1/2	3	3	No	Yes	Yes	No
1/2	1	5	3/2	Yes	Yes	No	No
1/2	5/4	6	6/5	Yes	Yes	No	No
1	1/4	1	8	No	Yes	Yes	No
1	1/2	2	4	No	Yes	Yes	No
1	1	3	2	No	Yes	Yes	No
1	2	5	1	Yes	Yes	No	No
3/2	1/4	1	10	No	Yes	Yes	Yes $\forall n$
3/2	1/2	1	5	No	Yes	Yes	Yes if $n \geq 3$
3/2	3/4	2	10/3	No	Yes	Yes	No
3/2	1	2	5/2	No	Yes	Yes	No
2	1/2	1	6	No	Yes	Yes	Yes $\forall$ n
2	1	2	3	No	Yes	Yes	Yes if $n \geq 4$
2	3/2	2	2	No	Yes	Yes	Yes if $n \geq 47$
2	2	3	3/2	No	Yes	Yes	No
2	3	4	1	Yes $\forall$ $n \geq 8$	Yes	No	No

The stability analysis presented in this section casts doubt on the claim that the class of solutions developed in the previous chapter result in physically reasonable distributions which break Buchdahl's tenuity boundary. Above we present results for a variety of equations of state, demonstrating that the results suggest only those solutions which satisfy $2M/R < 8/9$ and the dominant energy condition are stable.

It must be remarked that the investigations presented in this section are only very preliminary and are meant to serve as a starting point for future, more in-depth research in to the problem of anisotropic fluid stability. There exist non-linear barotropes which break the Buchdahl bound, and we do not make any claims about their stability. However, using the template above, given an equilibrium solution, we may relatively easily find the corresponding linear operator and perform various tests on those, too.

For perturbations of isotropic fluid models, that the eigenfrequency of the lowest order radial mode is real valued is a necessary and sufficient condition for stability against all kinds of small adiabatic oscillations [27]. For anisotropic matter $p_r \neq p_t$ may result in instability for perturbations which break the spherical symmetry. Thus we may also need to consider non-radial perturbative stability for those models which are ruled stable under radial perturbations.

Chapter 7

Discussion

The main work of this thesis was motivated by a desire to examine spherically symmetric hydrostatic fluids which break the Buchdahl bound. Initially, the work aimed at investigating a solution generating algorithm developed by Professor Lake [5] which produces all spherically symmetric static perfect fluid solutions to the Einstein field equations.

The problem of algorithmically generating spacetimes corresponding to spheres in hydrostatic equilibrium goes back at least as far as Tolman [20]. The more recent developments in this area of research, including the work in this thesis and much of the solution generating techniques in the literature [38, 39, 40, 41] have been largely influenced by Lake either directly or indirectly, see [5, 34, 42]. Of course, much of our research is made possible by developments in computer algebra software, including the GRTensorIII Maple package [6] which has been incredibly useful.

One may confidently argue that the nature of anisotropic matter in general relativity is a subject which has not yet been studied deeply enough to be considered thoroughly understood. Well known solution textbooks neglect to include various exact anisotropic matter solutions which have been present in the literature for decades [43]. As mentioned in chapter two, it has been speculated for decades that high density matter may be anisotropic. Thus, having broad techniques for developing novel, well-behaved solutions which satisfy our sensibility criteria is an important method by which we may understand and investigate various claims and hypotheses made throughout the literature.

In chapter five, we presented an analysis of an infinite class of hydrostatic anisotropic spherical solutions to Einstein's field equations. Using a particularly interesting generating function $\Phi(r)$, we found infinitely many previously unknown analytical closed-form solutions which satisfied our criteria for physical reasonability. We examined the character of these solutions under various choices of the independent parameters n, χ and Ω used in generating the solutions.

An important goal of our research was to investigate the quantity $2M/R$, defined by the radius at which the fluid solution junctions to a vacuum Schwarzschild exterior. In chapter three, the ubiquitous inequality due to Buchdahl was recovered as a limiting case (corresponding to pressure isotropy) of a more general procedure using less restrictive assumptions. We discovered that for the class of barotropic elastic fluids examined in this analysis, the tenuity was bounded from above and below and verified that our algorithmically generated solutions were in agreement with these bounds.

To investigate further the physicality of our solutions we examined a model of acoustic wave propagation in general relativity [33]. Having a framework for determining sound wave propagation speeds in arbitrary directions with arbitrary polarizations is important, and drawing grand conclusions based on sound speeds is only useful provided the sound speed calculation is robust. We used the model based on the Carter-Quintana elastic fluid formulation presented in chapter four to examine the transverse sound speeds in spherically symmetric matter configurations. We determined that for the equations of state with $\chi = 2$, the transverse sound speeds only remained subluminal if the dominant energy condition was satisfied.

Finally, in chapter six we presented a procedure for determining the perturbative stability of anisotropic matter distributions with general equations of state. This procedure was applied to our linear barotropic solutions to investigate their stability under radial perturbations which break the condition of staticity. Though our investigations were limited by computational constraints, the cases tested indicated that only those solutions which satisfied both the Buchdahl bound and the energy conditions were stable. Much more work needs to be completed to satisfactorily address the problem of whether there exist stable matter configurations with surface tenuity in excess of that found for the constant density Schwarzschild interior solution. To rephrase the problem more explicitly: an important and open question in relativistic fluid mechanics is whether or not there exist stable, static fluid spheres which exceed that tenuity value which saturates Buchdahl's bound. The research presented in chapter six suggests for our class of solutions that the answer is in the negative, however one must be careful not to extrapolate too broadly from these results. Our research at present only serves as a basis for hypothesizing that such stable configurations do not exist.

Appendix

A.1 List of Parameters and Definitions

$\mathcal{M}$	- A manifold, typically 4D, representing the locus of events called spacetime.
x^α, y^i	- Coordinates, typically $\alpha = 0, 1, 2, 3$, and $i = 1, 2, 3$.
$T_q\mathcal{M}, T_q^*\mathcal{M}$	- The tangent and cotangent spaces of $\mathcal{M}$ at the point q.
$,\nu, ;\nu$	- Partial and covariant differentiation with respect to the coordinates x^ν.
u^α	- Reserved for the tangent vector to a timelike curve $x^\alpha(\lambda)$, with $u^\alpha = \frac{dx^\alpha}{d\lambda}$.
$g_{\alpha\beta}, \eta_{\alpha\beta}$	- The general metric tensor and the metric tensor corresponding to flat space.
ds^2	- The line element, representing infinitesimal coordinate displacements in $\mathcal{M}$.
$R_{\alpha\beta\gamma\delta}, R_{\alpha\beta}, R$	- The Riemann and Ricci tensors and the Ricci scalar.
$G^\alpha_\beta, T^\alpha_\beta$	- The Einstein tensor $R^\alpha_\beta - \frac{1}{2}R\delta^\alpha_\beta$ and the energy momentum tensor.
Λ	- The cosmological constant appearing in the field equations.
t, r, θ, ϕ	- Coordinates used throughout the thesis for spherically symmetric spacetimes.
$\Phi(r,t), \lambda(r,t)$	- Metric functions describing the spherically symmetric spacetime geometry.
$m(r,t)$	- The 'effective gravitational mass' $m(r,t) = \frac{r}{2}(1 - e^{-2\lambda})$.
$\rho(r)$	- Energy density in the comoving frame.
$p_r(r), p_t(r)$	- Radial and tangential pressure as measured by a comoving observer.

Σ	- Typically a boundary surface across which junction conditions are satisfied.
R	- A boundary radius, representing the value r at which $p_r(R) = 0$.
M	- The effective gravitational mass at the boundary radius, $m(R)$.
α	- The 'tenuity', $2M/R$.
Ω, χ	- Constant parameters in the barotropic equation of state $p_r + \chi p_t = \Omega\rho$.
u, v, Δ	- Parametric functions based on the quantities m, p_r and p_t.
α, β, H	- Functions of u and v used in constraining the tenuity in chapter 3.1.
$x, y, w(x, y)$	- Parametric functions used in 3.2 related to m and p_r.
$\beta(r)$	- An auxiliary variable $\beta = 2\ln(r)$ used in 3.2 to simplify the field equations.
p^{ab}, E^{abcd}	- The pressure and relativistic Hadamard tensors as defined in [31].
λ^a	- A vector normal to the wavefront Σ representing a sound wave in chapter 4.
v^2_{pq}	- Squared sound speed with p and q defining wave direction and polarization.
a, b	- Functions of r used in algorithmically generating anisotropic fluid solutions
n, α	- Parameters used in defining the generating function Φ used in chapter 5.
f, g, h_a, k_a	- Functions of r defined in chapter 5 to simplify our analytic mass function.
ρ_c, p_c	- The energy density and pressure evaluated at $r = 0$.
$\xi(r, t)$	- The Lagrangian displacement used in our stability analysis.
$\delta f(r, t)$	- A small time dependent change in the quantity f away from equilibrium.
σ	- Eigenfrequency corresponding to an oscillatory mode of the perturbed fluid.

A.2 Modelling Elastic Fluids in General Relativity: The Carter-Quintana Formalism

A novel (and widely cited) method of describing elastic fluids is outlined by Carter and Quintana in a 1972 publication [31]. To begin this section I will present a basic overview of what will be referred to hereafter as the CQ formalism. The main reason for emphasizing this work is that it provides a theoretical framework for calculating the speed of sound in anisotropic fluids.

The CQ formalism is essentially a 3+1 foliation of spacetime (see e.g. chapter 4 of [8]), utilizing a differentiable projection $\mathcal{P}$ relating the 4-D spacetime manifold $\mathcal{M}$ to a 3-D 'material space' $\mathcal{X}$.

$$\mathcal{P} : \mathcal{M} \longmapsto \mathcal{X} \tag{1}$$

The points in the material space $\mathcal{X}$ are to be thought of as the particles constituting the fluid, such that the inverse of the projection map $\mathcal{P}^{-1}$ acting on a point $q \in \mathcal{X}$ is the timelike world-line of the particle associated with q (equivalently each point $p \in \mathcal{M}$ along a given world line is projected to the same point in $\mathcal{X}$). $\mathcal{P}$ is thus defined such that the kernel of the pushforward $\mathcal{P}_* : T_p\mathcal{M} \longmapsto T_{\mathcal{P}(p)}\mathcal{X}$ consists of those vectors which are collinear with the tangent to the flow lines. Assume that $\mathcal{M}$ is locally charted by coordinates x^a, $a = 0, 1, 2, 3$, and that $\mathcal{X}$ is locally charted by coordinates y^A, $A = 1, 2, 3$. The projection $\mathcal{P}$ is such that any point in a differentiably embedded spacelike hypersurface in $\mathcal{M}$, say $\mathcal{J}$, has a neighbourhood $\mathcal{U} \subset \mathcal{J}$ such that the mapping $\mathcal{P} : \mathcal{U} \longmapsto \mathcal{X}$ is a differentiable embedding.

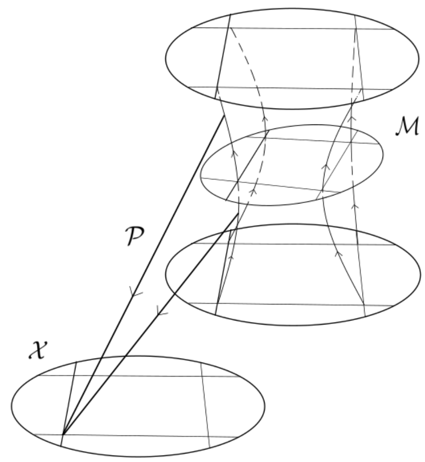

Figure A.1: The Carter-Quintana formalism utilizes a differentiable projection $\mathcal{P}$ relating the pseudo-Riemannian spacetime manifold $\mathcal{M}$ to the material space $\mathcal{X}$.

A coordinate patch x^a on $\mathcal{M}$ is referred to as comoving if there exists a relationship between it and a fixed coordinate patch y^A on $\mathcal{X}$ by the constraint that when $p = p(x^0, x^1, x^2, x^3) \in \mathcal{M}$, then $\mathcal{P}(p) = \mathcal{P}(p)(y^1, y^2, y^3)$, where $y^1 = x^1$, $y^2 = x^2$, $y^3 = x^3$. The coordinate patch is referred to as properly comoving if $x^0 = \tau$ where τ is a proper time parameter increasing along the particle world lines.

Note that the projection $\mathcal{P}$ distinguishes a fundamental field of timelike flow vectors that are tangent to the world lines in $\mathcal{M}$ and normalized such that with respect to some basis $u^a u_a = -1$. One can subsequently define a metric tensor on the subspaces spanned by those basis vectors which are orthogonal to the flow line tangents (or equivalently a projection operator which projects arbitrary vectors in to their components in the subspace orthogonal to $\mathbf{u}$).

$$\gamma_{ab} = g_{ab} + u_a u_b \tag{2}$$

Note that $\gamma^{ac}\gamma_{bc} = \gamma^{a}_{\ b}$, and that contractions of this subspace metric with the flow tangent vectors yields zero. Thus any tensor for which contraction with $\mathbf{u}$ on all indices yields zero, the metric γ_{ab} can be used in place of g_{ab} for the raising and lowering of indices. Such tensors are referred to in the CQ formalism as 'orthogonal spacetime tensors'. The induced metric γ measures distance between neighbouring particles in their local rest frame and is referred to as the "local state of strain".

The projection $\mathcal{P}$ introduces a pullback $\mathcal{P}^*$ from the space of covariant vectors at a fixed point $q \in \mathcal{X}$, $T^*_q\mathcal{X}$, to the space of covariant vectors at any fixed point $p \in \mathcal{M}$, $T^*_p\mathcal{M}$, where $p = \mathcal{P}^{-1}(q)$. Notice that this mapping is an isomorphism onto to the subspace of covectors in $T^*_p\mathcal{M}$ orthogonal to the flow tangent vectors. Similarly, the set of contravariant vectors orthogonal to $\mathbf{u}$ at a point $p \in \mathcal{M}$ is isomorphic to the set of contravariant vectors spanning $T_{\mathcal{P}(p)}\mathcal{X}$. Thus in general, the projection $\mathcal{P}$ induces a canonical isomorphism, say $\tilde{\mathcal{P}}$ between general mixed orthogonal spacetime tensors and their corresponding projection in to the material space.

$$\tilde{\mathcal{P}} : T(p) \longleftrightarrow \tilde{\mathcal{P}}(T)(q) \tag{3}$$

Above, $T(p)$ is a general mixed orthogonal spacetime tensor, $\tilde{\mathcal{P}}(T)$ is the corresponding projection in to the material space $\mathcal{X}$ (referred to as a material tensor), and $q = \mathcal{P}(p)$.

To extend the isomorphism above from tensors at fixed points to general tensor fields, CQ introduce the concept of "material constancy". A tensor field T is called materially constant if the image of T in $\mathcal{X}$ under the isomorphism is the same everywhere along a given world line (i.e. $\tilde{\mathcal{P}}(T(p))(q)$ is the same $\forall\ p \in \mathcal{P}^{-1}(q)$). Thus $\tilde{\mathcal{P}}$ induces a canonical isomorphism from the space of material tensor fields on $\mathcal{X}$ to the space of materially constant orthogonal spacetime tensor fields on $\mathcal{M}$.

If a spacetime tensor field T at a point $p \in \mathcal{M}$ is orthogonal but not materially constant, there exists a correspondence between $T(p)$ and a "material tensor field function of time" on $\mathcal{X}$ by

$$\tilde{\mathcal{P}} : T(p) \longleftrightarrow \tilde{\mathcal{P}}(T)(q, \tau), \tag{4}$$

where $q = \mathcal{P}(p)$, and $\tau = \tau(p)$ is a scalar proper time coordinate function on $\mathcal{M}$. Note that such a correspondence is non-canonical as it depends on the choice of τ. However, we may canonically define the convected differential of an orthogonal spacetime tensor field, $dT(p)$, which is the differential of the tensor field due to transport by a distance $d\tau$ along the world line through $p \in \mathcal{M}$ and satisfies the equation

$$\tilde{\mathcal{P}}(dT(p)) = d\tilde{\mathcal{P}}(T)(q, \tau). \tag{5}$$

Above, $d\tilde{\mathcal{P}}(T)(q, \tau)$ is the corresponding "material differential" defined at the fixed point $q = \mathcal{P}(p) \in \mathcal{X}$. $d\tilde{\mathcal{P}}(T)(q, \tau)$ is the differential of the material tensor field function of time at q, due to the variation $d\tau$ in τ evaluated at $\tau = \tau(p)$. Finally, we arrive at the definition of the "convected derivative" of an orthogonal tensor field on $\mathcal{M}$,

$$[T]^{\cdot}(p) = \frac{dT(p)}{d\tau}. \tag{6}$$

$[T]^{\cdot}(p)$ is zero if and only if the tensor field T is materially constant. This definition of convected derivative was extended by CQ to apply to general (not strictly orthogonal) spacetime tensor fields by introducing the concept of a convected 4-frame (see [31]). The result is that for a general mixed spacetime tensor field, the convected derivative has components

$$[T^{b\ldots}_{a\ldots}]^{\cdot} = \dot{T}^{b\ldots}_{a\ldots} - T^{c\ldots}_{a\ldots}(u^b_{;c} + u^b\dot{u}_c) - \cdots + T^{b\ldots}_{c\ldots}(u^c_{;a} + u^c\dot{u}_a) + \ldots . \tag{7}$$

The covariant derivative of the flow field admits a fundamental decomposition in terms of the orthogonal tensors ω_{ab} and θ_{ab} as

$$u_{a;b} = \theta_{ab} + \omega_{ab} - \dot{u}_a u_b, \tag{8}$$

where $\omega_{ab} = \omega_{[ab]}$ and $\theta_{ab} = \theta_{(ab)}$ are the "rotation" and "strain rate" tensors respectively. The strain rate tensor is further decomposed in terms of the trace free "shearing rate" tensor σ_{ab} and the "expansion" scalar $\theta = \theta^a_a$,

$$\theta_{ab} = \sigma_{ab} + \frac{1}{3}\theta\gamma_{ab}. \tag{9}$$

A convected 4-frame is a tetrad basis spanning the tangent space in $\mathcal{M}$ such that three of the contravariant vectors and three of the covariant vectors are materially constant, orthogonal vector fields, equivalent under the isomorphism $\tilde{\mathcal{P}}$ to the three contravariant and covariant vectors spanning the tangent space of $\mathcal{X}$. The remaining contravariant and covariant basis elements of the 4-frame are u^a and u_a.

The CQ formalism presents a theory of elastic fluids formulated in terms of orthogonal spacetime tensor field "functions of strain". A tensor field T is called a function of strain if, along each world line, its components with respect to a convected tetrad are well defined functions of the (six independent) components of the induced metric γ_{cd} with respect to the same convected tetrad.

A matter distribution that is isolated from any external fields will be called perfectly elastic if it obeys the following criteria [31]:

1. There is no material energy transport relative to the local rest frames of the medium. In other words, u^a is an eigenvector of the energy momentum tensor T^{ab}.

2. The energy momentum tensor due to the matter distribution is divergence free, $T^{ab}{}_{;b} = 0$.

3. Along any given world line, or equivalently for any point $q \in \mathcal{X}$, the quantities ρ and p^{ab} (as defined below) are well defined functions of strain.

From condition 1, we may immediately write the energy momentum tensor in terms of a scalar quantity ρ, and an orthogonal, symmetric tensor p^{ab} called the pressure tensor. ρ is the eigenvalue associated with the flow vector u^a and is called the "energy density".

$$T^{ab} = \rho u^a u^b + p^{ab} \tag{10}$$

Projecting the conservation equation $T^{ab}{}_{;b} = 0$ on to the surfaces orthogonal to the flow lines yields three independent equations relating the acceleration $\dot{u}^\alpha$ to the spatial gradients of the pressure tensor,

$$\left(\rho\gamma^{ab} + p^{ab}\right)\dot{u}_b + p^{cd}{}_{;b}\gamma^a_c\gamma^b_d = 0. \tag{11}$$

Upon specifying the energy density and pressure tensor as functions of strain, we obtain a complete system of equations for the three independent components of the flow vector u^a. However, the conservation equations contain a fourth independent equation obtained by contracting $T^{ab}{}_{;b}$ with the flow vector. The expression one obtains,

$$\dot{\rho} + \rho\theta + p^{ab}\theta_{ab} = 0, \tag{12}$$

must be identically satisfied, otherwise our system of equations is overdetermined. Using the identity

$$\frac{d}{d\tau}(\gamma_{ab}) = [\gamma_{ab}]^{\cdot} = 2\theta_{ab}, \tag{13}$$

we can write the so-called "consistency relation" above in terms of convected differentials along the flow lines

$$d\rho = -\frac{1}{2}(\rho\gamma^{ab} + p^{ab})d(\gamma_{ab}). \tag{14}$$

An important corollary is that for an equation of state given as some function of strain, $\rho = \rho(\gamma_{ab})$, the six independent components of p^{ab} are automatically restricted such that the consistency relation is satisfied,

$$p^{ab} = -\frac{\partial\rho}{\partial(\gamma_{ab})} - \rho\gamma^{ab}. \tag{15}$$

A.3 An Algorithm for the Summation of N Non-Interacting Perfect Fluids

In this brief addendum the goal is to introduce an extension of the work due to the Brazilian relativist Patricio Letelier [26] in which a sum of two non-interacting perfect fluids was shown to reduce to a familiar decomposition of an anisotropic fluid with two equal principal stresses. This section of the appendix follows Letelier closely and thus uses a signature of -2.

In the situation discussed by Letelier the stress energy tensor was taken as

$$T^{\mu\nu}(u_1, u_2) = t^{\mu\nu}(u_1) + t^{\mu\nu}(u_2) \tag{16}$$

where $t^{\mu\nu}(u_i) = (\rho_i + p_i)u_i^\mu u_i^\nu - p_i g^{\mu\nu}$, and by assumption $t^{\mu\nu}(u_i)_{;\mu} = 0$. The flow vectors u_i^μ are all timelike. The procedure is as follows; consider the term

$$(\rho_1 + p_1)u_1^\mu u_1^\nu + (\rho_2 + p_2)u_2^\mu u_2^\nu \tag{17}$$

and note the invariance under the transformations

$$u_1^\mu \longmapsto v_1^{\star\mu} = \cos(\alpha)u_1^\mu + \left(\frac{\rho_2 + p_2}{\rho_1 + p_1}\right)^{1/2} \sin(\alpha)u_2^\mu \tag{18}$$

$$u_2^\mu \longmapsto k_1^{\star\mu} = \cos(\alpha)u_2^\mu - \left(\frac{\rho_2 + p_2}{\rho_1 + p_1}\right)^{-1/2} \sin(\alpha)u_1^\mu. \tag{19}$$

i.e. $T^{\mu\nu}(u_1, u_2) = T^{\mu\nu}(v_1^\star, k_1^\star)$. Imposing the constraint that the newly transformed vectors are orthogonal (i.e. one is spacelike and one is timelike) determines the angular parameter α in terms of the pressures and energy densities and the inner product of the original timelike vectors.

$$v_1^{\star\mu} k_{1\star\mu} = 0 \implies \tan(2\alpha) = 2u_1^\mu u_{2\mu} \left(\frac{[(\rho_1 + p_1)(\rho_2 + p_2)]^{1/2}}{(\rho_1 + p_1) - (\rho_2 + p_2)}\right) \tag{20}$$

It can then be shown that $v_1^{\star\mu}$ is timelike and that $k_1^{\star\mu}$ is spacelike. We then normalize the newly generated vectors as

$$v_1^\mu \equiv \frac{v_1^{\star\mu}}{(v_1^{\star\nu} v_{1\nu}^\star)^{1/2}} \tag{21}$$

$$k_1^\mu \equiv \frac{k_1^{\star\mu}}{(-k_1^{\star\nu} k_{1\nu}^\star)^{1/2}} \tag{22}$$

and make the following definitions as in Letelier's work:

$$\bar{\rho}_1 \equiv T^{\mu\nu} v_{1\mu} v_{1\nu} = (\rho_1 + p_1) v_1^{\star\alpha} v_{1\alpha}^\star - (p_1 + p_2) \tag{23}$$

$$p_{r1} \equiv T^{\mu\nu} k_{1\mu} k_{1\nu} = (p_1 + p_2) - (p_2 + \rho_2) k_1^{\star\alpha} k_{1\alpha}^\star \tag{24}$$

$$p_{t1} \equiv p_1 + p_2 \tag{25}$$

Then we may rewrite the decomposition of the stress energy tensor in the following form:

$$T^{\mu\nu} = (\bar{\rho}_1 + p_{t1}) v_1^\mu v_1^\nu + (p_{r1} - p_{t1}) k_1^\mu k_1^\nu - p_{t1} g^{\mu\nu} = \bar{\rho}_1 v_1^\mu v_1^\nu + S_1^{\mu\nu} \tag{26}$$

where

$$S_1^{\mu\nu} = (p_{r1} - p_{t1}) k_1^\mu k_1^\nu - p_{t1} (g^{\mu\nu} - v_1^\mu v_1^\nu). \tag{27}$$

Therefore, the summation of two non-interacting perfect fluids yields a decomposition equivalent to that for our familiar anisotropic fluid in spherical symmetry wherein $T^\mu_\nu = \mathrm{diag}[-\bar{\rho}_1, p_{r1}, p_{t1}, p_{t1}]$.

One may extend this mathematical problem algorithmically by investigating the summation of N non-interacting perfect fluids:

$$T^{\mu\nu} = \sum_{i=1}^{N} t^{\mu\nu}(u_i) \tag{28}$$

Notice that we have $2i$ unknowns in terms of the physical parameters ρ_i and p_i. Furthermore, there are $3i$ unknowns in the components of the individual timelike fluid flow vectors u_i^μ and the metric constitutes 10 unknowns. There are 10 field equations, supplemented by the supposition that $\exists\ 4i$ individual conservation equations $t^{\mu\nu}(u_i)_{;\mu} = 0$. If we specify an equation of state for each of the non-interacting fluids $f_i(\rho_i, p_i) = 0$ then we have a well defined system of $5i + 10$ equations in $5i + 10$ unknowns.

For the problem of three interacting fluids, we then have the energy momentum tensor

$$T^{\mu\nu} = (\bar{\rho}_1 + p_{t1})v_1^\mu v_1^\nu + (p_{r1} - p_{t1})k_1^\mu k_1^\nu - p_{t1}g^{\mu\nu} + (\rho_3 + p_3)u_3^\mu u_3^\nu - p_3 g^{\mu\nu} \tag{29}$$

and as before we consider the term

$$(\bar{\rho}_1 + p_{t1})v_1^\mu v_1^\nu + (\rho_3 + p_3)u_3^\mu u_3^\nu \tag{30}$$

and make the transformations:

$$v_1^\mu \longmapsto v_2^{\star\mu} = \cos(\alpha_2)v_1^\mu + \left(\frac{\rho_3 + p_3}{\bar{\rho}_1 + p_{t1}}\right)^{1/2} \sin(\alpha_2)u_3^\mu \tag{31}$$

$$u_3^\mu \longmapsto k_2^{\star\mu} = \cos(\alpha_2)u_3^\mu - \left(\frac{\rho_3 + p_3}{\bar{\rho}_1 + p_{t1}}\right)^{-1/2} \sin(\alpha_2)v_1^\mu. \tag{32}$$

Again, we impose that the newly generated vectors are orthogonal resulting in an expression for the angular parameter α_2.

$$\tan(2\alpha_2) = \frac{[(\bar{\rho}_1 + p_{t1})(\rho_3 + p_3)]^{1/2}}{(\bar{\rho}_1 + p_{t1}) - (\rho_3 + p_3)} 2v_1^\mu u_{3\mu} \tag{33}$$

As previous it can then be shown that $v_2^{\star\mu}$ is timelike and that $k_2^{\star\mu}$ is spacelike and we normalize the newly generated vectors as

$$v_2^\mu \equiv \frac{v_2^{\star\mu}}{(v_2^{\star\nu} v_{2\nu}^\star)^{1/2}} \tag{34}$$

$$k_2^\mu \equiv \frac{k_2^{\star\mu}}{(-k_2^{\star\nu} k_{2\nu}^\star)^{1/2}} \tag{35}$$

and make the following definitions

$$\bar{\rho}_2 \equiv T^{\mu\nu} v_{2\mu} v_{2\nu} = (\bar{\rho}_1 + p_{t1}) v_2^{\star\alpha} v_{2\alpha}^\star - (p_{t1} + p_3) \tag{36}$$

$$p_{r2} \equiv T^{\mu\nu} k_{2\mu} k_{2\nu} = (p_{t1} + p_3) - (p_3 + \rho_3) k_2^{\star\alpha} k_{2\alpha}^\star \tag{37}$$

$$p_{t2} \equiv p_{t1} + p_3 \tag{38}$$

Then we may rewrite the decomposition of the stress energy tensor which represents the summation of 3 non-interacting perfect fluids in the following form:

$$T^{\mu\nu} = (\bar{\rho}_2 + p_{t2}) v_2^\mu v_2^\nu + (p_{r1} - p_{t1}) k_1^\mu k_1^\nu + (p_{r2} - p_{t2}) k_2^\mu k_2^\nu - p_{t2} g^{\mu\nu} = \bar{\rho}_2 v_2^\mu v_2^\nu + S_2^{\mu\nu} \tag{39}$$

where

$$S_2^{\mu\nu} = (p_{r1} - p_{t1}) k_1^\mu k_1^\nu + (p_{r2} - p_{t2}) k_2^\mu k_2^\nu - p_{t2}(g^{\mu\nu} - v_2^\mu v_2^\nu). \tag{40}$$

It is easy to see how this procedure can be extended algorithmically to obtain the decomposition of N non-interacting fluids in terms of one timelike and $N-1$ spacelike vectors.

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