Certain Weyl Modules of Infinite Dimensional Lie Superalgebras

by

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A thesis submitted to the
Department of Mathematics and Statistics
in conformity with the requirements for
the degree of Doctor of Philosophy

Queen's University
Kingston, Ontario, Canada
May 2024

Abstract

The notion of a Weyl module for classical affine algebras, a type of infinite dimensional Lie algebra, was introduced in 2001 by Chari and Pressley. These modules are universal, finite dimensional highest weight modules. We expand these ideas to infinite-dimensional Lie superalgebras; in particular to Lie superalgebras of the form $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is isomorphic to one of the superalgebras $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}(1\,|\,1)$, and $\mathfrak{osp}(1\,|2)$ and A is $\mathbb{C}[t]$ or $\mathbb{C}[t, t^{-1}]$. We prove that these Weyl modules are universal, finite-dimensional, highest weight $\mathfrak{g} \otimes A$-modules. We find that the weights satisfy a recurrence relation, and in the case of $\mathfrak{g} = \mathfrak{sl}(1\,|\,1)$, provide a formula for the dimension of each weight space. We also provide irreducibility criteria and the structure of the composition series. We then study $\mathfrak{sl}(1\,|\,1) \otimes A$ modules generated by an infinite set of highest weight vectors, and with finite support. For these modules, we provide irreducibility criteria, find that the weights are of exponential-polynomial form, and provide a formula for the dimension of each weight space.

Acknowledgements

I would first like to thank my advisors, Ivan Dimitrov and David Wehlau, for taking me on as a student, guiding and supporting me through graduate school, introducing me to the world of representation theory, and giving me an interesting project to work on.

I also thank James Mingo, Charles Paquette, Hadi Salmasian, and Aaron Vincent for agreeing to serve on my Doctoral Thesis examination committee. I thank the faculty and staff of the Queen's Mathematics and Statistics department. I also thank Ping Li and Peter Taylor for teaching me how to be an educator, and Jennifer Read for helping me with all of my administrative tasks.

I have made many friends at Queen's who have made graduate school an enjoyable place to be, including Annika, Neil, Sasha, Henry, Ben, Somya, Anita, Bora, Deepanshu, Ananya, etc. Special thanks to my best friend Tariq for patiently listening to me tell him about algebra, and for helping me see the positive when things got hard. Finally, I thank my family, mom Karmela, late father Srđan, brother Alek, sister-in-law Yunesta, my grandparents Gojko, Finka, Mirjana, and Mihailo, my aunts Ljilja and Tanja, and my cousins for their unconditional love and support.

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List of Abbreviations and symbols

Let A be $\mathbb{C}[t]$ or $\mathbb{C}[t, t^{-1}]$ and let $\mathfrak{g}$ be a basic classical Lie superalgebra.

Notation regarding Lie superalgebras and their roots:

$\widehat{\mathfrak{g}}$	the current Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}[t]$
$\overline{\mathfrak{g}}$	the loop Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$
x_n	the element $x_n := x \otimes t^n \in \mathfrak{g} \otimes A$
$\mathfrak{h} \subset \mathfrak{g}$	a Cartan subalgebra of $\mathfrak{g}$
$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$	the triangular decomposition of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$
$\mathfrak{b} \subset \mathfrak{g}$	a Borel subalgebra of $\mathfrak{g}$
Δ	the root system for $\mathfrak{g}$
$\Delta_{\bar{0}}$ (resp. $\Delta_{\bar{1}}$)	the set of even (resp. odd) roots
Δ^+ (resp. Δ^-)	the set of positive (resp. negative) roots

$\Sigma \subset \Delta^+$	a base (or set of simple roots)
$\Sigma_{\bar{0}}$ (resp. $\Sigma_{\bar{1}}$)	the set of even (resp. odd) simple roots w. r. t. Σ
$\mathfrak{g}_\alpha$	the root space of $\alpha \in \mathfrak{h}^*$
$U(\mathfrak{g})$	the universal enveloping algebra of $\mathfrak{g}$
X_n	the element $X \otimes t^n \in U(\mathfrak{g} \otimes A)$ corresponding to $x \in \mathfrak{g}$
$e_\alpha, f_\alpha, h_\alpha$	the Chevalley generators with respect to the even root $\alpha \in \mathfrak{h}^*$ such that $\text{span}\{e_\alpha, f_\alpha, h_\alpha\} \cong \mathfrak{sl}_2(\mathbb{C})$
$x_\alpha, y_\alpha, z_\alpha$	the Chevalley generators with respect to the odd root $\alpha \in \mathfrak{h}^*$

Notation for elements of certain Lie superalgebras:

For $\mathfrak{g} = \mathfrak{sl}(2)$	$e = E_{12}, f = E_{21}, h = E_{11} - E_{22}$
For $\mathfrak{g} = \mathfrak{gl}(1\,\vert\,1)$	$x = E_{12}, y = E_{21}, h' = E_{11}, h'' = E_{22}$
For $\mathfrak{g} = \mathfrak{sl}(1\,\vert\,1)$	$x = E_{12}, y = E_{21}, z = E_{11} + E_{22}$
For $\mathfrak{g} = \mathfrak{osp}(1\,\vert\,2)$	$e = E_{23}, x = \frac{1}{\sqrt{2}}(E_{13} + E_{21}), f = E_{32}, y = \frac{1}{\sqrt{2}}(E_{12} - E_{31}),$ $h = E_{22} - E_{33}$

Notation regarding representations of $\mathfrak{g}$:

V_μ	For a $\mathfrak{g}$-module V and $\mu \in \mathfrak{h}^*$, $V_\mu = \{v \in V \,\vert\, h \cdot v = \mu(h)v \ \forall h \in \mathfrak{h}\}$
supp$V$	The support of V, defined to be supp$V := \{\mu \in \mathfrak{h}^* \,\vert\, V_\mu \neq 0\}$
$W(\Lambda, m)$	Weyl module; see Definitions 4.1.1, 5.1.1, 6.1.1
$L(\mathbf{a})$	the $\mathbb{Z}$-graded $\mathfrak{h} \otimes A$-module; see Definition 5.2.2
$L(\mathbf{a})\langle 0 \rangle$	the irreducible submodule of $L(\mathbf{a})$
$L(\mathbf{a})\langle j \rangle$	the vertical shift of $L(\mathbf{a})\langle 0 \rangle$ by integer j
$V(\mathbf{a})$	$\mathbb{Z} \times \mathbb{Z}$-graded $\mathfrak{g} \otimes A$-module; see equation 5.17
$V(\mathbf{a})\langle 0 \rangle$	the irreducible submodule of $V(\mathbf{a})$
$V(\mathbf{a})\langle j \rangle$	the vertical shift of $V(\mathbf{a})\langle 0 \rangle$ by integer j

For $\mathfrak{sl}_2 \otimes A$ and $\mathfrak{osp}(1|2) \otimes A$, $W(\Lambda, m)^{-2k} = \text{span}\{F_{j_1} \cdots F_{j_k} \cdot w_0 \,|\, j_n \in \mathbb{Z}\}$.

For $\mathfrak{osp}(1|2) \otimes A$, $W(\Lambda, m)^{-(2k+1)} = \text{span}\{Y_i F_{j_1} \cdots F_{j_k} \cdot w_0 \,|\, j_n, i \in \mathbb{Z}\}$.

For $\mathfrak{sl}(1|1) \otimes A$, $W(\Lambda, m)^{-k} = \text{span}\{Y_{j_1} \cdots Y_{j_k} \cdot w_0 \,|\, j_n \in \mathbb{Z}\}$.

Chapter 1

Introduction

Sopus Lie used groups of symmetries as a way to study geometry. Lie groups were introduced as the groups of transformations which preserved certain geometric properties of an object. For every Lie group, there is a corresponding Lie algebra. The classification of finite-dimensional simple, complex Lie algebras was initiated by Killing and completed by Cartan in his 1894 dissertation. It states that a finite-dimensional simple complex Lie algebra belongs to one of four infinite families: $A(n)$, $B(n)$, $C(n)$, or $D(n)$, or is one of the five exceptional Lie algebras: F_4, G_2, E_6, E_7, or E_8.

In the early 20th century, Cartan classified simple infinite-dimensional Lie algebras of vector fields over a finite-dimensional space. After this, the study of infinite-dimensional Lie algebras was not very active until the late 1960s. A general theory of infinite-dimensional Lie algebras is not known at this time. However, four different classes have been studied intensely, one of which is the class of Kac-Moody Lie algebras. In the late 1960s, Kac and Moody introduced these independently. They removed the condition on the Cartan matrix which requires the Weyl group to be finitely generated. This resulted in the generalized Cartan matrix [Kac90] and led to a class

of infinite-dimensional Lie algebras called the Kac-Moody algebras.

An important subclass of Kac-Moody algebras is the affine Lie algebras, which correspond to a specific type of generalized Cartan matrix. They can also be constructed as central extensions of loop Lie algebras. Affine Lie algebras are well understood, as many concepts from the theory of finite-dimensional Lie algebras, such as the Cartan subalgebra, root systems, and the Weyl group, carry over to affine Lie algebras. Interestingly, affine Lie algebras appeared in physics literature on conformal field theory, statistical physics, and string theory [Car05].

In the 20th century, quantum mechanics was discovered as a deformation of classical mechanics, and the symmetries of quantum mechanical systems were a natural subject to study. Quantum groups emerged in the 1980s as a tool to study these symmetries. The naive way to relate quantum groups to Lie algebras would be to try and deform semisimple Lie algebras. However, semisimple Lie algebras are rigid objects, so one has to look at a similar but larger structure, namely Hopf algebras. Given a Lie algebra $\mathfrak{g}$, one can construct the quantized enveloping algebra $U_q(\mathfrak{g})$ such that classical enveloping algebra $U(\mathfrak{g})$ is recovered when $q \to 1$. The quantized enveloping algebra of $\mathfrak{sl}_2$ first appeared in the work of Kulish and Reshetikhin in 1981. A few years later, Drinfeld [Dri86] and Jimbo [Jim85] created an object called a quantum group as a deformation of a Kac–Moody algebra, in the category of Hopf algebras.

Another generalization of Lie algebras is Lie superalgebras. As Lie algebras were initially motivated by symmetries, Lie superalgebras were motivated by the study of supersymmetry in physics, which is the symmetry between particles called bosons and fermions. The prefix "super" comes from the physical motivation; however, mathematically, Lie superalgebras are $\mathbb{Z}_2$-graded Lie algebras. In 1977, Kac completely

classified the finite-dimensional Lie superalgebras in [Kac77]. He sorted them into two categories, classical and Cartan. The classical ones were further sorted into basic and strange types. Remarkably, the classification of the basic classical Lie superalgebras was very close to the Cartan-Killing classification of finite-dimensional simple Lie algebras. In particular, basic classical Lie superalgebras make up four infinite families and three exceptional Lie superalgebras. For the complete classification, refer to Theorem 2.1.19.

The representation theory of simple finite-dimensional Lie algebras is well understood. In 1914, Cartan classified the finite-dimensional irreducible representations of simple complex Lie algebras and proved that all such representations must have a highest weight and a highest weight vector. For Kac-Moody algebras, the notion of integrable modules was introduced. Integrable modules are not necessarily finite-dimensional, but they are subject to certain finiteness conditions. In the 1970s, Bernstein, Gelfand, and Gelfand introduced the category $\mathcal{O}$, which is a category of representations of semisimple Lie algebras. These are integrable modules with the additional condition of being finitely generated [Hum08]. In the paper [Cha86], Chari introduced the category $\tilde{\mathcal{O}}$ of representations of affine Lie algebras. Chari found conditions on irreducible integrable modules of affine Lie algebras, which placed them in the category $\mathcal{O}$ or $\tilde{\mathcal{O}}$. In the same year in [CP86], Chari and Pressley classified the finite-dimensional irreducible representations of affine Lie algebras and showed that such representations must be highest weight representations.

Highest weight representations appear in the representation theory of quantum groups as well. For a Kac-Moody algebra $\mathfrak{g}$, Lusztig proved in [Lus88] that certain irreducible highest weight modules can be deformed to a module over $U_q(\mathfrak{g})$.

Chari and Pressley went on to study the class of all finite-dimensional representations of complex affine Lie algebras in [CP01]. A particular class of these representations is the evaluation module over the loop Lie algebra. They showed that any simple, finite-dimensional loop Lie algebra representation is isomorphic to such a tensor of evaluation modules. Chari and Pressley defined the Weyl module for affine Lie algebras as a larger class of integrable modules, proving that they are universal highest weight finite-dimensional modules.

Kac studied a certain class of finite-dimensional representations of basic Lie superalgebras, which he called typical. Typical representations are defined such that they split in any finite-dimensional representation [Kac06]. Kac explicitly constructed typical representations for all finite-dimensional basic classical Lie superalgebras. He constructed a highest weight module over a basic classical Lie superalgebra, and found necessary and sufficient conditions for a typical representation to be highest weight. Penkov and Serganova constructed a general theory of finite-dimensional representations of any finite-dimensional Lie superalgebra. They proved that any two Borel subalgebras of a Lie superalgebra, which contain the same Cartan subalgebra, can be obtained from each other by a chain of reflections. They defined odd reflections, and proved that an odd or real reflection must be associated to either (i) an isotropic odd root, (ii) a non-isotropic odd root, or (iii) an even root [PS94]. Moreover, they stated that the Chevalley generators corresponding to an isotropic odd root, a non-isotropic odd root, or an even root generate a rank 1 subalgebra isomorphic to $\mathfrak{sl}(1|1)$, $\mathfrak{osp}(1|2)$, or $\mathfrak{sl}_2$, respectively.

In [CLS19], Calixto, Lemay, and Savage replaced the affine Lie algebra in [CP01] by a Lie superalgebra of the form $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a basic classical Lie superalgebra,

or $\mathfrak{g} = \mathfrak{sl}(n,n)$ where $n \geq 2$, and A is a finitely generated unital associative $\mathbb{C}$-algebra. They studied Weyl modules associated with these algebras and proved some fundamental properties of these modules, such that they are universal and finite-dimensional. Weyl modules of Lie superalgebras were also studied in [Nao12], [TG15], [BCM22], [BCM19].

This thesis aims to blend the approaches of Weyl modules and reflections. As a first step, we study the local Weyl modules over the current algebras associated to the rank 1 Lie superalgebras: $\mathfrak{sl}_2$, $\mathfrak{sl}(1\,|\,1)$, and $\mathfrak{osp}(1\,|\,2)$. As a special case of the algebras studied in [CLS19], we restrict the algebra A to be the ring of polynomials or Laurent polynomials over $\mathbb{C}$ in one variable. Restricting to this special case allows one to obtain more precise results about the structure of these modules, such as the character, irreducibility criteria, and the filtration. Note that in this thesis we are always working with local Weyl modules.

In [CLS19], Proposition 4.5 states that each weight space of a local Weyl module is finite dimensional, and Theorem 4.12 states that the local Weyl module is finite dimensional. Our results state the precise dimension of each local Weyl module. This thesis also contains results about the form of the highest weight, irreducibility criteria which depends on the highest weight, and the structure of the submodules of $W(\Lambda, m)$, which did not appear in [CLS19].

In the case of $\mathfrak{sl}_2$, results about the local Weyl module of $\mathfrak{sl}_2 \otimes A$ are mostly known. The approach taken for in this thesis, using properites of matrices of highest weights is novel. Moreover, the technical details of these results do not appear in the literature.

For a Lie superalgebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a Weyl module $W(\Lambda, m)$ is defined as a quotient of the Verma module $M(\Lambda)$ by an ideal depending on $m \in \mathbb{Z}_{>0}$.

The highest weight $\Lambda \in (\mathfrak{h} \otimes A)^*$ induces a sequence $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$ where $c_n = \Lambda(z_n)$ and a matrix $[c_{i+j}]_{i,j\in\mathbb{Z}}$ where the entries come from $\mathbf{c}$. When $A = \mathbb{C}[t]$, we just set $c_n = 0$ for all $n \in \mathbb{Z}_{<0}$. The following two results capture this information when $\mathfrak{g} = \mathfrak{sl}(1\,|\,1)$.

Theorem 1.0.1. *Let $m \in \mathbb{Z}_{>0}$ and let $W(\Lambda, m)$ be as in Definition 5.1.1. Then*

1. *$W(\Lambda, m) \neq 0$ and only if $c_{k+m} = \alpha_1 c_{k+m-1} + \alpha_2 c_{k+m-2} + \cdots + \alpha_m c_k$ for all $k \in \mathbb{Z}_{>0}$ and for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$.*

2. *If $W(\Lambda, m) \neq 0$ then $\dim W(\Lambda, m)^{-k} = \binom{m}{k}$ for $0 \leq k \leq m$.*

3. *If $W'(\Lambda, m)$ is a $\mathfrak{g} \otimes A$-module generated by a nonzero vector w which satisfies the relations (5.2), then there exists a $\mathfrak{g} \otimes A$-submodule $V \subseteq W(\Lambda, m)$ such that $W'(\Lambda, m) \cong W(\Lambda, m)/V$.*

The structure of the weights allows us to construct a basis in such a way that makes computing the character simple. Moreover, the structure of the weights plays a key role in determining the irreducibility of $W(\Lambda, m)$.

Proposition 1.0.2. *Suppose that the sequence $\mathbf{c}$ associated to the module $W(\Lambda, m)$ satisfies an order $k \leq m$ linear recurrence relation with constant coefficients, and does not satisfy any linear recurrence relations of lesser degree. Then $W(\Lambda, m)$ is simple if and only if $k = m$. If $k < m$, then $W(\Lambda, m)$ has the following filtration:*

$$0 = W_{m-k} \subset W_{m-k-1} \subset \cdots \subset W_1 \subset W_0 = W(\Lambda, m)$$

where $W_i/W_{i+1} \cong \binom{m-k}{i} W(\Lambda, k)[-i]$.

For a semisimple Lie algebra $\mathfrak{g}$, the category $\mathcal{O}$ of $U(\mathfrak{g})$-modules, of Bernstein, Gelfand and Gelfand [BGG76], is defined such that $V \in \mathcal{O}$ if and only (i) $V = \bigoplus_{\lambda\in\mathfrak{h}} V_\lambda$,

(ii) $\mathfrak{n}^+$ acts locally nilpotently on V, and (iii) V is finitely generated over $U(\mathfrak{g})$. Now consider the affine algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$. In [Cha86], Chari introduced analogue of $\mathcal{O}$ called the category $\tilde{\mathcal{O}}$ of $\tilde{\mathfrak{g}}$-modules. The category $\tilde{\mathcal{O}}$ is defined as $V \in \tilde{\mathcal{O}}$ if and only (i) $cV = 0$, (ii) $V = \bigoplus_{\lambda \in \mathfrak{h}} V_\lambda$, and (iii) the support of V is contained in a finite union of cones.

When $\mathfrak{g} = \mathfrak{sl}_2$, one may construct an irreducible $\tilde{\mathfrak{g}}$-module $N(\varphi)$ for every exponential-polynomial function $\varphi : \mathbb{Z} \to \mathbb{C}$. In [Wil08], Wilson obtained a character formula for the modules $N(\varphi)$. Using this formula, along with results of Chari [Cha86], and Billig and Zhao [BZ04], Wilson derived a formula for the character of the irreducible modules in the category $\tilde{\mathcal{O}}$ with finite-dimensional weight spaces.

We do something similar when $\mathfrak{g} = \mathfrak{sl}(1 \,|\, 1)$. Consider the infinite dimensional weight $\mathfrak{sl}(1 \,|\, 1) \otimes \mathbb{C}[t, t^{-1}]$-modules with finite dimensional weight spaces. These are modules $V(\mathbf{a})$ which are induced from an $\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$-module generated by an infinite string of linearly independent highest weight vectors $\{w_n \,|\, n \in \mathbb{Z}\}$ where $h \otimes t^i \cdot w_n = a_i w_{n+i}$ and a_i comes from the sequence of complex numbers $\mathbf{a} = (a_i)_{i \in \mathbb{Z}}$. We prove some facts about the sequence $\mathbf{a}$.

Proposition 1.0.3. *Let $V(\mathbf{a})$ be as in (5.17) and let $A = [a_{i+j}]_{i,j \in \mathbb{Z}}$. The following are equivalent:*

1. *The set $\{k \in \mathbb{Z} \mid V(\mathbf{a})^{(k,l)} \neq 0\}$ is finite.*

2. *$V(\mathbf{a})$ has finite dimensional graded subcomponents.*

3. *The rank of A is finite.*

4. *The sequence $\mathbf{a} = \{a_n\}$ is exponential-polynomial.*

We find that every such module can be decomposed as a direct sum of irreducible modules, illustrated in the following proposition.

Proposition 1.0.4. *If $\{a_n\}_{n\in\mathbb{Z}}$ is exponential-polynomial of degree r, then*

1. $$L(\mathbf{a}) = \bigoplus_{j=0}^{r-1} L(\mathbf{a})\langle j\rangle,$$

   where $L := L(\mathbf{a})\langle 0\rangle$ is an irreducible $\overline{\mathfrak{h}}$-module as in Lemma 5.2.1 and $L(\mathbf{a})\langle j\rangle \cong L$ for $j = 0, \ldots, r-1$.

2. $$V(\mathbf{a}) = \bigoplus_{j=0}^{r-1} V(\mathbf{a})\langle j\rangle,$$

   where $V(\mathbf{a})\langle 0\rangle = V(L)$.

Furthermore, we find an explicit basis and compute the character formula for these modules.

Proposition 1.0.5. *Let $\mathbf{a} = \{a_n\}_{n\in\mathbb{Z}}$ be exponential-polynomial of degree r and let $A = [a_{i+j}]_{i,j\in\mathbb{Z}}$. Suppose* $\operatorname{rank} A = N$. *Then*

1. *$V(\mathbf{a})^{(-k,l)}$ has basis $\{Y_{i_1}\cdots Y_{i_k}\cdot w_{l-(i_1+\cdots+i_k)} \,|\, 0 \le i_1 < \cdots < i_k \le N-1\}$, so* $\dim V(\mathbf{a})^{(-k,l)} = \binom{N}{k}$, *and*

2. *$V(\mathbf{a})\langle j\rangle^{(-k,l)}$ has basis $\{Y_{i_1}\cdots Y_{i_k}\cdot w_{l-(i_1+\cdots+i_k)} \,|\, 0 \le i_1 < \cdots < i_k \le N-1$ and $l-(i_1+\cdots+i_k) = j \mod r\}$.*

For the remainder of the thesis, let $A = \mathbb{C}[t]$ or $\mathbb{C}[t, t^{-1}]$. The thesis begins by recalling the structure theory of finite-dimensional Lie superalgebras. Then we study

the finite-dimensional modules of the current and loop algebras $\mathfrak{sl}_2 \otimes A$ in Chapter 4. One such module is defined in [CP01] and [CLS19]. We use this definition to prove existence and finite-dimensionality. In Chapters 5 and 6, we then move on to study finite-dimensional modules of $\mathfrak{sl}(1|1) \otimes A$ and $\mathfrak{osp}(1|2) \otimes A$ in an analogous way. In Chapter 5, we study a class of infinite-dimensional $\mathfrak{sl}(1\,1) \otimes A$-modules mentioned above. We find a direct sum decomposition, compute the dimension of each weight space, and write down an explicit basis for these modules. Chapter 7 concludes the thesis with some remarks on future directions.

Chapter 2

Preliminaries

First we introduce some definitions and preliminary concepts. We review highest weight theory of Lie algebras and Lie superalgebras, as well as the classification of finite-dimensional classical Lie superalgebras. We compare and contrast reflections of Lie algebras versus Lie superalgebras. At the end of the chapter, we define current and loop algebras, and exponential-polynomial functions.

2.1 Preliminaries and Definitions

Throughout this document we will work over the complex numbers $\mathbb{C}$. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ denote the ring of two elements.

2.1.1 Structure Theory

Definition 2.1.1. A vector superspace V is a $\mathbb{Z}_2$-graded vector space: $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The dimension of a vector superspace V is the tuple $\dim V = (\dim V_{\bar{0}} \mid \dim V_{\bar{1}})$. We call elements of $V_{\bar{0}}$ even and elements of $V_{\bar{1}}$ odd. A homogeneous element $v \in V_i$ has

parity $|v| = i$, where $i \in \{\bar{0}, \bar{1}\}$ [Mus12]. The superdimension of V is defined to be $\mathrm{sdim} V := \dim V_{\bar{0}} - \dim V_{\bar{1}}$. We use the notation $\mathbb{C}^{m|n} := \mathbb{C}^m \oplus \mathbb{C}^n$, where $\mathbb{C}^m$ is the even subspace and $\mathbb{C}^n$ is the odd subspace.

Definition 2.1.2. A subspace of a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace $W = W_{\bar{0}} \oplus W_{\bar{1}} \subseteq V$ with a compatible $\mathbb{Z}_2$ grading. That is, $W_i \subseteq V_i$ for $i \in \mathbb{Z}_2$ [CW12].

Definition 2.1.3. A superalgebra $\mathcal{A}$ (also called a $\mathbb{Z}_2$-graded algebra) is a vector superspace $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ together with a multiplication satisfying $\mathcal{A}_\alpha \mathcal{A}_\beta \subseteq \mathcal{A}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$.

Definition 2.1.4. A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ together with a $\mathbb{Z}_2$-graded bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ satisfying the following axioms: for homogeneous elements $a, b, c \in \mathfrak{g}$,

1. $[a, b] = -(-1)^{|a|\cdot|b|}[b, a]$ (Skew-supersymmetry);
2. $(-1)^{|a|\cdot|c|}[a, [b, c]] + (-1)^{|b|\cdot|a|}[b, [c, a]] + (-1)^{|c|\cdot|b|}[c, [a, b]] = 0$ (Super Jacobi identity) [CW12].

Definition 2.1.5. Let $\mathfrak{g}$ be a Lie superalgebra.

1. Let V, W be subspaces of $\mathfrak{g}$. Then $[V, W]$ is the subspace $[V, W] := \mathrm{span}\{[v, w] \,|\, v \in V, w \in W\} \subseteq \mathfrak{g}$.
2. The Lie superalgebra $\mathfrak{g}$ is abelian if and only if $[\mathfrak{g}, \mathfrak{g}] = 0$.

Remark 2.1.6. For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, its even part $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and its odd part $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module. Indeed, if $a, b, c \in \mathfrak{g}_{\bar{0}}$ then the usual minus signs appear in Definition 2.1.4, proving that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra. If $a, b \in \mathfrak{g}_{\bar{0}}$ and $c \in \mathfrak{g}_{\bar{1}}$, still

no signs appear in the graded Jacobi identity in Definition 2.1.4 which is equivalent to $\mathfrak{g}_{\bar{1}}$ being a $\mathfrak{g}_{\bar{0}}$-module under the adjoint action.

In particular, if $\mathfrak{g}_{\bar{1}} = 0$, then $\mathfrak{g}$ is a Lie algebra. Also note that if $\mathfrak{g}_{\bar{0}} = 0$, then for all $x, y \in \mathfrak{g} = \mathfrak{g}_{\bar{1}}$, $[x, y] \in \mathfrak{g}_{\bar{0}} = 0$. So $[\mathfrak{g}, \mathfrak{g}] = 0$. i.e., $\mathfrak{g}$ is abelian.

Example 2.1.7. This example is taken from page 3 of [CW12]. Let $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ be an associative superalgebra. If $\mathcal{A}$ is equipped with the supercommutator

$$[a, b] := ab - (-1)^{|a||b|}ba$$

for homogeneous $a, b \in \mathcal{A}$, then $\mathcal{A}$ becomes a Lie superalgebra.

Example 2.1.8. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. We can choose an ordered basis for $V_{\bar{0}}$ and $V_{\bar{1}}$ such that when combined, we get a homogeneous ordered basis for V. We will parametrize this basis by the set $I(m|n) = \{\bar{1}, \ldots, \bar{m}; 1, \ldots, n\}$ with total order $\bar{1} < \cdots < \bar{m} < 1 < \cdots < n$. Then $\mathrm{End}(V) = \mathrm{End}(\mathbb{C}^{m|n})$ is endowed with an induced $\mathbb{Z}_2$-grading $\mathrm{End}(V) = \mathrm{End}_{\bar{0}}(V) \oplus \mathrm{End}_{\bar{1}}(V)$, where

$$\mathrm{End}_{\alpha}(V) := \{a \in \mathrm{End}(V) \,|\, a(V_{\beta}) \subseteq V_{\alpha+\beta}\},$$

and so $\mathrm{End}(V)$ becomes an associative superalgebra [Kac77]. $\mathrm{End}(V)$ forms a Lie superalgebra when equipped with the supercommutator. This Lie superalgebra is called the general linear Lie superalgebra and is denoted by $\mathfrak{gl}(V)$ [CW12]. When $V = \mathbb{C}^{m|n}$, we write $\mathfrak{gl}(m|n) := \mathfrak{gl}(\mathbb{C}^{m|n})$.

Definition 2.1.9. Let $\mathfrak{g}$ be a Lie superalgebra. Then a $\mathfrak{g}$-module is a vector space V together with a bilinear map

$$\mathfrak{g} \times V \to V$$

$$(x, v) \mapsto xv$$

such that $[x, y]v = x(yv) - (-1)^{|x||y|}y(xv)$ for all homogeneous $x, y \in \mathfrak{g}\,,\ v \in V$ [Mus12].

Definition 2.1.10. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie superalgebras. A homomorphism of Lie superalgebras is a linear map $f : \mathfrak{g} \to \mathfrak{g}'$ which preserves the $\mathbb{Z}_2$-grading, such that

$$f([a, b]) = [f(a), f(b)] \quad \forall\, a, b \in \mathfrak{g}.$$

Definition 2.1.11. The adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is defined by

$$\mathrm{ad}(a)(b) := [a, b] \quad \forall\, a, b \in \mathfrak{g}.$$

The map ad is a homomorphism of Lie superalgebras. This homomorphism is an action of $\mathfrak{g}$ on itself, called the adjoint action.

Definition 2.1.12. A Lie superalgebra is called simple if it contains no nontrivial ideals [Kac77].

2.1.2 The classical Lie superalgebras

In [Kac77], Kac classified the simple Lie superalgebras. This classification is divided into two groups: classical Lie superalgebras, and nonclassical Lie superalgebras. The classical subalgebras are those whose even subalgebra has a completely reducible

adjoint representation. The classical Lie superalgebras are further divided into two parts: basic and strange. The nonclassical Lie superalgebras are often called simple Lie superalgebras of Cartan type [Kac06]. In this thesis, we only focus on the classical Lie superalgebras.

Definition 2.1.13. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called classical if $\mathfrak{g}$ is simple and if $\mathfrak{g}_{\bar{1}}$ is completely reducible as a $\mathfrak{g}_{\bar{0}}$-module [Sch06]. Note that a simple Lie superalgebra $\mathfrak{g}$ is classical if and only if $\mathfrak{g}_{\bar{0}}$ is reductive [Mus12].

Definition 2.1.14. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace and let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. A bilinear form

$$B(\cdot, \cdot) : V \to V$$

is called even if $B(V_{\bar{i}}, V_{\bar{j}}) = 0$ when $\bar{i} + \bar{j} = \bar{1}$, and odd if $B(V_{\bar{i}}, V_{\bar{j}}) = 0$ when $\bar{i} + \bar{j} = \bar{0}$. The form B is called $\mathfrak{g}$-invariant if $B([a, b], c) = B(a, [b, c])$ for all $a, b, c \in \mathfrak{g}$.

Definition 2.1.15. An even bilinear form is called supersymmetric if $B|_{V_{\bar{0}} \times V_{\bar{0}}}$ is symmetric and $B|_{V_{\bar{1}} \times V_{\bar{1}}}$ is skew-symmetric; it is called skew-supersymmetric if $B|_{V_{\bar{0}} \times V_{\bar{0}}}$ is skew-symmetric and $B|_{V_{\bar{1}} \times V_{\bar{1}}}$ is symmetric.

Definition 2.1.16. The Killing form on a Lie superalgebra $\mathfrak{g}$ is defined as

$$K(x, y) = \mathrm{str}(\mathrm{ad}x\, \mathrm{ad}y)\,,$$

where str is the supertrace, defined in Definition 2.1.21

Recall that the Killing form on a simple Lie algebra is necessarily nondegenerate, (see Theorem 5.1 in [Hum12].) This is not the case for simple Lie superalgebras. In

fact, classical Lie superalgebras can be sorted into two classes based on the Killing form: (i) the Killing form is nondegenerate, or (ii) the Killing form is identically zero [Rit05]. This leads us to the following two definitions.

Definition 2.1.17. A finite dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called basic classical if:

1. $\mathfrak{g}$ is a simple Lie superalgebra;
2. $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra;
3. There exists a nondegenerate even supersymmetric invariant bilinear form on $\mathfrak{g}$ [CWN17].

In other words, a basic classical Lie superalgebra is a classical Lie superalgebra with the additional requirement of the existence of a nondegenerate even supersymmetric invariant bilinear form.

Definition 2.1.18. The classical Lie superalgebras which are not basic are called strange [FSS96].

Recall that a Lie algebra $\mathfrak{g}$ is automatically a Lie superalgebra by setting $\mathfrak{g}_{\bar{0}} = \mathfrak{g}$ and $\mathfrak{g}_{\bar{1}} = 0$. In this case, $\mathfrak{g}$ is isomorphic to one of the simple Lie algebras listed in the classification of simple complex Lie algebras, found in Theorem 9.26 of [FH13]. Theorem 2.1.19, due to Kac and found on page 44 of [Kac77], summarizes the classification of the finite-dimensional classical Lie superalgebras that are not Lie algebras. The basic classical Lie superalgebras can be grouped as follows: four infinite families $A(m,n), B(m,n), C(n)$, and $D(m,n)$, which are similar to the Cartan series A_n, B_n, C_n, and D_n from the classification of simple Lie algebras; the exceptional Lie

superalgebras, which are made up of the 40-dimensional $F(4)$, the 31-dimensional $G(3)$, and a family of 17-dimensional exceptional Lie superalgebras $D(2,1;a)$. And the strange classical are grouped in the following two infinite families: $\mathfrak{p}(n)$ and $\mathfrak{q}(n)$.

Theorem 2.1.19. *(Kac) Let $\mathfrak{g}$ be a finite-dimensional classical Lie superalgebra such that $\mathfrak{g}$ is not a Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the following Lie superalgebras.*

A Lie superalgebra belonging to one of the infinite families:

1. *A(m,n): $\mathfrak{sl}(m+1|n+1)$ with $m > n \geq 1$;*
2. *A(m,m): $\mathfrak{sl}(m+1,m+1)/(I_{2m+2})$ with $m \geq 1$;*
3. *B(m,n): $\mathfrak{osp}(2m+1|2n)$ with $m \geq 0, n > 0$;*
4. *C(n): $\mathfrak{osp}(2|2n-2)$ with $n \geq 2$;*
5. *D(m,n): $\mathfrak{osp}(2m|2n)$ with $m \geq 2, n \geq 1$;*

The exceptional Lie superalgebras:

6. *G(3), a simple algebra of dimension 31;*
7. *F(4), a simple algebra of dimension 40;*
8. *$D(2,1;\alpha)$ where $\alpha \neq 0, -1$.*

The "strange" Lie superalgebras:

9. *p(n), $n \geq 2$;*
10. *q(n), $n \geq 2$;*

A Lie superalgebra of type 1. - 8. is basic classical, meaning that it admits a non-degenerate even supersymmetric bilinear form, while an algebra of type 9. or 10. does not. Notice that there is a distinction between the families $A(m,n)$ where $m \neq n$ and $A(m,m)$. The Lie superalgebra $\mathfrak{sl}(m+1,m+1)$ contains the one-dimensional ideal consisting of the matrices λI_{2m+2}, hence it is not simple. If we quotient by this ideal, we get the simple Lie superalgebra $\mathfrak{sl}(m+1,m+1)/(I_{2m+2})$ which belongs to the family $A(m,m)$, for $m > 0$ [Kac06]. $D(2,1;\alpha)$ is a one-parameter family of l7-dimensional Lie superalgebras.

We briefly describe some Lie superalgebras listed in Theorem 2.1.19.

The general linear Lie superalgebras

The Lie superalgebra $\mathfrak{gl}(m|n) = \mathfrak{gl}(\mathbb{C}^m \oplus \mathbb{C}^n)$ can be realized as

$$\mathfrak{gl}(m|n) := \left\{ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \;\middle|\; \begin{array}{ll} A \in M_{m\times m}(\mathbb{C}), & B \in M_{m\times n}(\mathbb{C}), \\ C \in M_{n\times m}(\mathbb{C}), & D \in M_{n\times n}(\mathbb{C}) \end{array} \right\}.$$

The even part of $\mathfrak{gl}(m|n)$ is made up of all matrices with $B = C = 0$, while the odd part is made up of all matrices with $A = D = 0$. The superbracket of $\mathfrak{gl}(m|n)$ is determined by $[a,b] := ab - (-1)^{|a||b|}ba$, where $a, b \in \mathfrak{gl}(m|n)$ and ab denotes usual matrix multiplication.

Example 2.1.20. Let $\mathfrak{g} = \mathfrak{gl}(1|1)$. We may choose a basis of $\mathfrak{g}$ to be $\{x, y, h', h''\}$ where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, h'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\mathfrak{g}_{\bar{0}} = \mathrm{span}_{\mathbb{C}}\{h', h''\}$ and $\mathfrak{g}_{\bar{1}} = \mathrm{span}_{\mathbb{C}}\{x, y\}$.

The special linear Lie superalgebras

Definition 2.1.21. Let $\mathfrak{g} \in \mathfrak{gl}(m|n)$ be as above. The supertrace is defined as

$$\mathrm{str}(g) := \mathrm{tr}(A) - \mathrm{tr}(D),$$

where $\mathrm{tr}(X)$ is the usual trace of a square matrix X.

Consider the subspace

$$\mathfrak{sl}(m|n) := \{g \in \mathfrak{gl}(m|n) \,|\, \mathrm{str}(g) = 0\}.$$

This subspace is a subalgebra of $\mathfrak{gl}(m|n)$ called the special linear Lie superalgebra.

Example 2.1.22. Let $x, y, h', h'' \in \mathfrak{gl}(1|1)$ be as in Example 2.1.20. Then $z := h' + h''$ is in the centre of $\mathfrak{g}$, and $\mathrm{str}(z) = \mathrm{str}(x) - \mathrm{str}(y) = 0$. The special linear Lie algebra $\mathfrak{sl}(1|1)$ has a basis $\{x, y, z\}$.

When $m \neq n$ and $m + n \geq 2$, then $\mathfrak{sl}(m|n)$ is simple. Let $I_{m|n}$ denote the $(m + n) \times (m + n)$ identity matrix. When $m = n$, $\mathrm{str}(I_{m|m}) = 0$, so $\mathfrak{sl}(m|m)$ has a non-trivial center generated by $I_{m|m}$. When $m \geq 2$, $\mathbb{C}I_{m|m}$ is a non-trivial ideal, and $\mathfrak{sl}(m|m)/\mathbb{C}I_{m|m}$ is simple. The special Lie superalgebras belong to the following

families:

$$A(m\,|\,n) = \begin{cases} \mathfrak{sl}(m+1, n+1) & \text{if } m \neq n, m, n \geq 0, \\ \mathfrak{psl}(m+1, m+1) := \mathfrak{sl}(m+1, m+1)/\mathbb{C}I_{m+1|m+1} & \text{if } m = n > 0. \end{cases}$$

The orthosymplectic Lie superalgebras

Definition 2.1.23. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. Let B be a non-degenerate supersymmetric bilinear form on V. Note that since $B|_{V_{\bar{1}} \times V_{\bar{1}}}$ is skew-symmetric, then $\dim V_{\bar{1}} = 2n$ for some $n \in \mathbb{Z}_{\geq 0}$. Let $\dim V_{\bar{0}} = m$. The Lie superalgebra $\mathfrak{osp}(m|2n)$ is defined as $\mathfrak{osp}(m|2n) := \mathfrak{osp}(V)_{\bar{0}} \oplus \mathfrak{osp}(V)_{\bar{1}}$, where

$$\begin{aligned} \mathfrak{osp}(V)_{\bar{0}} &:= \{g \in \mathfrak{gl}(V)_{\bar{0}} \,|\, B(g(x), y) = B(x, g(y)) \,\forall\, x, y \in V\}, \\ \mathfrak{osp}(V)_{\bar{1}} &:= \{g \in \mathfrak{gl}(V)_{\bar{1}} \,|\, B(g(x), y) = -B(x, g(y)) \,\forall\, x, y \in V\}\ . \end{aligned}$$

and is called orthosymplectic.

The orthosymplectic Lie superalgebras are distinguished by three cases [Kac77]:

$$\begin{aligned} B(m, n) &= \mathfrak{osp}(2m+1|2n) \quad \text{for } m \geq 0, n \geq 1, \\ C(n) &= \mathfrak{osp}(2|2n-2) \quad \text{for } n \geq 2, \\ D(m, n) &= \mathfrak{osp}(2m|2n) \quad \text{for } m \geq 2, n \geq 1. \end{aligned}$$

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a basic complex Lie superalgebra. Recall that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra. The following definitions are compatible with this.

Definition 2.1.24. A Cartan subalgebra of $\mathfrak{g}$ is defined to be a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\alpha \in \mathfrak{h}^*$. Denote the root space of α as

$$\mathfrak{g}_\alpha := \{g \in g \,|\, [h, g] = \alpha(h)g \quad \forall h \in h\}.$$

Definition 2.1.25. The root system for $\mathfrak{g}$ is denoted by Δ and defined as

$$\Delta := \{\alpha \in \mathfrak{h}^* \,|\, \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\} \ .$$

Then $\Delta_{\bar{0}} := \{\alpha \in \mathfrak{h}^* \,|\, \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0, \alpha \neq 0\}$ denotes the set of even roots, while $\Delta_{\bar{1}} := \{\alpha \in \mathfrak{h}^* \,|\, \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0, \alpha \neq 0\}$ denotes the set of odd roots [CW12].

Theorem 2.1.26. *Let $\mathfrak{g}$ be a basic Lie superalgebra with a Cartan subalgebra $\mathfrak{h}$. Theorem 1.18 from [CW12] states that*

(i) $\dim \mathfrak{g}_\alpha = 1$ *for* $\alpha \in \Delta$.

(ii) $\Delta = -\Delta, \Delta_{\bar{0}} = -\Delta_{\bar{0}}$, *and* $\Delta_{\bar{1}} = -\Delta_{\bar{1}}$.

(iii) For $\alpha \in \Delta$, $k\alpha \in \Delta$ *for an integer* $k \neq \pm 1$ *if and only if* α *is an odd root such that* $(\alpha, \alpha) \neq 0$. *In this case,* $k = \pm 2$.

Let $E = \text{span}_{\mathbb{R}}\Delta$ be the real vector space spanned by the roots Δ. Since Δ is a finite set, we may choose a total order $\geq$ on the elements in Δ. Then we may extend this total order $\geq$ to E in a way that is compatible with the vector space structure of E. i.e., if $x, x', y, y' \in E$ are such that $x \geq y$ and $x' \geq y'$ and $c \in \mathbb{R}_{>0}$, then $x + x' \geq y + y'$, $-y \geq -x$, and $cx \geq cy$ [CW12].

Definition 2.1.27. By Theorem 2.1.26, if $\alpha \in \Delta$, then $-\alpha \in \Delta$. Since $0 \notin \Delta$, then $|\Delta| = 2n$ for some integer $n \geq 1$. Fix a total order $\geq$ on the elements of Δ such that the set $\{\alpha > 0\} \subset \Delta$ contains exactly n elements. The positive system Δ^+ is defined as $\Delta^+ := \{\alpha \in \Delta \,|\, \alpha > 0\}$ for the given total ordering on E. Elements of Δ^+ are referred to as positive roots, and $\Delta = \Delta^+ \cup (-\Delta^+)$.

Definition 2.1.28. Given a positive system Δ^+, the fundamental system $\Sigma \subset \Delta^+$ is defined as

$$\Sigma := \{\alpha \in \Delta^+ \,|\, \alpha \text{ cannot be expressed as a sum of two elements of } \Delta^+\}.$$

Elements of Σ are referred to as simple roots.

The system Σ forms a basis of the space $\mathfrak{h}^*$ and every root $\alpha \in \Delta^+$ can be expressed as $\alpha = \sum_i k_i \alpha_i$ where $\alpha_i \in \Sigma$ and all of the k_i are non-negative integers.

2.1.3 Invariant Bilinear Forms

In this section we will only consider the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$.

The supertrace gives rise to the non-degenerate supersymmetric invariant bilinear form

$$\begin{aligned} (\cdot,\cdot): \quad \mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n) &\to \mathbb{C} \\ (a,b) &\mapsto \operatorname{str}(ab)\ . \end{aligned}$$

Here ab denotes matrix multiplication. If we restrict this form to $\mathfrak{h} \subset \mathfrak{g}$, the Cartan subalgebra of diagonal matrices, we get a non-degenerate symmetric invariant bilinear

form on $\mathfrak{h}$:

$$(E_{ii}, E_{jj}) = \begin{cases} 1 & \text{if } \overline{1} \leq i = j \leq \overline{m} \\ -1 & \text{if } 1 \leq i = j \leq n \\ 0 & \text{if } i \neq j \end{cases} .$$

We denote the basis of $\mathfrak{h}^*$ dual to $\{E_{\bar{i},\bar{i}}, E_{jj}\}$ by $\{\delta_i, \epsilon_j\} := \{\epsilon_{\bar{i}}, \epsilon_j\}$. The form $(\cdot, \cdot)$ restricted to $\mathfrak{h}$ induces a non-degenerate bilinear form on $\mathfrak{h}^*$, which we also denote by $(\cdot, \cdot)$:

$$(\epsilon_i, \epsilon_j) = \begin{cases} 1 & \text{if } \overline{1} \leq i = j \leq \overline{m} \\ -1 & \text{if } 1 \leq i = j \leq n \\ 0 & \text{if } i \neq j \end{cases} .$$

2.1.4 Rank 1 Lie superalgebras

The rank 1 Lie algebra $\mathfrak{sl}_2$ plays an important role in the representation theory of Lie algebras. There are three rank 1 Lie superalgebras: $\mathfrak{sl}_2$, $\mathfrak{sl}(1|1)$, and $\mathfrak{osp}(1|2)$, which play a similar role in the super setting. Proposition 3 in [PS94] shows that the representation theory of a basic classical Lie superalgebra can be studied by using the rank 1 Lie subsuperalgebras.

Definition 2.1.29. A root $\alpha \in \Delta$ is called isotropic if $(\alpha, \alpha) = 0$ [CW12]. Note that the Killing form is positive definite on the Cartan subalgebra, so an isotropic root is necessarily odd. We denote the set of isotropic odd roots by

$$\overline{\Delta}_{\bar{1}} = \{\alpha \in \Delta_{\bar{1}} \,|\, (\alpha, \alpha) = 0\} = \{\alpha \in \Delta_{\bar{1}} \,|\, 2\alpha \notin \Delta\},$$

where the second equality follows from Theorem 2.1.26.

Definition 2.1.30. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\alpha \in \mathfrak{h}^*$. Then the coroot of α, denoted by $h_\alpha \in \mathfrak{h}$, is defined by

$$(h_\alpha, h) = \alpha(h) \text{ for all } h \in \mathfrak{h}.$$

We denote by e_α and f_α the root vectors of α and $-\alpha$, so that $[e_\alpha, f_\alpha] = h_\alpha$. The elements $e_\alpha, f_\alpha, h_\alpha$ are called Chevalley generators, and they generate a subalgebra isomorphic to one of $\mathfrak{sl}_2, \mathfrak{sl}(1|1)$, or $\mathfrak{osp}(1|2)$, depending on whether α is even, α is odd and isotropic, or α is odd and 2α is even, respectively.

Example 2.1.31. Let $\{H_{\overline{i}\overline{i}}, H_{jj} \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$ be a basis of $\mathfrak{h}$ and $\{\delta_{\overline{i}}, \epsilon_j \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$ the dual basis of $\mathfrak{h}^*$. Then the root system of $\mathfrak{gl}(m|n)$ is $\Delta = \Delta_{\bar{0}} \oplus \Delta_{\bar{1}}$, where

$$\Delta_{\bar{0}} = \{\pm(\delta_{\overline{i}} - \delta_{\overline{j}}), \pm(\epsilon_k - \epsilon_l) \,|\, 1 \leq i < j \leq m, 1 \leq k < l \leq n\},$$
$$\Delta_{\bar{1}} = \{\pm(\delta_{\overline{i}} - \epsilon_k) \,|\, 1 \leq i \leq m, 1 \leq k \leq n\}.$$

If we take the standard Borel subalgebra $\mathfrak{b}$ of all upper triangular matrices, this gives us a corresponding standard positive root system

$$\Delta^+ = \{\delta_{\overline{i}} - \delta_{\overline{j}}, \epsilon_k - \epsilon_l, \delta_{\overline{i}} - \epsilon_k \,|\, 1 \leq i < j \leq m, 1 \leq k < l \leq n\},$$

and standard fundamental system

$$\Sigma = \{\delta_{\overline{1}} - \delta_{\overline{2}}, \ldots, \delta_{\overline{m-1}} - \delta_{\overline{m}}, \delta_{\overline{m}} - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n\}.$$

Since $\delta_{\overline{m}} - \epsilon_1 \in \Delta_{\bar{1}}$ and $(\delta_{\overline{m}} - \epsilon_1, \delta_{\overline{m}} - \epsilon_1) = 0$,then $\delta_{\overline{m}} - \epsilon_1$ is an odd isotropic root.

Definition 2.1.32. The $\epsilon\delta$ sequence of an ordered fundamental system

$$\{\epsilon_{i_1} - \epsilon_{i_2}, \epsilon_{i_2} - \epsilon_{i_3}, \ldots, \epsilon_{i_{m+n-2}} - \epsilon_{i_{m+n-1}}, \epsilon_{i_{m+n-1}} - \epsilon_{i_{m+n}}\}$$

is obtained by taking the sequence $\epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_{m+n-1}}, \epsilon_{i_{m+n}}$, then identifying $\epsilon_{\bar{i}} = \delta_i$, and finally dropping the indices.

Hence the standard Borel subalgebra corresponds to the $\epsilon\delta$ sequence $\underbrace{\delta \cdots \delta}_{m}\underbrace{\epsilon \cdots \epsilon}_{n}$, while the opposite Borel subalgebra corresponds to the $\epsilon\delta$ sequence $\underbrace{\epsilon \cdots \epsilon}_{n}\underbrace{\delta \cdots \delta}_{m}$. This tells us that the opposite fundamental system is

$$\Sigma^{op} = \{\epsilon_n - \epsilon_{n-1}, \cdots, \epsilon_2 - \epsilon_1, \epsilon_1 - \delta_m, \delta_m - \delta_{m-1}, \ldots, \delta_2 - \delta_1\}.$$

Theorem 2.1.33. *(V. Kac, [Kac06]) A basic classical Lie superalgebra $\mathfrak{g}$ satisfies the following properties:*

1. $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha\right)$;
2. $\dim \mathfrak{g}_\alpha = 1$, *for* $\alpha \in \Delta$, *except for* $A(1,1)$;
3. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ *if and only if* $\alpha, \beta, \alpha + \beta \in \Delta$;
4. *If* $\alpha \in \Delta$ *(or* $\Delta_{\bar{0}}$, *or* $\Delta_{\bar{1}}$, *or* $\overline{\Delta}_{\bar{1}}$*), then so is* $-\alpha$;
5. $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *for* $\alpha \neq -\beta$, *the form* $(\cdot, \cdot)$ *determines a non-degenerate pairing of* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_\alpha$, *and the restriction of* $(\cdot, \cdot)$ *on* $\mathfrak{h}$ *is non-degenerate;*
6. $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)h_\alpha$;

7. *The bilinear form on* $\mathfrak{h}^*$ *defined by* $(\alpha, \beta) = (h_\alpha, h_\beta)$ *is a non-degenerate* W*-invariant form, where* W *is the Weyl group of* $\mathfrak{g}_{\bar{0}}$*;*

8. $k\alpha \in \Delta$ *for* $\alpha \neq 0, k \neq \pm 1$ *if and only if* $\alpha \in \Delta_{\bar{1}}$*, and* $(\alpha, \alpha) \neq 0$*; in this case* $k = \pm 2$*.*

The properties of Theorem 2.1.33, except for the ones dealing specifically with odd roots, are analogous to the properties of simple finite-dimensional Lie algebras.

2.1.5 The Universal Enveloping Lie Superalgebra

Definition 2.1.34. (The Universal Enveloping Lie Superalgebra) A pair $(U(\mathfrak{g}), \iota)$, where $U(\mathfrak{g})$ is an associative superalgebra and $\iota : \mathfrak{g} \mapsto U(\mathfrak{g})$ is a homomorphism of Lie superalgebras, is called the universal enveloping superalgebra of a Lie superalgebra $\mathfrak{g}$ if for any other such pair $(U'(\mathfrak{g}), \iota')$ there is a unique homomorphism $\theta : U(\mathfrak{g}) \mapsto U'(\mathfrak{g})$ for which $\iota' = \theta \circ \iota$ [Kac77].

For a Lie superalgebra, a universal enveloping algebra exists and is unique up to isomorphism [CW12]. As in the case of Lie algebras, there is a version of the Poincaré-Birkhoff-Witt (PBW) Theorem for Lie superalgebras.

Theorem 2.1.35. *(The Poincaré-Birkhoff-Witt (PBW) Theorem for Lie Superalgebras). Let* $\{x_1, x_2, \ldots, x_n\}$ *be a graded basis of* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$*. Then the set*

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \,|\, \alpha_j \in \mathbb{Z} \text{ if } x_i \in \mathfrak{g}_{\bar{0}}, \text{ and } \alpha_j \in \{0, 1\} \text{ if } x_i \in \mathfrak{g}_{\bar{1}}\}$$

is a basis of $U(\mathfrak{g})$*.*

Any representation of a Lie superalgebra $\mathfrak{g}$ can be thought of as a representation of $U(\mathfrak{g})$, and vice versa. Since $U(\mathfrak{g})$ is associative, while $\mathfrak{g}$ is not necessarily, then we use $U(\mathfrak{g})$ and the Poincaré-Birkhoff-Witt theorem as tools when studying representations of $\mathfrak{g}$. Throughout the thesis, we interchange $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules.

2.1.6 Highest Weight Theory and Simple Reflections

Unlike in the case of Lie algebras, bases of a root system Δ are not always W-conjugate (where W is the Weyl group) because of the existence of odd roots [CW12]. Hence, unlike in the case of semisimple Lie algebras, for a Lie superalgebra $\mathfrak{g}$ there is in general more than one conjugacy class of Borel subalgebras and more than one Dynkin diagram. However, in the case of classical simple Lie superalgebras or Lie superalgebras of Type A, there are only a finite number of conjugacy classes of Borel subalgebras [Mus12]. We will be working with these types of Lie superalgebras in this document.

Definition 2.1.36. Suppose that $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$. Then $\mathfrak{b}$ is a Borel subalgebra if $\mathfrak{b}$ is maximal with the following properties:

1. $\mathfrak{b}_{\bar{0}}$ is a Borel subalgebra of $\mathfrak{g}_{\bar{0}}$, and
2. $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra.

This definition extends the definition of a Borel subalgebra of a Lie algebra. The motivation for studying Borel subalgebras of Lie superalgebras comes from the representation theory of Lie algebras. In the representation theory of Lie algebras, a Borel subalgebra of a Lie algebra $\mathfrak{g}$ is a maximal solvable subalgebra $\mathfrak{b}$. When we fix a base of a root system, we may decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \ , \tag{2.1}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}^{\pm} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{\pm\alpha}$. Then the Borel subalgebra associated to this root system is $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ [Ser00]. This Borel subalgebra has an abelian quotient algebra $\mathfrak{b}/\mathfrak{n}^+ \cong \mathfrak{h}$. Hence any $\lambda \in \mathfrak{h}^*$ gives rise to a one dimensional $\mathfrak{b}$-module where $\mathfrak{h}$ acts by scalar multiplication by the weight λ and $\mathfrak{n}^+$ acts trivially. This $\mathfrak{b}$-module is usually denoted by $\mathbb{C}_\lambda$. In fact, $\mathbb{C}_\lambda$ is a $U(\mathfrak{b})$-module, so we may extend $\mathbb{C}_\lambda$ to be a $U(\mathfrak{g})$-module as follows.

Define $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. $M(\lambda)$ is a $U(\mathfrak{g})$-module under the natural left action of $U(\mathfrak{g})$. i.e., for $x \in \mathfrak{g}$,

$$x \cdot (1 \otimes v_\lambda) = x \otimes v_\lambda = \begin{cases} x \otimes v_\lambda & \text{if } x \in \mathfrak{n}^- \\ 1 \otimes \lambda(x) v_\lambda & \text{if } x \in \mathfrak{h} \\ 0 & \text{if } x \in \mathfrak{n}^+ \end{cases} .$$

By the PBW theorem, $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$. So as a $U(\mathfrak{n}^-)$-module, $M(\lambda) \cong U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$ [Hum12]. $M(\lambda)$ is called the Verma module.

Furthermore, $M(\lambda)$ has a unique maximal proper submodule $I(\lambda)$ and a unique irreducible quotient $V(\lambda) = M(\lambda)/I(\lambda)$. The Verma module $M(\lambda)$ is the universal highest weight module of weight λ. This is summarized in the following two theorems:

Theorem 2.1.37. *(A [Hum12]) Let V, W be cyclic modules each generated by a single maximal vector v^+ and w^+, respectively, of weight λ. If V and W are irreducible, then they are isomorphic.*

Theorem 2.1.38. *(B [Hum12]) Let $\lambda \in \mathfrak{h}^*$. Then there exists an irreducible cyclic module $V(\lambda)$ of weight λ, such that $V(\lambda) = U(\mathfrak{g}) \cdot v^+$ for a maximal weight vector v^+ of weight λ.*

Before we move onto the highest weight theory of Lie superalgebras, we will recall the role which simple reflections play in the representation theory of simple Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra and let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra with positive roots Δ^+ and corresponding base (simple roots) $\sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$. Then $\mathfrak{b} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha\right)$ and for each simple root α, the elements $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ generate a subalgebra isomorphic to $\mathfrak{sl}_2$. We call this subalgebra the $\mathfrak{sl}_2$-triple $\{e_\alpha, h_\alpha, f_\alpha\}$. Let V be an irreducible $\mathfrak{g}$-module. It is well known that $V = V(\lambda)$ for a unique $\lambda \in \mathcal{P} := \left\{ \lambda \in \mathfrak{h} \;\middle|\; 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\geq 0} \text{ for all } 1 \leq i \leq l \right\}$ if and only if V is finite-dimensional.

This is proven using the following scheme. First it is shown that if V is a finite-dimensional irreducible $\mathfrak{g}$-module, then V is a highest weight module of weight λ for some $\lambda \in \mathfrak{h}^*$. We also require the following definition.

Definition 2.1.39. For a $\mathfrak{g}$-module V, the support of V is $\mathrm{Supp}V := \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$.

1. Since V is finite-dimensional, there is a common eigenvector for $\mathfrak{h}$, so the submodule $V' = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ is non-zero. Since V is irreducible, then $V = V'$, hence V is a weight module.

2. Next, it is shown that V is a highest weight module. Since V is finite-dimensional, then $\mathrm{Supp}V$ must be a finite set and $\dim V_\mu$ must be finite for all $\mu \in \mathrm{Supp}V$. Finite support implies that there exists a weight $\lambda \in \mathrm{Supp}V$ such that $\lambda + \alpha \notin \mathrm{Supp}V$ for all $\alpha \in \Delta^+$. Let $w \in V_\lambda$. Since for all $\alpha \in \Delta$ and all $\mu \in \mathrm{Supp}V$, $e_\alpha V_\mu \subseteq V_{\mu+\alpha}$, then $e_\alpha \cdot w = 0$. And since $hV_\lambda \subseteq V_\lambda$ for all $h \in \mathfrak{h}$, then $h \cdot w = \lambda(h)w$ for all $h \in \mathfrak{h}$. Hence by the PBW theorem, $V'' := U(\mathfrak{g}) \cdot w = U(\mathfrak{n}^-) \cdot w \subseteq V$ is a submodule. Since V is irreducible, then $V'' = V$. So V is indeed a highest

weight module generated by w with relations $e_\alpha \cdot w = 0$ and $h \cdot w = \lambda(h)w$ for all $h \in \mathfrak{h}$.

Next we will show λ is dominant integral, i.e., $\lambda \in \mathcal{P}$.

1. Recall $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, the Verma module with highest weight λ , where $\mathbb{C}_\lambda$ is the $\mathfrak{b}$-module $\mathbb{C} \cdot w$. It is well known that any highest weight $\mathfrak{g}$-module is a quotient of $M(\lambda)$, and that $M(\lambda)$ has a unique maximal proper submodule $I(\lambda)$ and unique irreducible quotient $V(\lambda)$. So V is a quotient of $M(\lambda)$, and since V is irreducible, then $V = V(\lambda)$.

2. Finally, we show that λ is dominant integral. Indeed, let w be the highest weight of V and take any $\mathfrak{sl}_2$-triple $\{e_\alpha, h_\alpha, f_\alpha\}$ in $\mathfrak{g}$. Since V is finite-dimensional, then $U(\text{span}\{e_\alpha, h_\alpha, f_\alpha\}) \cdot w$ is also finite-dimensional. So by $\mathfrak{sl}_2$-theory, $m_\alpha \in \mathbb{Z}$ and $(f_\alpha)^{m_\alpha+1} \cdot w = 0$ for all $\alpha \in \Delta^+$. In particular, $m_{\alpha_i} \in \mathbb{Z}_{\geq 0}$ for all simple roots α_i.

So we have shown that if V is a finite-dimensional irreducible $\mathfrak{g}$-module, then $V \cong V(\lambda)$ where λ is dominant integral. Now, when is $V(\lambda)$ finite-dimensional? Using the symmetry of the Weyl group, we show that when λ is dominant integral, then $V(\lambda)$ is finite-dimensional. Suppose $V(\lambda)$ is a highest weight irreducible $\mathfrak{g}$-module with respect to the Borel subalgebra $\mathfrak{b}$ of highest weight λ generated by w, such that $m_i := 2\dfrac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq l$.

1. Fix $\alpha_i \in \Sigma$ and let $\{e_{\alpha_i}, h_{\alpha_i}, f_{\alpha_i}\}$ be the $\mathfrak{sl}_2$-triple associated to α_i. We can use the simple reflection $\sigma_i \in W$ to flip the base at root α_i. This gives us a new set of positive roots $(\Delta^+)' := \sigma_i(\Delta^+) = (\Delta^+ \setminus \{\alpha_i\}) \cup \{-\alpha_i\}$, a new base $\sigma_i(\Sigma) = (\Sigma \setminus \{\alpha_i\}) \cup \{-\alpha_i\}$, and hence a new Borel subalgebra $\mathfrak{b}' = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in (\Delta^+)'} \mathfrak{g}^\alpha\right)$. It also gives us a new $\mathfrak{sl}_2$- triple $\{e'_{\alpha_i} = e_{-\alpha_i}, h'_{\alpha_i} = h_{-\alpha_i}, f'_{\alpha_i} = f_{-\alpha_i}\}$. Take

$w' = f_{\alpha_i}^{m_i} \cdot w \in V^{\lambda - m_i\alpha_i}$. Then $e'_{\alpha_i} \cdot w' \in V^{\lambda - m_i\alpha_i - \alpha_i} = 0$. So w' is the highest weight vector of highest weight

$$\lambda' = \lambda - m_i\alpha_i = \lambda - 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i = \sigma_i(\lambda)$$

with respect to the Borel subalgebra $\sigma_i(\mathfrak{b})$. Furthermore, since the form $(\cdot, \cdot)$ is bilinear, then $2\frac{(\sigma_i(\lambda), \sigma_i(\alpha_j))}{(\sigma_i(\alpha_j), \sigma_i(\alpha_j))} = 2\frac{(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}_{\geq 0}$. So $V(\lambda)$ is a highest weight irreducible $\mathfrak{g}$-module with respect to the Borel subalgebra $\mathfrak{b}' = \sigma_i(\mathfrak{b})$ of highest weight $\lambda' = \sigma_i(\lambda)$ for each $1 \leq i \leq l$. Since the Weyl group is generated by the simple reflections $\{\sigma_i \,|\, 1 \leq i \leq l\}$, then in fact $V(\lambda)$ is a highest weight irreducible $\mathfrak{g}$-module with respect to the Borel subalgebra $\tilde{\mathfrak{b}} = \omega(\mathfrak{b})$ of highest weight $\tilde{\lambda} = \omega(\lambda)$ for each $\omega \in W$.

2. Let $\omega_o \in W$ be the longest element. That is, $\omega_0(\Delta^+) = \Delta^-$, and $\omega_0(\Sigma) = -\Sigma$, and $\omega_0(\mathcal{Q}) = -\mathcal{Q}$, etc., where $\mathcal{Q}^+ = \bigcup_\alpha \mathbb{Z}_{\geq 0}\alpha$. By the above, $V_{\mathfrak{b}}(\lambda) = V_{\mathfrak{b}^{op}}(\lambda^{op})$. Since $\mathrm{Supp}V_{\mathfrak{b}}(\lambda) \subset \lambda - \mathcal{Q}^+$ and $\mathrm{Supp}V_{\mathfrak{b}^{op}}(\lambda^{op}) \subset (\lambda^{op} - (\mathcal{Q}^+)^{op}) = (\lambda^{op} + \mathcal{Q}^+)^{op}$, then $\mathrm{Supp}V_{\mathfrak{b}}(\lambda) \subset (\lambda - \mathcal{Q}^+) \cap (\lambda^{op} + \mathcal{Q}^+)$, which is a finite set.

The key thing which allows us to go from $\mathfrak{b}$ to $\mathfrak{b}^{\mathrm{op}}$ is the fact that W acts transitively on the bases (sets of simple roots) of V. Hence, we can find a chain of adjacent Borel subalgebras and corresponding highest weights

$$\mathfrak{b} = \mathfrak{b}_1 \sim \mathfrak{b}_2 \sim \cdots \sim \mathfrak{b}_k = \mathfrak{b}^{op}$$

$$\lambda = \lambda_1 \sim \lambda_2 \sim \cdots \sim \lambda_k = \lambda^{op}.$$

In the case of Lie superalgebras this cannot be done by elements of the Weyl group

alone, so different methods must be used to go from a Borel subalgebra to its opposite. This is discussed in the following two sections.

2.1.7 Highest Weight Theory of Basic Lie Superalgebras

We can do something analogous with basic Lie superalgebras, but only using simple reflections will not be enough. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a basic Lie superalgebra, $\mathfrak{h}$ be the standard Cartan subalgebra, and Δ the standard root system. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ associated to Σ. That is $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+ \subset \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$. We will need to use the following lemma.

Lemma 2.1.40. *Let $\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$ be a finite-dimensional solvable Lie superalgebra such that $[\mathfrak{s}_{\bar{1}}, \mathfrak{s}_{\bar{1}}] \subseteq [\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{0}}]$. Then every finite-dimensional irreducible $\mathfrak{s}$-module is one-dimensional. A complete list of finite-dimensional irreducible $\mathfrak{s}$-modules is given by $\mathbb{C}_\lambda$, for $\lambda \in (\mathfrak{h}_{\bar{0}}/[\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{0}}])$ [CW12].*

Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$. As showed in Section 1.5.3 of [CW12], V contains a one-dimensional $\mathfrak{b}$-module of the form $\mathbb{C}_\lambda = \mathbb{C}v_\lambda$ for $\lambda \in \mathfrak{h}^*$ on which $\mathfrak{b}$ acts in an analogous way as in the case of Lie algebras. Specifically,

$$h \cdot v_\lambda = \lambda(h)v_\lambda \text{ and } x \cdot v_\lambda = 0, \ \forall h \in \mathfrak{h}, \ \forall x \in \mathfrak{n}^+.$$

By the PBW theorem, $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$. Since V is irreducible, then we must have that $V = U(\mathfrak{g})v_\lambda = U(\mathfrak{n}^-)v_\lambda$. Hence, using this and the PBW theorem, we see that that V admits a direct sum decomposition

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu. \tag{2.2}$$

Definition 2.1.41. 1. The weight λ is called the $\mathfrak{b}$-highest weight of V.

2. The space $\mathbb{C}_\lambda$ is called the $\mathfrak{b}$-highest weight space.

3. The vector v_λ is called the $\mathfrak{b}$-highest weight vector for V.

4. Modules satisfying (2.2) are called weight modules [Mus12].

When it is clear which Borel subalgebra we have chosen, the "$\mathfrak{b}$" will be dropped from these names.

Proposition 2.1.42. *Let $\mathfrak{g}$ be a basic Lie superalgebra with a Borel subalgebra $\mathfrak{b}$. Then every finite-dimensional irreducible $\mathfrak{g}$-module is a $\mathfrak{b}$-highest weight module [CW12].*

2.1.8 Odd and Real Reflections

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a basic Lie superalgebra with standard Cartan subalgebra $\mathfrak{h}$, corresponding Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, positive roots Δ^+, and corresponding base $\Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$. Let α be a simple root of $\mathfrak{g}$, and let $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$. The subalgebra generated by e_α and f_α is isomorphic to one of $\mathfrak{sl}_2, \mathfrak{sl}(1|1)$, or $\mathfrak{osp}(1|2)$. Can we use the same scheme as in Section 2.1.6 to classify irreducible finite-dimensional $\mathfrak{g}$-modules? In fact, we cannot always use this due to the existence of odd roots. Let V be an irreducible $\mathfrak{g}$-module. As in Section 2.1.6, Lemma 2.1.40 and 2.1.42 together imply that V is a highest weight module of highest weight λ for some $\lambda \in \mathfrak{h}^*$. Now suppose $V(\lambda)$ is a highest weight irreducible $\mathfrak{g}$-module with respect to the Borel subalgebra $\mathfrak{b}$ of highest weight λ generated by w, such that $m_i = 2\dfrac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq l$. Can we find a chain of adjacent Borel subalgebras to go from $\mathfrak{b}$ to $\mathfrak{b}^{op}$? We need some definitions first to be able to continue.

Definition 2.1.43. Let α be an even root. A real reflection r_α is a linear map on $\mathfrak{h}^*$ such that

$$r_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha \text{ for } \lambda \in \mathfrak{h}^*.$$

Definition 2.1.44. The Weyl group of $\mathfrak{g}$ is defined to be the Weyl group of $\mathfrak{g}_{\bar{0}}$.

Remark 2.1.45. The Weyl group W is generated by real reflections.

A real reflection is essentially a simple reflection in the case of Lie algebras. The real reflections r_α for $\alpha \in \Sigma$ generate the Weyl group W of $\mathfrak{g}_{\bar{0}}$.

1. If α is even, then e_α and f_α generate $\mathfrak{sl}_2$ and the real reflection r_α is equal to the simple reflection σ_α, so everything follows as in Section 2.1.6.

2. If α is odd and 2α is even, then e_α and f_α generate $\mathfrak{osp}(1|2)$. Since 2α is even, then we may use the real reflection $r_{2\alpha}$. Then $\Sigma_\alpha = r_{2\alpha}(\Sigma)$ and $\Delta^+_\alpha = r_{2\alpha}(\Delta^+) = (\Delta^+ \setminus \{\alpha, 2\alpha\}) \cup \{-\alpha, -2\alpha\}$ and everything else follows as in Section 2.1.6.

3. If α is odd and isotropic, then e_α and f_α generate $\mathfrak{sl}(1|1)$. In this case we cannot use real reflections since we do not have an even root. The solution is to use odd reflections. This was introduced by Penkov and Serganova [PS94], is described below, and is illustrated in Example 2.1.46.

   First send α to $-\alpha$ to get

$$\Delta^+_\alpha = (\Delta^+ \setminus \{\alpha\}) \cup \{-\alpha\}.$$

   To get the new fundamental system $r_\alpha(\Sigma)$, first we must take away α and include $-\alpha$ since by definition $\Sigma_\alpha \subset \Delta^+_\alpha$. This is not enough however, because this

does not give a new base, so more must be done. The new fundamental system corresponding to Δ_α^+ is

$$\Sigma_\alpha = \{\beta \in \Sigma \mid (\beta, \alpha) = 0,\ \beta \neq \alpha\} \cup \{\beta + \alpha \mid \beta \in \Sigma,\ (\beta, \alpha) \neq 0\} \cup \{-\alpha\}$$

and the new Borel subalgebra associated to Δ_α^+ is given by

$$\mathfrak{b}_\alpha = \mathfrak{h} \oplus \left(\bigoplus_{\beta \in \Delta_\alpha^+} \mathfrak{g}_\beta \right).$$

What the above means is that if $\lambda \in \Sigma$ and $\lambda \neq \alpha$, then the new base is constructed such that

$$\lambda \mapsto \begin{cases} \lambda & \text{if } (\lambda, \alpha) = 0 \\ \lambda + \alpha & \text{if } (\lambda, \alpha) \neq 0 \end{cases}.$$

If α is an odd isotropic root, and we set $r_\alpha(\Sigma) := \Sigma_\alpha$ and $r_\alpha(\Delta^+) := \Delta_\alpha^+$, we call this r_α an odd reflection.

Recall given a simple Lie algebra $\mathfrak{g}$ and Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$ and $\tilde{\mathfrak{b}} \subset \mathfrak{g}$, then there exists $\omega \in W$ such that $\mathfrak{b} = \omega\tilde{\mathfrak{b}}$. Since W is generated by simple reflections, this is not the case for basic Lie superalgebras. In general, there is more than one W-conjugacy class of Borel subalgebras.

Example 2.1.46. Let us illustrate the notion of odd reflections using the example of $\mathfrak{g} = \mathfrak{gl}(1|2)$. The Lie superalgebra $\mathfrak{gl}(1|2)$ has roots $\Delta_{\bar{0}} = \{\pm(\epsilon_1 - \epsilon_2)\}$ and $\Delta_{\bar{1}} = \{\pm(\delta_1 - \epsilon_1), \pm(\delta_1 - \epsilon_2)\}$, and the standard positive system for $\mathfrak{gl}(1|2)$ is $\Delta^+ = \{\epsilon_1 - \epsilon_2, \delta_1 - \epsilon_1, \delta_1 - \epsilon_2\}$ with corresponding fundamental system $\Sigma = \{\delta_1 - \epsilon_1, \epsilon_1 - \epsilon_2\}$. So

the opposite fundamental system is $\Sigma^{op} = \{\epsilon_2 - \epsilon_1, \epsilon_1 - \delta_1\}$. Recall that the Weyl group W is defined to be the Weyl group of $\mathfrak{g}_{\bar{0}}$, hence it is generated by the simple real reflection $\sigma_{\epsilon_1-\epsilon_2}$. i.e., $W = \langle\sigma_{\epsilon_1-\epsilon_2}\rangle = \{1, \sigma_{\epsilon_1-\epsilon_2}\}$.

If we apply $\sigma_{\epsilon_1-\epsilon_2}$ to Δ^+, we get $\Delta^+_{\epsilon_1-\epsilon_2} = \{\epsilon_2 - \epsilon_1, \delta_1 - \epsilon_1, \delta_1 - \epsilon_2\}$ and $\Sigma_{\epsilon_1-\epsilon_2} = \{\delta_1 - \epsilon_2, \epsilon_2 - \epsilon_1\}$, which is not the opposite fundamental system. This is why odd reflections are essential. To get to Σ^{op}, we first apply the odd reflection $r_{\delta_1-\epsilon_1}$ to Δ^+ which yields $(\Delta^+)' = \{\epsilon_1 - \epsilon_2, \epsilon_1 - \delta_1, \delta_1 - \epsilon_2\}$ and $\Sigma' = \{\epsilon_1 - \delta_1, \delta_1 - \epsilon_2\}$. Next we apply the odd reflection $r_{\delta_1-\epsilon_2}$ which yields $(\Delta^+)'' = \{\epsilon_1 - \epsilon_2, \epsilon_1 - \delta_1, \epsilon_2 - \delta_1\}$ and $\Sigma'' = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \delta_1\}$. Finally, we apply the simple real reflection $\sigma_{\epsilon_1-\epsilon_2}$ to get $\sigma_{\epsilon_1-\epsilon_2}((\Delta^+)'') = \{\epsilon_2 - \epsilon_1, \epsilon_1 - \delta_1, \epsilon_2 - \delta_1\}$ and $\sigma_{\epsilon_1-\epsilon_2}(\Sigma'') = \{\epsilon_2 - \epsilon_1, \epsilon_1 - \delta_1\}$.

Example: $\mathfrak{g} = \mathfrak{osp}(1|2)$

$$\text{Let } \mathfrak{g} = \mathfrak{osp}(1|2) = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ \beta & h & e \\ -\alpha & f & -h \end{pmatrix} \;\middle|\; \alpha, \beta, e, f, h \in \mathbb{C} \right\} \subset \mathfrak{gl}(1|2) \ .$$

The superbracket is given by $[a, b] := ab - (-1)^{|a||b|}ba$. Let

$$f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ .$$

The basis of $\mathfrak{g}$ is $\{e, f, h, x, y\}$ where $\text{span}\{e, f, h\} \cong \mathfrak{sl}_2(\mathbb{C})$ and the non-zero brackets are given by:

$$[e, y] = -x, [f, x] = -y, [x, x] = e, [x, y] = h, [h, x] = x, [y, y] = -f, [h, y] = -y$$
$$[e, f] = h, [h, e] = 2e, [h, f] = -2f \ .$$

Take the standard Cartan subalgebra $\mathfrak{h} = \text{span}\{h\}$ and the Borel subalgebra to be $\mathfrak{b} = \text{span}\{h, e, x\}$. Let $\{\delta\}$ be the corresponding dual basis in $\mathfrak{h}^*$. From the brackets above, we see that

$$x \in \mathfrak{g}_\delta, y \in \mathfrak{g}_{-\delta}, e \in \mathfrak{g}_{2\delta}, f \in \mathfrak{g}_{-2\delta}.$$

Hence with respect to $\mathfrak{h}$, the even roots are $\Delta_{\bar{0}} = \{\pm 2\delta\}$ and the odd roots are $\Delta_{\bar{1}} = \{\pm\delta\}$. The coroot of δ is $h_\delta \in \mathfrak{h}$. It is defined by $(h_\delta, h') = \delta(h')$ for all $h' \in \mathfrak{h}$. Since $\mathfrak{h} = \text{span}_{\mathbb{C}}(h)$, then $h_\delta = kh$ for some $k \in \mathbb{C}$. So, using the fact that $(h_\delta, h) = \delta(h)$ and $(h, h) = \text{str}(h^2) = 2 = 2\delta(h)$, then

$$\delta(h) = (h_\delta, h) = (kh, h) = 2k\delta(h) \ .$$

This implies that $h_\delta = \frac{1}{2}h$. The base is of the root system with respect to $\mathfrak{h}$ is clearly $\Sigma = \{\delta\}$. Since $(\delta, \delta) = 1 \neq 0$, then δ is a non-isotropic root, so the simple root δ is a unique, odd, non-isotropic root.

2.2 The current algebra and the loop algebra

Associated to any Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is the current algebra. This is the infinite-dimensional complex Lie algebra $\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t]$, with the Lie bracket

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n}, \quad x, y \in \mathfrak{g},\ n, m \in \mathbb{Z}_{\geq 0}\ .$$

The current algebra $\widehat{\mathfrak{g}}$ is $\mathbb{Z}_{\geq 0}$-graded via

$$\deg(x \otimes t^n) = n, \quad x \in \mathfrak{g},\ n \in \mathbb{Z}_{\geq 0}\ .$$

Likewise, associated to any Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is the loop algebra, given by $\overline{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, with the Lie bracket

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n},\ x, y \in \mathfrak{g},\ n, m \in \mathbb{Z}.$$

The loop algebra $\overline{\mathfrak{g}}$ is $\mathbb{Z}$-graded via

$$\deg(x \otimes t^n) = n,\ x \in \mathfrak{g}, \quad n \in \mathbb{Z}\ .$$

2.3 Exponential polynomial functions

Definition 2.3.1. For any $\lambda \in \mathbb{C}^\times$, the exponential function $\exp(\lambda) : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\exp(\lambda)(m) = \lambda^m,\ m \in \mathbb{Z},$$

which can be found in [Wil08].

Definition 2.3.2. A function $\varphi : \mathbb{Z} \to \mathbb{C}$ is called exponential-polynomial in one variable if it can be written as a finite sum of products of polynomial and exponential functions. i.e., if there exists $n \in \mathbb{Z}_{>0}$, polynomials $g_i \in \mathbb{C}[k]$, and scalars $\lambda_i \in \mathbb{C}^\times$ such that

$$\varphi(k) = \sum_{i=1}^{n} g_i(k) \exp(\lambda_i)(k),$$

which can be found in [Wil09]. From now on, we will simply say "exponential-polynomial function" or "exponential-polynomial" to refer to an exponential-polynomial function in one variable.

Lemma 2.3.3. *Let $r > 0$ be an integer and $\xi \in \mathbb{C}^\times$ an r^{th} root of unity. Define the function*

$$q_r = \sum_{i=0}^{r-1} \exp(\xi^i) \ .$$

Then

$$q_r(n) = \begin{cases} r & \text{if } n \equiv 0 \mod r \\ 0 & \text{otherwise} \end{cases} .$$

Proof. If $n \equiv 0 \mod r$, then $q_r(n) = \sum_{i=0}^{r-1} (\xi^i)^n = \sum_{i=0}^{r-1} 1 = r$.
If $n \equiv l \mod r$ where $l \neq 0$, then there exist $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ such that $0 < \beta < r - 1$ and $n = \alpha r + \beta$. So $\xi^n = \xi^{\alpha r + \beta} = \xi^\beta \neq 1$. Hence

$$q_r(n) = 1^n + \xi^n + \xi^{2n} + \cdots + \xi^{(r-1)n} = \frac{(\xi^n)^r - 1}{\xi^n - 1} = \frac{1 - 1}{\xi^\beta - 1} = 0.$$

□

Lemma 2.3.4. *Let $\varphi : \mathbb{Z} \to \mathbb{C}$ be a nonzero exponential polynomial function. Then*

there exist $n \in \mathbb{Z}_{>0}$ *and a unique maximal* $r > 0$ *such that*

$$\varphi = q_r \sum_{i=1}^{n} g_i \exp(\lambda_i)$$

for some polynomial functions $g_i : \mathbb{Z} \to \mathbb{C}$*, and scalars* $\lambda_i \in \mathbb{C}^\times$*, such that if* $\left(\frac{\lambda_i}{\lambda_j}\right)^r = 1$*, then* $i = j$*. In this case we say that* φ *is exponential polynomial of degree* r*, and write* $\deg\varphi = r$ *for the degree of* φ *[Wil08].*

The previous two lemmas show that if φ is exponential-polynomial of degree r, then $\varphi(k) = 0$ if $k \not\equiv 0 \mod r$.

Lemma 2.3.5. *Let* $\varphi : \mathbb{Z} \to \mathbb{C}$ *be a nonzero function. Then* φ *is exponential polynomial if and only if* φ *is the solution to a homogeneous linear recurrence relation with constant coefficients.*

Proof. Consider a homogeneous linear recurrence relation with constant coefficients

$$(*) \qquad x_{N+k} = \alpha_1 x_{N+k-1} + \cdots + \alpha_N x_k,\ \alpha_i \in \mathbb{C} \text{ and } \alpha_N \neq 0\ . \tag{2.3}$$

Suppose φ is the solution to $(*)$. The characteristic polynomial of $(*)$ is $p(t) = t^N - \alpha_1 t^{N-1} - \cdots - \alpha_N$. Let $r_1, \ldots, r_l$ be the roots of p with multiplicities $m_1, \ldots, m_l$, respectively. Then the general solution to $(*)$ is of the form

$$x_n = \sum_{i=1}^{l} (c_{i,1} r_i^n + \cdots + c_{i,m_i} n^{m_i} r_i^n) = \sum_{i=1}^{l} g_i(n) r_i^n,$$

where $g_i(n) = c_{i,1} r_i^n + \cdots + c_{i,m_i} r_i^n$ for some $c_{i,j} \in \mathbb{C}$ [GK90]. This general solution is precisely of exponential-polynomial form, and since φ is of the form of x_n, it must

too be exponential-polynomial.

Now suppose that φ is exponential polynomial. Let $\varphi(n) = \sum_{i=1}^{l} g_i(n) \exp(\lambda_i)(n)$ for some $g_i \in \mathbb{C}[n]$ and $\lambda_i \in \mathbb{C}$. Let $N := \sum_{i=1}^{l} \deg g_i$ and let

$$p(t) := \prod_{i=1}^{N} (t - \lambda_i)^{\deg g_i} = t^N - \alpha_1 t^{N-1} - \cdots - \alpha_N.$$

This is the characteristic equation of the linear recurrence relation

$x_{k+N} = \alpha_1 x_{k+N-1} + \cdots + \alpha_N x_k$ of which φ is a solution. □

Chapter 3

Finite dimensional graded components

Let $\mathfrak{g}$ be one of $\mathfrak{sl}_2, \mathfrak{sl}(1|1)$, or $\mathfrak{osp}(1|2)$, where the base field is $\mathbb{C}$. We will study finite dimensional modules over the current algebra $\hat{\mathfrak{g}}$, and the loop algebra $\overline{\mathfrak{g}}$, by defining a quotient of the Verma module $M(\Lambda)$. At the end of the Chapter, we discuss irreducible $\mathbb{Z}$-graded $\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$-modules, which will be used in Chapter 5.2.

3.1 Finite Dimensional $\mathfrak{g} \otimes A$-modules

Recall that $\mathfrak{g}$ is an $\mathfrak{h}$-module via the adjoint action.

Let $\mathfrak{g}_k := \{X \in \mathfrak{g} \mid [H, X] = kX \ \forall \ H \in \mathfrak{h}\}$. When $\mathfrak{g} = \mathfrak{sl}_2$, we have the decomposition $\mathfrak{sl}_2 = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$, where $\mathfrak{g}_k = 0$ for $k \neq 0, \pm 2$. From this decomposition we get the $\mathbb{Z}$-grading of $\mathfrak{sl}_2$ where $\deg(f) = -2$, $\deg(h) = 0$, and $\deg(e) = 2$. Similarly, when $\mathfrak{g} = \mathfrak{osp}(1\,|2)$ we have the decomposition $\mathfrak{osp}(1\,|\,2) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$, where $\mathfrak{g}_k = 0$ for $k \neq 0, \pm 1, \pm 2$, and the corresponding $\mathbb{Z}$-grading of $\mathfrak{g}$ where $\deg(f) = -2$, $\deg(y) = -1$, $\deg(h) = 0$, $\deg(x) = 1$, and $\deg(e) = 2$.

In the case of $\mathfrak{g} = \mathfrak{sl}(1\,|\,1)$ the Cartan subalgebra $\mathfrak{h} = \mathbb{C}I_2$ is central so we do not

get a $\mathbb{Z}$-grading of $\mathfrak{g}$ by viewing $\mathfrak{g}$ as an $\mathfrak{h}$-module. However, we may use the larger Lie superalgebra $\mathfrak{gl}(1\,|\,1) \supset \mathfrak{sl}(1\,|\,1)$. The Cartan subalgebra of $\mathfrak{gl}(1\,|\,1)$ is 2-dimensional and not central. By viewing $\mathfrak{gl}(1\,|\,1)$ as an $\mathfrak{h}$-module, we get the decomposition $\mathfrak{gl}(1\,|\,1) = \mathfrak{gl}(1\,|\,1)_{-1} \oplus \mathfrak{gl}(1\,|\,1)_0 \oplus \mathfrak{gl}(1\,|\,1)_1$ and corresponding $\mathbb{Z}$-grading of $\mathfrak{gl}(1\,|\,1)$. This allows us to obtain the decomposition $\mathfrak{sl}(1\,|\,1) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ where $\mathfrak{g}_k = 0$ for $k \neq 0, \pm 1$, from which we get the $\mathbb{Z}$-grading of $\mathfrak{sl}(1\,|\,1)$ where $\deg(y) = -1$, $\deg(z) = 0$, and $\deg(x) = 1$.

With respect to the above grading of $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{g} = \mathfrak{osp}(1\,|\,2)$, we may view any weight module over $\mathfrak{g}$ as a $\mathbb{Z}$-graded module. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$. Suppose $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M^\lambda$ is a weight $\mathfrak{g}$-module, and fix a weight $\lambda_0 \in \mathrm{supp}M$. For any homogeneous $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$, $[H, X] = kX$ where $k \in \{0, \pm 1, \pm 2\}$. Hence for any homogeneous $X \in \mathfrak{g}$, $X \cdot M^{\lambda_0} \subseteq M^{\lambda_0+k}$ where $k \in \{0, \pm 1, \pm 2\}$. So M can be viewed as a $\mathbb{Z}$-graded $\mathfrak{g}$-module, and $M = \bigoplus_{k\in\mathbb{Z}} M^{\lambda_0+k}$.

We extend the grading above to a $\mathbb{Z}$-grading on $U(\mathfrak{g} \otimes A)$ as follows. Suppose $X := X^1 \cdots X^n \in U(\mathfrak{g} \otimes A)$ is written in the PBW order. Then $\deg(X) = \sum_{i=1}^n \deg(X^i)$. We will use this grading to study finite dimensional $\mathbb{Z}$-graded $(\mathfrak{g} \otimes A)$-modules. Let M be such an irreducible module and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$, where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, be the triangular decomposition of $\mathfrak{g}$. Then as an $\mathfrak{h}$-module, M is semi-simple. i.e., $M = \bigoplus_{\mu\in\mathfrak{h}^*} M_\mu$ is a weight module. To achieve finite dimensionality, two conditions must be satisfied. The weight spaces need to have finite support, and each weight space must be finite-dimensional. Since M has finite support there is a largest $\mu_0 \in \mathrm{supp}M$ which we call the highest weight. Let $w_0 \in M^{\mu_0}$ be a highest weight vector. Recall that if $0 \neq X_n \in \mathfrak{n}^+ \otimes A$, then $X_n \cdot M^\mu \subseteq M^{\mu'}$, where $\mu < \mu'$. Since μ_0 is the highest weight, then necessarily $(\mathfrak{n}^+ \otimes A) \cdot w_0 = 0$. This will be the first condition for the Weyl module.

On the other hand, if $0 \neq X_n \in \mathfrak{n}^- \otimes A$ then $X_n \cdot M^\mu \subseteq M^{\mu'}$, where $\mu' < \mu$. Hence, $M^{\mu_0} = U(\mathfrak{h} \otimes A) \cdot w_0$. Let $N \subset M^{\mu_0}$ be a proper $U(\mathfrak{h} \otimes A)$-submodule. By the PBW theorem, $U(\mathfrak{g} \otimes A) \cong U(\mathfrak{n}^- \otimes A) \otimes U(\mathfrak{h} \otimes A) \otimes U(\mathfrak{n}^+ \otimes A)$, so $U(\mathfrak{g}) \cdot N \subset M$ is a proper $\mathfrak{g} \otimes A$-submodule. Since we assumed that M is an irreducible $(\mathfrak{g} \otimes A)$-module, then $N = 0$ and so M^{μ_0} must be an irreducible $U(\mathfrak{h} \otimes A)$-module. Since M^{μ_0} is the highest weight space of M, then $\dim M^{\mu_0} = 1$, i.e., $M^{\mu_0} = \mathbb{C}w_0$. This means that for all $h_n \in \mathfrak{h} \otimes A$, $h_n \cdot w = \lambda(h_n)w$ for some $\lambda \in (\mathfrak{h} \otimes A)^*$. This will be the second condition for the Weyl module.

Finally, to have finite support, for each Borel subalgebra $(\mathfrak{n}^+)' \oplus \mathfrak{h} = \mathfrak{b}' \subset \mathfrak{g}$ there must exist a weight $\lambda \in \mathfrak{h}^*$ such that $x \cdot v = 0$ for all $v \in M^\lambda$ and for all $x \in (\mathfrak{n}^+)'$. This will be the third condition. Such a module is called a Weyl module and is a highest weight module. It will be discussed in detail in the following chapters. Using generators and relations, we prove that each weight space of the Weyl module is finite-dimensional, and we derive a formula for the dimension of the module. This is summarized in Theorem 3.1.3.

Definition 3.1.1. Let $\mathfrak{g}$ be a Lie superalgebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Let A be a finitely generated unital associative $\mathbb{C}$-algebra and let $\Lambda \in (\mathfrak{h} \otimes A)^*$. Define the Verma module $M(\Lambda)$ as the cyclic $\mathfrak{g} \otimes A$-module with highest weight Λ generated by w_0 such that

$$(\mathfrak{n}^+ \otimes A) \cdot w_0 = 0, \tag{3.1}$$

$$(h \otimes a) \cdot w_0 = \Lambda(h \otimes a)w_0 \text{ for all } h \otimes a \in \mathfrak{h} \otimes A. \tag{3.2}$$

Let $I(\Lambda) \subset M(\Lambda)$ be the maximal submodule and $V(\Lambda) := M(\Lambda)/I(\Lambda)$ be the

irreducible quotient of $M(\Lambda)$.

In Chapters 3, 4, and 5, we define the Weyl module with generators and relations. Here we provide an equivalent definition using the Verma module.

Definition 3.1.2. Let $\mathfrak{g}$ be a rank k Lie superalgebra with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let A be a finitely generated unital associative $\mathbb{C}$-algebra, let $\Lambda \in (\mathfrak{h} \otimes A)^*$, and let $M(\Lambda)$ be the Verma module. Let $\Sigma = \{\alpha_1, \ldots, \alpha_k\}$ be a set of simple roots of $\mathfrak{g}$. For each $\alpha \in \Sigma_{\bar{0}}$ and for each $\beta \in \Sigma_{\bar{1}}$, let $\{f_\alpha, h_\alpha, e_\alpha\}$ and $\{y_\beta, z_\beta, x_\beta\}$ be the Chevalley generators, respectively. For $\mathbf{m} = (m_1, \ldots, m_k) \in (\mathbb{Z}_{>0})^k$, let $I(\Lambda, m) \subset M(\Lambda)$ be a submodule generated by $\{(F_{\alpha_i} \otimes 1)^{m_i+1} \cdot w_0 \,|\, \alpha_i \in \Sigma_{\bar{0}}\} \cup \{(Y_{\alpha_i} \otimes 1)(Y_{\alpha_i} \otimes t) \cdots (Y_{\alpha_i} \otimes t^{m_i}) \cdot w_0 \,|\, \alpha_i \in \Sigma_{\bar{1}}\}$. Then $W(\Lambda, m) = M(\Lambda)/I(\Lambda, m)$ is a local Weyl module of $\mathfrak{g} \otimes A$.

There are two remarks to make about $W(\Lambda, m)$. The first is that $W(\Lambda, m)$ is irreducible if and only if $I(\Lambda, m)$ is a maximal submodule. The second is that $W(\Lambda, m) = 0$ if and only if $I(\Lambda, m) = W(\Lambda)$.

The following theorem is a summary of some of the results in Chapters 3, 4, and 5.

Theorem 3.1.3. *Let $\mathfrak{g}$ be one of $\mathfrak{sl}_2$, $\mathfrak{sl}(1\,|\,1)$, or $\mathfrak{osp}(1\,|\,2)$ and let $A = \mathbb{C}[t]$ or $A = \mathbb{C}[t, t^{-1}]$. Let $V(\Lambda)$ be as in Definition 3.1.1. Then the following are equivalent:*

(i) $\dim V(\Lambda) < \infty$

(ii) $\mathrm{supp} V(\Lambda) < \infty$

(iii) $\dim V(\Lambda)^\mu < \infty$ *for all* $\mu \in \mathrm{supp} V(\Lambda)$

(iv) There exists an ideal $I \subset A$ *such that* $\dim(A/I) < \infty$ *and* $(\mathfrak{h} \otimes I) \cdot w_0 = 0$.

3.1.1 Connections to other research

Weyl modules for the affine Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus c\mathbb{C}$, where $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra, were first introduced by Chari and Pressley in [CP01]. Around 20 years later, Calixto, Lemay and Savage replaced $\mathfrak{g}$ by a Lie superalgebra in [CLS19]. In particular, they studied Weyl modules over Lie superalgebras of the form $\mathfrak{g}\otimes_{\mathbb{C}} A$, where $\mathfrak{g}$ is a basic Lie superalgebra, or $\mathfrak{g}=\mathfrak{sl}(n,n)$, $n\geq 2$, and A is an associative commutative unital $\mathbb{C}$-algebra, and proved that these modules are finite-dimensional.

3.2 $\mathbb{Z}\times\mathbb{Z}$-graded $\mathfrak{g}\otimes A$-modules

3.2.1 Irreducible $\mathbb{Z}$-graded $\overline{\mathfrak{h}}$-modules

Recall that for a Lie superalgebra $\mathfrak{l}$, the loop algebra is defined as $\bar{\mathfrak{l}}:=\mathfrak{l}\otimes\mathbb{C}[t,t^{-1}]$. In the previous section, we described $\mathfrak{g}\otimes A$-modules generated by a one dimensional $\overline{\mathfrak{h}}$-module. The next type of module we study is generated by an irreducible $\mathbb{Z}$-graded $\overline{\mathfrak{h}}$-module with finite-dimensional graded components. In contrast to the previous section, these modules are not necessarily finite-dimensional. From these modules we then induce $\mathbb{Z}\times\mathbb{Z}$-graded $\overline{\mathfrak{g}}$-modules. The next proposition shows for such an irreducible $\overline{\mathfrak{h}}$-module, each graded component is either trivial or one-dimensional.

Proposition 3.2.1. *Let L be an irreducible $\mathbb{Z}$-graded $\overline{\mathfrak{h}}$-module such that*

$$L=\bigoplus_{i\in\mathbb{Z}} L_i$$

where $\dim L_i<\infty$ for all $i\in\mathbb{Z}$. Then

1. $\dim L_i \leq 1$, *and*

2. *the set* $I = \{i \in \mathbb{Z} \mid \dim L_i = 1\} \subset \mathbb{Z}$ *is equal to* $I = i_0 + (r)$ *for some* $i_0 \in \mathbb{Z}$ *and* $r \geq 0$.

Remark 3.2.2. We may think of L as the $\overline{\mathfrak{h}}$-module $t^{i_0}\mathbb{C}[t^r, t^{-r}]$ with the action $h_n \cdot t^{i_0+rm} = a(n)t^{i_0+rm+n}$, where $m, n \in \mathbb{Z}$, and $a : \mathbb{Z} \to \mathbb{C}$ is some exponential polynomial function of degree r.

Proof. For the first part, we can consider $U(\overline{\mathfrak{h}})$ as a $\mathbb{Z}$-graded algebra by letting $\deg(Z_{k_1} \cdots Z_{k_s}) = k_1 + \cdots + k_s$. Since L is an irreducible $\overline{\mathfrak{h}}$-module, then L is also an irreducible $U(\overline{\mathfrak{h}})$-module. Fix $i \in \mathbb{Z}$ and consider L_i. Then $U(\overline{\mathfrak{h}})^0 \cdot L_i \subseteq L_i$, so L_i is an irreducible $U(\overline{\mathfrak{h}})^0$-module. Suppose $L_i \neq 0$. Since L_i is a $U(\overline{\mathfrak{h}})^0$-module, then each $h \in U(\overline{\mathfrak{h}})^0$ is a linear operator on L_i and hence has an eigenvector. Let $h' \in U(\overline{\mathfrak{h}})^0$ and consider L_i', the corresponding eigenspace. Since $U(\overline{\mathfrak{h}})^0$ is commutative, L_i' is invariant under the action of any $h \in U(\overline{\mathfrak{h}})^0$. Hence $L_i' \subseteq L$ is a $U(\overline{\mathfrak{h}})^0$-submodule. Since L_i is an irreducible $U(\overline{\mathfrak{h}})^0$-module, then $L_i' = L_i$. By Lemma 2.1.40, $\dim L' = 1$, hence $\dim L_i = 1$.

For the second part, suppose $L \neq 0$. Then there exists $i_0 \in \mathbb{Z}$ such that $L_{i_0} \neq 0$. There are two cases. In the first case, $L_i = 0$ for all $i \in \mathbb{Z} \setminus \{i_0\}$. Then $I = i_0 + (0)$. In the second case, there exists at least one element $j \in I$ such that $j \neq i_0$ and $L_j \neq 0$. We can choose j such that the distance $r := |i_0 - j|$ is minimal. Without loss of generality, suppose $j > i_0$, i.e., $j = i_0 + r$. Then $L_{i_0+1} = \cdots = L_{i_0+r-1} = 0$. By part a), $\dim L_{i_0} = \dim L_j = 1$. Choose a basis vector $0 \neq v_{i_0}$ of L_{i_0}. Since $L_j \neq 0$ and L is an irreducible $\overline{\mathfrak{h}}$-module, there exists $X \in \overline{\mathfrak{h}}^r$ such that $Xv_{i_0} \neq 0$. Choose the vector $v_{i_0+r} := Xv_{i_0} \in L_{i_0+r}$ to be a basis vector of L_{i_0+r}. We will use strong induction on N, where N is such that $i_0, i_0 + r, i_0 + 2r, \ldots, i_0 + Nr \in I$, to prove that $i_0 + (r) \subseteq I$.

By above we know that the base case of $N = 1$ is true. Now suppose there exists $N \in \mathbb{Z} > 0$ such that $i_0 + kr \in I$ for all $0 \leq k \leq N$. As in the base case, choose a basis vector $v_{i_0} \in L_{i_0}$. Since L is an irreducible $\overline{\mathfrak{h}}$-module, there exists $X \in \overline{\mathfrak{h}}^r$ such that $v_{i_0+r} := Xv_{i_0} \in L_{i_0+r}$ is non-zero. Similarly, there exists $Y \in \overline{\mathfrak{h}}^{-r}$ such that $Yv_{i_0+r} = v_{i_0}$. Consider $X^k v_{i_0} \in L_{i_0+kr}$ for $k = 0, \ldots, N$. Since the elements of $U(\overline{\mathfrak{h}})$ commute, then $0 = X^k v_{i_0}$ implies that $0 = Y^k X^k v_{i_0} = (YX)^k v_{i_0} = v_{i_0}$, which is a contradiction. Hence we may fix a basis of L_{i_0+kr} for $k = 0, \ldots, N$, to be $\{v_{i_0+kr}\}$, where $v_{i_0+kr} := X^k v_{i_0}$. This also implies that $i_0 - r, i_0 - 2r, \ldots, i_0 - Nr \in I$. Since otherwise, if $Y^k v_{i_0} \in L_{i_0-kr} = 0$ for $1 \leq k \leq N$, then, since elements of $U(\overline{\mathfrak{h}})$ commute,

$$0 = Y^k v_{i_0} = X^k Y^k v_{i_0} = Y^k X^k v_{i_0} = Y^k v_{i_0+kr} = v_{i_0}.$$

Moreover, we know that XYv_{i_0+Nr} and YXv_{i_0-Nr} are both non-zero. Now suppose that $Xv_{i_0+Nr} = 0$ and $Yv_{i_0-Nr} = 0$. Then

$$0 = Xv_{i_0+Nr} = YXv_{i_0+Nr} = XYv_{i_0+Nr}, \text{ and}$$
$$0 = Yv_{i_0-Nr} = XYv_{i_0-Nr} = YXv_{i_0-Nr},$$

which are both contradictions. Hence $i_0 \pm (N+1)r \in I$, and by induction $i_0 + (r) \subseteq I$. Now we will show that $i_0 + kr + l \notin I$ for all $k \in \mathbb{Z}$ and $1 \leq l \leq r - 1$. By way of contradiction, suppose there exists $k \in \mathbb{Z}$ and $1 \leq l \leq r - 1$ such that $L_{i_0+kr+l} \neq 0$. Choose a basis vector v_{i_0+kr+l} of L_{i_0+kr+l}. Since $L_{i_0+kr} \neq 0$ and L is an irreducible $\overline{\mathfrak{h}}$-module, then there exists $Z \in \overline{\mathfrak{h}}^{-l}$ such that $Zv_{i_0+kr+l} = v_{i_0+kr}$. Then

$X^k Z v_{i_0+kr+l} = v_{i_0}$ is non-zero. Since elements of $U(\overline{\mathfrak{h}})$ commute,

$$0 \neq X^k Z v_{i_0+kr+l} = Z(X^k v_{i_0+kr+l}) \ .$$

This implies that $0 \neq X^k v_{i_0+kr+l} \in L_{i_0+l}$, which contradicts $L_{i_0+l} = 0$. Hence $I = i_0 + (r)$ for some $r \geq 0$. □

Chapter 4

The Lie algebra $\mathfrak{sl}_2 \otimes A$

Let A be $\mathbb{C}[t]$ or $\mathbb{C}[t, t^{-1}]$. Before going to the case of Weyl modules of current and loop Lie superalgebras, we will first study the Weyl module of the Lie algebra $\mathfrak{sl}_2 \otimes A$. We will then use the techniques from this section to study the Weyl module of $\mathfrak{sl}(1|1) \otimes A$ in Chapter 5 and $\mathfrak{osp}(1|2) \otimes A$ in Chapter 6.

4.1 Highest weight modules of $\mathfrak{sl}_2 \otimes A$

For $X \in \mathfrak{g}$, define $X_n := X \otimes t^n \in \mathfrak{g} \otimes A$. Recall that $\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ and recall the definition of the Verma module $M(\Lambda)$ from Definition 3.1.1.

Definition 4.1.1. Let $\mathbf{a} := (a_n)_{n\in\mathbb{Z}}$, where $a_n \in \mathbb{C}$. Let $\Lambda \in (\mathfrak{h} \otimes A)^*$ be such that $\Lambda(h_n) = a_n$. Define $I(\Lambda, m) \subset M(\Lambda)$ to be the submodule generated by $F_0^{m+1} \cdot w_0$. Then

$$W(\Lambda, m) := M(\Lambda)/I(\Lambda, m)\,. \tag{4.1}$$

$W(\Lambda, m)$ is called a Weyl module of $\mathfrak{sl}_2 \otimes A$.

More informally, $W(\Lambda, m)$ is the cyclic highest weight $U(\mathfrak{sl}_2 \otimes A)$-module generated by w_0 and defined by the relations

$$\begin{aligned} H_n \cdot w_0 &= \Lambda(h_n) w_0 \quad \forall\, H_n \in U(\mathfrak{h} \otimes A)\,, \\ (\mathfrak{n}^+ \otimes \mathbb{C}[t]) \cdot w_0 &= 0\,, \\ F_0^{m+1} \cdot w_0 &= 0\ . \end{aligned} \tag{4.2}$$

Note that here we abuse notation by using w_0 as the generating vector of both $M(\Lambda)$ and $W(\Lambda, m)$.

Remark 4.1.2. We would like w_0 to be non-zero. However, Theorem 4.1.3 states that $W(\Lambda, m) = 0$ if certain conditions hold. In this case $w_0 = 0$.

The rest of this section is devoted to proving the following theorem.

Theorem 4.1.3. *Let $m \in \mathbb{Z}_{>0}$. Let $W(\Lambda, m)$ be the $U(\mathfrak{sl}_2 \otimes A)$-module generated by a vector w_0 with relations (4.2). Then*

1. *$W(\Lambda, m) \neq 0$ if and only if there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{C}^*$ with $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for all n .*

2. *If $W(\Lambda, m) \neq 0$ then $\dim W(\Lambda, m) = 2^m$.*

3. *If $W'(\Lambda, m)$ is a $(\mathfrak{sl}_2 \otimes A)$-module generated by a vector w_0 which satisfies the relations (4.2), then there exists an $(\mathfrak{sl}_2 \otimes A)$-submodule $M \subseteq W(\Lambda, m)$ such that $W'(\Lambda, m) \cong W(\Lambda, m)/M$.*

We start by proving the existence of $W(\Lambda, m)$ given that $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$. To show that such a module exists (i.e., is non-zero), we will prove that $W(\Lambda, m)$ is equal to the tensor product of evaluation modules, which are

defined below. Suppose that for all $n \geq 0$, $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$. Let V be a highest weight $\mathfrak{sl}_2(\mathbb{C})$-module with highest weight 1, generated by v_i. For $\alpha_i \in \mathbb{C}$ let $V_i(\alpha_i)$ be a highest weight $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-module generated by $v_{i,0}$ with action $x_n \cdot v_{i,0} = \alpha_i^n (x \cdot v_i)$ for all $v_i \in V_i(\alpha_i)$ and for all $x_n \in \mathfrak{sl}_2 \otimes \mathbb{C}[t]$. Note that this action evaluates the polynomial t^n at α_i. We refer to $V_i(\alpha_i)$ as the evaluation module at α_i. Now consider

$$W := \bigotimes_{i=1}^{m} V_i(\alpha_i), \tag{4.3}$$

the tensor product of evaluation modules. We claim that W satisfies relations (4.2). Indeed, since W is generated by $v_{1,0} \otimes \cdots \otimes v_{m,0}$, then

1.

$$\begin{aligned} H_n \cdot v_{1,0} \otimes \cdots \otimes v_{m,0} &= \sum_{i=1}^{m} v_{1,0} \otimes \cdots \otimes (H_n \cdot v_{i,0}) \otimes \cdots \otimes v_{m,0} \\ &= \sum_{i=1}^{m} v_{1,0} \otimes \cdots \otimes (\alpha_i^n v_{i,0}) \otimes \cdots \otimes v_{m,0} \\ &= \sum_{i=1}^{m} \alpha_i^n (v_{1,0} \otimes \cdots \otimes v_{m,0}) \\ &= \left(\sum_{i=1}^{m} \alpha_i^n \right) v_{1,0} \otimes \cdots \otimes v_{m,0}. \end{aligned}$$

2. Moreover, since $E_n \cdot v_{i,0} = 0$ for all $i = 1, \ldots, m$, then $E_n \cdot (v_{1,0} \otimes \cdots \otimes v_{m,0}) = 0$.

3. Finally, since $F_0^{m+1} \cdot v_{i,0} = 0$ for all $i = 1, \ldots, m$, then

$$\begin{aligned} & F_0^{m+1} \cdot (v_{1,0} \otimes \cdots \otimes v_{m,0}) \\ =& (F_0^{m+1} \cdot v_{1,0}) \otimes v_{2,0} \cdots \otimes v_{m-1,0} \otimes v_{m,0} + \cdots + v_{1,0} \otimes v_{2,0} \otimes \cdots \otimes v_{m-1,0} \otimes (F_0^{m+1} \cdot v_{m,0}) \\ =& 0. \end{aligned}$$

So W satisfies the conditions of the Weyl module. Hence it must be a quotient of the Weyl module thus proving the existence of the Weyl module. Now we will assume a module which satisfies the conditions of (4.2) exists and prove that properties 1., 2., and 3. of Theorem 4.1.3 hold.

Remark 4.1.4. We introduce the following notation for elements of $U(\mathfrak{sl}(2) \otimes A)$. Let $I = \{i_1, \dots, i_n\}, J = \{j_1, \dots, j_n\}$ be multisets of integers. We will write $E_I := E_{i_1} \cdots E_{i_n}$ and $F_J := F_{j_1} \cdots F_{j_n}$. For any $x \in \mathfrak{sl}(2)$ and $i, j \in \mathbb{Z}$, the elements X_i, X_j commute in $U(\mathfrak{sl}(2) \otimes A)$, hence this notation is well-defined.

Proposition 4.1.5. *Let $m \in \mathbb{Z}_{\geq 0}$. Then $F_0^{m+1} \cdot w_0 = 0$ implies $F_I \cdot w_0 = 0$ for any multiset of integers $I = \{i_1, i_2, \dots, i_{m+1}\}$.*

Proof. We will use induction on n, where n is the size of the multiset $\{i_1, i_2, \dots, i_n\}$ to show that $F_{i_1} \cdots F_{i_n} F_0^{m+1-n} \cdot w_0 = 0$. When $n = 1$,

$$\begin{aligned} 0 &= H_{i_1} F_0^{m+1} w_0 \\ &= F_0^{m+1} H_{i_1} w_0 - \sum_{k=0}^{m} F_0^k [H_{i_1}, F_0] F_0^{m-k} w_0 \\ &= 0 - 2(m+1) F_0^m F_{i_1} w_0 \ . \end{aligned}$$

Hence $F_0^m F_{i_1} w_0 = 0$ for any $i_1 \geq 0$.

Now suppose the result holds for some $n \geq 1$. Since $F_{i_n} \cdots F_{i_1} F_0^{m+1-n} w_0 = 0$, then

$$\begin{aligned} 0 &= H_{i_{n+1}} F_{i_n} \cdots F_{i_1} F_0^{m+1-n} w_0 \\ &= F_{i_n} \cdots F_{i_1} H_{i_{n+1}} F_0^{m+1-n} w_0 - 2 \sum_{r=1}^{n} F_{i_{n+1}+i_r} F_{i_n} \cdots \widehat{F_{i_r}} \cdots F_{i_1} F_0^{m+1-n} w_0 \\ &= F_{i_n} \cdots F_{i_1} H_{i_{n+1}} F_0^{m+1-n} w_0 - 0 \end{aligned}$$

$$= F_{i_n} \cdots F_{i_1} F_0^{m+1-n} H_{i_{n+1}} w_0 - \sum_{k=0}^{m-n} F_{i_n} \cdots F_{i_1} F_0^k [H_{i_{n+1}}, F_0] F_0^{m-n-k} w_0$$

$$= a_{i_n+1} F_{i_n} \cdots F_{i_1} F_0^{m+1-n} w_0 - 2(m+1-n) F_{i_{n+1}} \cdots F_{i_1} F_0^{m-n} w_0$$

$$= -2(m+1-n) F_{i_{n+1}} \cdots F_{i_1} F_0^{m+1-(n+1)} w_0.$$

Where $F_{i_1} \cdots F_{i_{n+1}} F_0^{m-n} w_0 = F_{i_n} \cdots F_{i_1} F_0^k F_{i_{n+1}} F_0^{m-n-k} w_0$, since the F_i's commute. In particular, when $n = m+1$, then $F_{i_1} \cdots F_{i_{m+1}} w_0 = 0$ for all $i_1, \ldots, i_{m+1} \in \mathbb{Z}$. $\square$

Remark 4.1.6. We introduce the following notation regarding partitions. For a partition $\lambda \vdash n$, let $s(\lambda)$ denote the length of λ. Then we may write $\lambda = (\lambda_1, \ldots, \lambda_s)$. Let $\lambda! := \lambda_1! \cdots \lambda_s!$ and $(\lambda - 1)! := (\lambda_1 - 1)! \cdots (\lambda_s - 1)!$. For a partition $\lambda \vdash n$ and an integer $1 \leq i \leq n$, let μ_i be the number of parts equal to i. i.e., $\mu_i = \#\{\lambda_j \,|\, 1 \leq j \leq s \text{ and } \lambda_j = i\}$. Take the multisets $I = \{i_1, \ldots, i_n\}$ and $J = \{j_1, \ldots, j_n\}$, where $i_k, j_l \in \mathbb{Z}_{\geq 0}$. Then for any partition $\lambda \vdash n$, $\lambda = (\lambda_1, \ldots, \lambda_s)$, we have the set of decompositions of $I \cup J$

$$K_{I,J}^{\lambda} := \{K = \{K_1, \ldots, K_s\} \,|\, K_i = I_i \sqcup J_i,\ I_1 \sqcup \cdots \sqcup I_s = I, J_1 \sqcup \cdots \sqcup J_s = J, |I_i| = |J_i| = \lambda_i\}.$$

Notice that

$$|K_{I,J}^{\lambda}| = \frac{\binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2} \cdots \binom{n-\lambda_1-\cdots-\lambda_{s-1}}{\lambda_s}}{\mu_1! \cdots \mu_n!} \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2} \cdots \binom{n-\lambda_1-\cdots-\lambda_{s-1}}{\lambda_s}$$
$$= \frac{n!n!}{\lambda!\lambda!\mu_1! \cdots \mu_n!} .$$

Let

$$a_{I,J}^{\lambda} = a^{\lambda} := \sum_{K \in K_{I,J}^{\lambda}} a_{\sum_{K_1}} \cdots a_{\sum_{K_s}}, \tag{4.4}$$

where the subscript I, J is dropped when is it clear which sets I, J are being used. Hence after fixing sets I, J of size n, for each $\lambda \vdash n$, a^λ is a sum of $\frac{n!n!}{\lambda!\lambda!\mu_1!\cdots\mu_n!}$ terms.

Example 4.1.7. For example, if $I = \{i_1, i_2, i_3\}$, $J = \{j_1, j_2, j_3\}$, and $\lambda = (2, 1)$, then

$$\begin{aligned} a^\lambda_{I,J} &= a_{i_1+i_2+j_1+j_2}a_{i_3+j_3} + a_{i_1+i_2+j_1+j_3}a_{i_3+j_2} + a_{i_1+i_2+j_2+j_3}a_{i_3+j_1} \\ &\quad + a_{i_1+i_3+j_1+j_2}a_{i_2+j_3} + a_{i_1+i_3+j_1+j_3}a_{i_2+j_2} + a_{i_1+i_3+j_2+j_3}a_{i_2+j_1} \\ &\quad + a_{i_2+i_3+j_1+j_2}a_{i_1+j_3} + a_{i_2+i_3+j_1+j_3}a_{i_1+j_2} + a_{i_2+i_3+j_2+j_3}a_{i_1+j_1} \ . \end{aligned}$$

Let $p_k := \sum_i x_i^k$ denote the k^{th} power sum symmetric polynomial.

Let $e_n := \sum_{i_1<i_2<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$ denote the n^{th} elementary symmetric function.

Let M_n be the matrix

$$M_n := \begin{bmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & \cdots & & n-1 \\ p_n & p_{n-1} & \cdots & & p_1 \end{bmatrix} \tag{4.5}$$

and let M'_n be the matrix

$$M'_n := \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & a_{n-2} & \cdots & & n-1 \\ a_n & a_{n-1} & \cdots & & a_1 \end{bmatrix} \tag{4.6}$$

Suppose we restrict the p_k's and e_n's to m variables. Then $e_n = 0$ for all $n \geq m+1$,

and from page 136 of [Gou99] we get the identity $\det M_m = (-1)^m m! e_m$.

Recall the Newton-Girard formulas

$$\begin{aligned} ke_k &= \textstyle\sum_{i=1}^{k} (-1)^{i+1} p_i e_{k-i} \quad \text{if } k \leq m \ , \\ p_k &= \textstyle\sum_{i=1}^{m} (-1)^{i+1} e_i p_{k-i} \quad \text{if } k > m \ . \end{aligned}$$

from Section 10.12 of [Sér12]. The second formula is equivalent to $p_{m+k} = \sum_{i=1}^{m} (-1)^{i+1} e_i p_{m+k-i}$ for $k > 0$, meaning that power sums satisfy a recurrence relation of at least order m. And since $e_0 = 1$, the first formula can be rewritten as $p_k = \sum_{i=1}^{k-1} (-1)^{i+k+1} p_i e_{k-i} + (-1)^{k+1} k e_k$ when $k \leq m$. Hence the minimal homogeneous recurrence relation satisfied by the p_k's is of order m. Since $e_n = \sum_{k=0}^{n} \alpha_k p_k$, for some $\alpha_k \in \mathbb{C}$, setting $0 = \det M_m$ gives a recurrence relation on the p_k's. Proposition 4.1.8 tells us the necessary and sufficient condition such that $\det M'_n = 0$. Hence, if this condition is satisfied, the sequence $\mathbf{a}$ will satisfy the same recurrence relations as the power sums in m variables. In particular, the sequence $\mathbf{a}$ will satisfy a recurrence relation of minimal order m.

Proposition 4.1.8. *Let $n \geq m+1$ and let $I = \{i_1, \ldots, i_n\}, J = \{j_1, \ldots, j_n\}$ be multisets of integers. Then $E_I F_J \cdot w_0 = 0$ if and only if $\det M'_n = 0$.*

To prove Proposition 4.1.8 we first need some lemmas.

Lemma 4.1.9. *For any $n \in \mathbb{Z}_{\geq 0}$, and any multisets of integers $I = \{i_1, \ldots, i_n\}$, $J = \{j_1, \ldots, j_n\}$, we have*

$$E_I F_J w_0 = \sum_{\lambda \vdash n} (-1)^{s(\lambda)+n} \lambda! (\lambda - 1)! a^{\lambda}_{I,J} w_0, \tag{4.7}$$

where $a^{\lambda}_{I,J}$ is as in equation (4.4). In particular,

$$E_{i_1} \cdots E_{i_n} F_0^n w_0 = (*)_n \, (i_1, \ldots, i_n) \ ,$$

where

$$(*)_n (i_1, \ldots, i_n) := \sum_{\lambda \vdash n} (-1)^{s(\lambda)+n} \frac{(\lambda - 1)!}{\mu_1! \cdots \mu_s!} a^\lambda w_0. \tag{4.8}$$

Lemma 4.1.9 along with Proposition 4.1.8 give us information about the sequence **a**.

Proof. Fix $\lambda \vdash n$, $\lambda = (\lambda_1, \ldots, \lambda_s)$. Let a^λ be the coefficient of

$$a := a_{i_1+\cdots+i_{\lambda_1}+j_1+\cdots j_{\lambda_1}} a_{i_{\lambda_1+1}+\cdots+i_{\lambda_1+\lambda_2}+j_{\lambda_1+1}+\cdots+j_{\lambda_1+\lambda_2}} \cdots$$
$$a_{i_{\lambda_1+\cdots+\lambda_{s-1}+1}+\cdots+j_{\lambda_1+\cdots+\lambda_{s-1}+1}+\cdots+j_{\lambda_1+\cdots+\lambda_s}}$$

in the PBW basis order of $E_{i_n} \cdots E_{i_n} F_{j_1} \cdots F_{j_n} w_0$. To get from $E_{i_n} \cdots E_{i_n} F_{j_1} \cdots F_{j_n} w_0$ to $a^\lambda w_0$, the E_i's must be passed over the F_j's. First consider $E_{i_1} F_{j_1} \cdots F_{j_n}$. Since $E_i F_j = F_j E_i + H_{i+j}$ and $H_i F_j = F_j H_i - 2F_{i+j}$, then when $E_{i_1} F_{j_1} \cdots F_{j_n}$ is written in the order of the PBW basis, two things occur:

1. E_i and F_{j_r} are replaced by H_{i+j_r}.

2. E_i, F_{j_r}, and F_{j_t} are merged into $-2F_{i+j_r+j_t}$.

Now to find the factor of the coefficient due to $a_{i_1+\cdots+i_{\lambda_1}+j_1+\cdots j_{\lambda_1}}$, we must apply 2. from above $\lambda_1 - 1$ times. So we get

$$\begin{aligned}\binom{\lambda_1}{2}\binom{\lambda_1 - 1}{2} \cdots \binom{2}{2}(-2)^{\lambda_1 - 1} &= \frac{\lambda_1!}{2!(\lambda_1 - 2)!} \frac{(\lambda_1 - 1)!}{2!(\lambda_1 - 3)!} \cdots \frac{2!}{2!0!}(-2)^{\lambda_1 - 1} \\ &= (-1)^{\lambda_1 - 1} \lambda_1! (\lambda_1 - 1)! \ .\end{aligned}$$

Similarly, to find the factor of the coefficient due to $a_{i_{\lambda_1+1}+\cdots+i_{\lambda_1+\lambda_2}+j_{\lambda_1+1}+\cdots j_{\lambda_1+\lambda_2}}$, we must apply 2. from above $\lambda_2 - 1$ times; and so on. Hence,

$$\begin{aligned} a^\lambda &= (-1)^{\lambda_1-1}\lambda_1!(\lambda_1-1)(-1)^{\lambda_2-1}\lambda_2!(\lambda_2-1)\cdots(-1)^{\lambda_s-1}\lambda_s!(\lambda_s-1) \\ &= (-1)^{n-s}\lambda!(\lambda-1)! = (-1)^{l(\lambda)+n}\lambda!(\lambda-1)! \ . \end{aligned}$$

□

Lemma 4.1.10. *For any $n \in \mathbb{Z}_{\geq 0}$, $i \in \mathbb{Z}$, and multiset of integers $J = \{j_1, \ldots, j_n\}$, we have*

$$H_i F_J w_0 = a_i F_J w_0 - 2\sum_{k=1}^{n} F_{i+j_k} F_{J-\{j_k\}} w_0 \ .$$

Proof. When $n = 1$, then $H_i F_{j_1} w_0 = F_{j_1} H_i w_0 - 2F_{i+j_1} w_0 = a_i F_{j_1} w_0 - 2F_{i+j_1} w_0$. Now assume the equation holds for some $n \geq 1$. Let $J = \{j_1, \ldots, j_{n+1}\}$. Then

$$H_i F_J w_0 = H_i F_{j_{n+1}} F_{J-\{j_{n+1}\}} w_0 = F_{j_{n+1}} H_i F_{J-\{j_{n+1}\}} w_0 - 2F_{i+j_{n+1}} F_{J-\{j_{n+1}\}} w_0 \ .$$

By induction

$$F_{j_{n+1}} H_i F_{J-\{j_{n+1}\}} w_0 = a_i F_J w_0 - 2\sum_{k=1}^{n} F_{i+j_k} F_{J-\{j_k\}} w_0.$$

Hence

$$\begin{aligned} H_i F_J w_0 &= a_i F_J w_0 - 2\sum_{k=1}^{n} F_{i+j_k} F_{J-\{j_{n+1}\}} w_0 - 2F_{i+j_{n+1}} F_{J-\{j_{n+1}\}} w_0 \\ &= a_i F_J w_0 - 2\sum_{k=1}^{n+1} F_{i+j_k} F_{J-\{j_k\}} w_0 \ . \end{aligned}$$

□

Lemma 4.1.11. *For any $n \in \mathbb{Z}_{\geq 2}$, $i \in \mathbb{Z}$, and multiset of integers $J = \{j_1, \ldots, j_n\}$, we have*

$$E_i F_J w_0 = \sum_{r=1}^{n} a_{i+j_r} F_{J-\{j_k\}} w_0 - 2 \sum_{\substack{s,t=1 \\ s<t}}^{n} F_{i+j_s+j_t} F_{J-\{j_s,j_t\}} w_0 \ .$$

In particular, $E_i F_j^n \cdot w_0 = n a_{i+j} F_j^{n-1} \cdot w_0 - 2\binom{n}{2} F_{i+2j} F_j^{n-2} \cdot w_0$.

Proof. When $n = 2$, $E_i F_{j_1} F_{j_2} w = a_{i+j_2} F_{j_1} w_0 + a_{i+j_1} F_{j_2} w - 2F_{i+j_1+j_2} w_0$. Now assume the equation holds for some $n \geq 2$. Let $J = \{j_1, \ldots, j_{n+1}\}$. Then

$$E_i F_J w_0 = F_{j_{n+1}} E_i F_{J-\{j_{n+1}\}} w_0 + H_{i+j_{n+1}} F_{J-\{j_{n+1}\}} w_0 \ .$$

By induction,

$$F_{j_{n+1}} E_i F_{J-\{j_{n+1}\}} w_0 = \sum_{r=1}^{n} a_{i+j_r} F_{J-\{j_r\}} w_0 - 2 \sum_{\substack{s,t=1 \\ s<t}}^{n} F_{i+j_s+j_t} F_{J-\{j_s,j_t\}} w_0,$$

and by Lemma 4.1.11

$$H_{i+j_{n+1}} F_{J-\{j_{n+1}\}} w_0 = a_{i+j_{n+1}} F_{J-\{j_{n+1}\}} w_0 - 2 \sum_{s=1}^{n} F_{i+j_s+j_{n+1}} F_{J-\{j_s,j_{n+1}\}} w_0.$$

Combining these gives the desired result. □

Lemma 4.1.12. *Recall the matrices M and M' defined in (4.5) and (4.6), respectively. Then* $\det M'_n \cdot w_0 = (*)_n(1, \ldots, 1)$, *where* $(*)_n(1, \ldots, 1)$ *is as in (4.8).*

Proof. We proceed by induction. When $n = 1$, clearly $\det M'_1 \cdot w_0 = a_1 \cdot w_0 = (*)_1(1)$.

Now consider

$$\det M'_{n+1} = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ a_{n-1} & a_{n-2} & \cdot & & n-1 & 0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & n \\ a_{n+1} & a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix} .$$

Let $(M'_{n+1})_k$ denote the matrix obtained by deleting the $n+1^{\text{th}}$ row and k^{th} column of M'_{n+1}. Then by expanding along the bottom row, we have

$$\det M'_{n+1} = \sum_{k'=1}^{n+1} (-1)^{n+1+k'} a_{n+2-k'} \det(M'_{n+1})_{k'} \ .$$

Letting $k = k' - 1$, we have

$$\det M'_{n+1} = \sum_{k=0}^{n} (-1)^{n+k} a_{n+1-k} \det(M'_{n+1})_{k+1} = \sum_{k=0}^{n} \frac{a_{n+1-k}\, n!}{k!} \det M'_k \ .$$

Since $0 \leq k \leq n$, then $\det M'_k = (*)_k(1, \ldots, 1)$. So

$$\begin{aligned} \det M'_{n+1} &= \sum_{k=0}^{n} a_{n+1-k} \frac{n!}{k!} \left(\sum_{\lambda \vdash k} (-1)^{s(\lambda)+k} \frac{k!}{\lambda_1 \cdots \lambda_s \mu_1! \cdots \mu_k!} a_{\lambda_1} \cdots a_{\lambda_s} \right) \\ &= \sum_{k=0}^{n} a_{n+1-k} \left(\sum_{\lambda \vdash k} (-1)^{s(\lambda)+k} \frac{n!}{\lambda_1 \cdots \lambda_s \mu_1! \cdots \mu_k!} a_{\lambda_1} \cdots a_{\lambda_s} \right) \ . \end{aligned}$$

The above includes all partitions of $n+1$ because for any fixed k, we get the partitions $\lambda' = (n+1-k, \lambda_1, \ldots, \lambda_s)$,where $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition of k.

Now consider $\lambda \vdash n+1$. With $\lambda = (\lambda_1^{\mu_{\lambda_1}}, \ldots, \lambda_r^{\mu_{\lambda_r}})$ where $\lambda_1, \ldots, \lambda_r$ are all distinct.

Then

$$
\begin{aligned}
&= \sum_{i=1}^{r} a_{\lambda_i} \left(\frac{(-1)^{l(\lambda)+n} n!}{\lambda_1^{\mu_1} \cdots \lambda_i^{\mu_i - 1} \cdots \lambda_r^{\mu_r} \mu_1! \cdots (\mu_i - 1)! \cdots \mu_n!} \right) a_{\lambda_1}^{\mu_{\lambda_1}} \cdots a_{\lambda_i}^{\mu_{\lambda_i}-1} \cdots a_{\lambda_r}^{\mu_{\lambda_r}} \\
&= (-1)^{l(\lambda)+n} \sum_{i=1}^{r} \left(\frac{n! \mu_i \lambda_i}{\lambda_1^{\mu_1} \cdots \lambda_r^{\mu_r} \mu_1! \cdots \mu_n!} \right) a_{\lambda_1}^{\mu_{\lambda_1}} \cdots a_{\lambda_i}^{\mu_{\lambda_i}-1} \cdots a_{\lambda_r}^{\mu_{\lambda_r}} \\
&= (-1)^{l(\lambda)+n} \sum_{i=1}^{r} \left(\frac{(n+1)!}{\lambda_1^{\mu_1} \cdots \lambda_r^{\mu_r} \mu_1! \cdots \mu_{n+1}!} \right) a_{\lambda_1}^{\mu_{\lambda_1}} \cdots a_{\lambda_i}^{\mu_{\lambda_i}-1} \cdots a_{\lambda_r}^{\mu_{\lambda_r}}.
\end{aligned}
$$

So

$$
\begin{aligned}
\det M'_{n+1} w_0 &= \sum_{k=0}^{n} (-1)^{n+1+k} a_{n+1-k} \left(\sum_{\lambda \vdash k} (-1)^{s(\lambda)} \frac{n!}{\lambda_1 \cdots \lambda_t \mu_1! \cdots \mu_k!} a_{\lambda_1} \cdots a_{\lambda_t} \right) w_0 \\
&= \sum_{\lambda \vdash n+1} (-1)^{n+1} (-1)^{l(\lambda)} \frac{(n+1)!}{\lambda_1 \cdots \lambda_s \mu_1! \cdots \mu_{n+1}!} a_{\lambda_1} \cdots a_{\lambda_s} w_0 \\
&= \sum_{\lambda \vdash n+1} (-1)^{l(\lambda)+n+1} \frac{(n+1)!}{\lambda_1 \cdots \lambda_s \mu_1! \cdots \mu_{n+1}!} a_{\lambda_1} \cdots a_{\lambda_s} w_0 \\
&= (*)_{n+1}(1, \ldots, 1).
\end{aligned}
$$

□

Proof of Proposition 4.1.8. The backwards direction is clear. Since we assumed the module defined in 4.1.1 exists, then we have that $F_0^n w_0 = 0$. By Proposition 4.1.5, $F_0^n w_0 = 0$ implies $0 = F_{j_0} F_{j_1} \cdots F_{j_n} \cdot w_0 = E_{i_0} E_{i_1} \cdots E_{i_n} F_{j_0} F_{j_1} \cdots F_{j_n} \cdot w_0$.
Now for the forward direction, assume that for all $n \geq m$, $E_{i_0} E_{i_1} \cdots E_{i_n} F_{j_0} F_{j_1} \cdots F_{j_n} \cdot w_0 = 0$. By Lemma 4.1.9, for all $n \geq m+1$,

$$
0 = E_1^n F_0^n w_0 = \sum_{\lambda \vdash n} (-1)^{s(\lambda)+n} \frac{n!}{\lambda_1 \cdots \lambda_s \mu_1! \cdots \mu_s!} a_{\lambda_1} \cdots a_{\lambda_s} w_0 = (*)_n(1, \ldots, 1) \ .
$$

By Lemma 4.1.12, $(*)_n(1,\dots,1) = \det M'w_0$.

Since $(*)_n(1,\dots,1) = \det M'w_0$, then the a_n's satisfy the same relations as the p_k's. Hence if $\det M'_n \cdot w_0 = 0$, then the sequence $\mathbf{a}$ satisfies a homogeneous linear recurrence relation of order m, and does not satisfy a homogeneous linear recurrence relation of order less than m.

Proposition 4.1.13. *Let $m \in \mathbb{Z}_{\geq 0}$ and consider the sequence $a = \{a_n \,|\, n \in \mathbb{Z}\}$. Then $\det M' = 0$ if and only if $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for some $\alpha_1,\dots,\alpha_m \in \mathbb{C}^*$ and the sequence $\mathbf{a}$ satisfies a homogeneous recurrence relation of minimal order m.*

Proof. Suppose $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for some $\alpha_1,\dots,\alpha_m \in \mathbb{C}^*$ and let W be as in (4.3). Then $a_n = \alpha_1^n + \cdots + \alpha_m^n$ implies that $W(\Lambda, m)$ is a quotient of W, since W satisfies the defining relations of $W(\Lambda, m)$. Hence by Propositions 4.1.5 and 4.1.8, $\det M'_n = 0$. Going in the other direction, suppose $\det M'_n = 0$. Let $k \geq 0$ and let p_k be the power-sum symmetric polynomial of degree k in m variables $\alpha_1,\dots,\alpha_m$. By Lemma 4.1.12 the a_n's satisfy the same relations as the p_k's, which satisfy the homogeneous linear recurrence relation $p_k = \sum_{j=1}^{m}(-1)^{j-1}e_j p_{k-j}$ for $k \geq m$ and do not satisfy a homogeneous recurrence relation of order less than m. This means the minimal characteristic polynomial must be of order m. Let $l \geq 0$. This gives the recurrence relation

$$a_{l+m} = e_1 a_{l+m-1} - e_2 a_{l+m-2} + \cdots + (-1)^{m-1} e_m a_l.$$

The minimal characteristic polynomial of this recurrence relation is

$$\begin{aligned} p(t) &= t^m - e_1 t^{m-1} - \cdots + (-1)^{m-1} e_m \\ &= t^m - (\alpha_1 + \alpha_2 + \cdots + \alpha_m) t^{m-1} + \cdots + (-1)^{m-1} \alpha_1 \alpha_2 \cdots \alpha_m. \end{aligned}$$

Hence, for all $n \geq m+1$, $a_n = \alpha_1^n + \cdots + \alpha_m^n$, where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}^*$ are distinct roots of $p(t)$. $\square$

To summarize, we assumed that the module $W(\Lambda, m)$ exists and using the three propositions above we showed that this implies that $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}^*$. Hence part 1. of Theorem 4.1.3 is proven.
Next, since we have shown that $W(\Lambda, m) = W$, and W is finite dimensional, then $W(\Lambda, m)$ is finite dimensional. However, even if we did not know this, we could still prove finite dimensionality. It can be shown that $F_n w_0 \in \operatorname{span}\{F_0 w_0, F_1 w_0, \ldots, F_{m-1} w_0\}$ for all $n \geq m$ and $W(\Lambda, m) = \operatorname{span}\{F_{i_0} F_{i_1} \cdots F_{i_k} w_0 \,|\, i_0, i_1, \ldots, i_k \in \mathbb{Z}_{\geq 0}\}$. Then we see that $W(\Lambda, m) = \operatorname{span}\{F_{i_0} F_{i_1} \cdots F_{i_k} w_0 \,|\, i_0 \leq i_1 \leq \ldots \leq i_k \leq m-1\}$. Since $W(\Lambda, m)$ has finite support and each weight space is finite dimensional, then $W(\Lambda, m)$ is finite dimensional. Hence Part 2. of Theorem 4.1.3 is proven.

Lemma 4.1.14. *Let $n \geq 2$, and $j_1, \ldots, j_n \in \mathbb{Z}$. Then $E_0^{n-1} F_{j_1} \cdots F_{j_n} \cdot w_0 = \sum_{\lambda \vdash n} c_\lambda a^\lambda F^\lambda \cdot w_0$, where $c_\lambda \in \mathbb{C}$ and in particular $c_{(n)} = (-1)^{n-1} n!(n-1)!$. Furthermore, for a fixed $1 \leq k \leq n$, the coefficient of $a_{j_k} F_{j_1 + \cdots + \widehat{j_k} + \cdots + j_n} \cdot w_0$ is $(-1)^{n-2}(n-1)!(n-1)!$.*

Proof. Consider $E_0^{n-1} F_{j_1} \cdots F_{j_n} \cdot w_0$. By Lemma 4.1.11, $c_{(n)} F^{(n)} \cdot w_0$ is obtained by repeatedly collapsing $F_i F_j$ into $-2F_{i+j}$. This process has to be done $n-1$ times, and each time one must select two elements F_i, F_j. In the first step, we select two elements from $F_{j_1} \cdots F_{j_n}$. There are $\binom{n}{2}$ choices. In the next step, we select two elements from $n-1$ choices, and so on. Naturally, we stop when we only have two elements to choose from. Each time we collapse $F_i F_j$ into $-2F_{i+j}$, a factor of -2 appears. Hence the coefficient belonging to $F_n \cdot w_0 = F_{j_1 + \cdots + j_n} \cdot w_0$ is $(-2)^{n-1} \prod_{k=2}^{n} \binom{k}{2} = (-1)^{n-1} n!(n-1)!$.
For the second assertion, notice that when E_0 acts on a vector of the form $F_{i_1} \cdots F_{i_t} \cdot w_0$,

one of two things occurs. We get either a term in the form of $a_l F_{i_1} \cdots \widehat{F_l} \cdots F_{i_t} \cdot w_0$ where $l \in \{i_1, \ldots, i_t\}$, or we get a term in the form of $-2F_{r+s}F_{i_1} \cdots \widehat{F_r} \cdots \widehat{F_s} \cdots F_{i_t} \cdot w_0$ where $r, s \in \{i_1, \ldots, i_t\}$. To get a vector which is a multiple of $a_{j_k} F_{j_1+\cdots+\widehat{j_k}+\cdots+j_n} \cdot w_0$, we want the first outcome to happen once, and the second outcome to happen $n-2$ times. There is only one way for the first outcome to happen. When the second outcome happens, we merge $F_r F_s$ into F_{r+s}. Since we have a total of $n-1$ $F_i's$ to merge, and each time we must choose two, there are a total of $\prod_{k=2}^{n-1} \binom{k}{2}$ ways for the second outcome to happen. Each time the second outcome happens, we get a factor of -2. Furthermore, there are $n-1$ ways to choose in which order to get step 1 and step 2. Hence the coefficient of $a_{j_k} F_{j_1+\cdots+\widehat{j_k}+\cdots+j_n} \cdot w_0$ is $1 \cdot (-2)^{n-2} \prod_{k=2}^{n-1} \binom{k}{2} (n-1) = (-1)^{n-2}(n-1)!(n-1)!$ □

The following Lemma is used to prove finite dimensionality.

Lemma 4.1.15. *$F_n w_0 \in \operatorname{span}\{F_0 w_0, \ldots, F_{m-1} w_0\}$ for all $n \geq m$.*

Proof. Consider $E_0^m F_0 F_1^m \cdot w_0$. By Lemma 4.1.14,

$$\begin{aligned} 0 \;= E_0^m F_0 F_1^m \cdot w_0 \;&= \textstyle\sum_{i=0}^{m-2} c_i F_i \cdot w_0 + (-1)^m (m) m! m! a_1 F_{m-1} \cdot w_0 \\ &\quad + ((-1)^{m-1} m! m! a_0 + (-1)^m (m+1)! m!) F_m \cdot w_0 \; . \end{aligned}$$

Since $a_0 = m$, by simplifying, we get $(-1)^{m+1} m! m! F_m \cdot w_0 = \sum_{i=0}^{m-1} c_i F_i \cdot w_0$ and so $F_m \cdot w_0 \in \operatorname{span}\{F_0 w_0, \ldots, F_{m-1} w_0\}$. Now let $n \geq m$ and suppose that $F_i \cdot w_0 \in \operatorname{span}\{F_0 w_0, \ldots, F_{m-1} w_0\}$ for all $i \leq n$. Then

$$0 = E_0^m F_0 F_1^{m-1} F_{n+2-m} \cdot w_0 = \sum_{i=0}^{n} c_i F_i \cdot w_0 + (-1)^{m-1} m! m! (a_0 - m - 1) F_{n+1} \cdot w_0.$$

By the inductive hypothesis, $F_i \cdot w_0 \in \operatorname{span}\{F_0 w_0, \ldots, F_{m-1} w_0\}$ for all $0 \leq i \leq n$. Hence $F_{n+1} \cdot w_0 \in \operatorname{span}\{F_0 w_0, \ldots, F_{m-1} w_0\}$. □

Lemma 4.1.16. *Suppose that* $F_0^{m+1} \cdot w_0 = 0$. *Then* $H_n \cdot (F_0^m \cdot w_0) = -a_n F_0^m \cdot w_0$ *for all* $n \in \mathbb{Z}$.

Proof. Since $F_0^{m+1} \cdot w_0 = 0$, then by Lemma 4.1.11,

$$0 = E_n F_0^{m+1} \cdot w_0 = (m+1) a_n F_0^m \cdot w_0 - 2\binom{m+1}{2} F_n F_0^{m-1} \cdot w_0$$

for any $n \in \mathbb{Z}$. i.e., $F_n F_0^{m-1} \cdot w_0 = \frac{1}{m} a_n F_0^m \cdot w_0$.
Then for any $n \in \mathbb{Z}$, by Lemma 4.1.10

$$H_n \cdot (F_0^m \cdot w_0) = a_n F_0^m \cdot w_0 - 2m F_n F_0^{m-1} \cdot w_0 = -a_n F_0^m \cdot w_0 \ .$$

□

Proposition 4.1.17. *The module* $W(\Lambda, m)$ *is finite dimensional. In particular,*

(i) $\dim W(\Lambda, m) = 2^m$.

(ii) $\dim W(\Lambda, m)^{-2k} = \dim W(\Lambda, m)^{-2(m+1-k)}$ *when* $0 \leq k \leq \frac{m}{2}$.

Proof. To prove (i), we know that $W = \bigotimes_{i=1}^m V_i(\alpha_i)$. Since $\dim V_i(\alpha_i) = 2$, (i) follows. Note that this is consistent with results in [Nao12] and [TG15].
To prove (ii), by Lemma 4.1.15, $F_n w_0 \in \operatorname{span}\{F_0 w_0, \ldots, F_{m-1} w_0\}$ for all $n \geq m$ and $W(\Lambda, m)^{-2} = \operatorname{span}\{F_0 w_0, \ldots, F_{m-1} w_0\}$. We show that the vectors $\{F_0 w_0, \ldots, F_{m-1} w_0\}$ are linearly independent. By way of contradiction, suppose not. Then there exist $c_0, \ldots, c_{m-1} \in \mathbb{C}$, not all 0, such that

$$\sum_{i=0}^{m-1} c_i F_i \cdot w_0 = 0. \tag{4.9}$$

Let $n \in \mathbb{Z}$. Acting on (4.9) by E_n, we get

$$\left(\sum_{i=0}^{m-1} c_i a_{n+i}\right) w_0 = 0$$

implying that the sequence $\mathbf{a}$ satisfies a homogeneous recurrence relation of degree less than m, which contradicts Proposition 4.1.13. Hence $\dim W(\Lambda, m)^{-2} = m$ and $\{F_0 w_0, \ldots, F_{m-1} w_0\}$ is a basis for $W(\Lambda, m)^{-2}$.

Now consider $W(\Lambda, m)^{-2k}$ for each $k \geq 1$. This space is made up of vectors of the form $F_{j_1} \cdots F_{j_k} \cdot w_0$ where $j_n \in \mathbb{Z}$. Since each $F_i \cdot w_0$ can be written as a linear combination of $F_0 w_0, \ldots, F_{m-1} w_0$, then $W(\Lambda, m)^{-2k} = \operatorname{span}\{F_{j_1} \cdots F_{j_k} \cdot w_0 \,|\, 0 \leq j_1 \leq \cdots \leq j_k \leq m-1\}$. When $k \geq m+1$, then $\dim W(\Lambda, m)^{-2k} = 0$ by the definition of $W(\Lambda, m)$.

Now consider the vector $v := F_0^m \cdot w_0$. Since $H_n \cdot v = -a_n v$, $E_n^{m+1} \cdot v = 0$, and $F_n \cdot v = 0$ for all $n \in \mathbb{Z}$, then v is a highest weight vector which satisfies relation (4.2) with respect to the opposite Borel subalgebra. So $U(\mathfrak{g} \otimes A) \cdot v$ generates a Weyl algebra isomorphic to $W(\Lambda, m)$. This implies that $\dim W(\Lambda, m)^{-2m} = 1$ and in general, that $\dim W(\Lambda, m)^{-2k} = \dim W(\Lambda, m)^{-2(m+1-k)}$ when $0 \leq k \leq \frac{m}{2}$.

□

Chapter 5

The Lie superalgebra $\mathfrak{sl}(1|1) \otimes A$

5.1 Highest Weight Modules

Let A be $\mathbb{C}[t]$ or $\mathbb{C}[t, t^{-1}]$ and for $g \in \mathfrak{g}$, define $g_n := g \otimes t^n \in \mathfrak{g} \otimes A$. We denote elements of the universal enveloping algebra using capital letters. For example, if $x \in \mathfrak{g}$, then $X \in U(\mathfrak{g})$. In the first section of this chapter we study Weyl modules over $\mathfrak{sl}(1|1) \otimes A$. In the second section, we study infinite-dimensional $\mathbb{Z} \times \mathbb{Z}$-graded $\mathfrak{sl}(1|1) \otimes A$-modules which satisfy certain finiteness conditions.

5.1.1 The algebra $\mathfrak{sl}(1|1) \otimes A$

When studying Weyl modules of $\mathfrak{sl}_2 \otimes A$, we started with a cyclic highest weight module M and then imposed a finite dimensionality condition. A cyclic highest weight module of $\mathfrak{sl}_2 \otimes A$ is generated by a vector w_0 such that $E_n \cdot w_0 = 0$ and $H_n \cdot w_0 = a_n w_0$ for some $a_n \in \mathbb{C}$. Since $w_0, F_0 w_0, F_0^2 w_0, \ldots$ are in distinct weight spaces, they are linearly independent if they are non-zero. Hence the condition that M is finite dimensional

implies that $F_0^m w_0 = 0$ for some $m \in \mathbb{Z}_{\geq 0}$. This gives us the definition of the Weyl module over $\mathfrak{sl}_2 \otimes A$. What is the analogue of this definition if we study Weyl modules over $\mathfrak{sl}(1|1) \otimes A$? A basis of $\mathfrak{sl}(1|1)$ is $\{x, y, z\}$. We cannot require the condition that $Y_0^{m+1} w_0 = 0$ for some $m \in \mathbb{Z}$. Since y has odd parity, $0 = [Y_0, Y_0] = 2Y_0^2$, i.e., $Y_0^2 \cdot w_0 = 0$ is automatic. So if we imposed the condition that $Y_0^{m+1} w_0 = 0$, it would not give us any new structure on the module. The next natural step is to try the condition $Y_{i_0} Y_{i_1} \cdots Y_{i_m} \cdot w_0 = 0$ for all $y_{i_0}, y_{i_1}, \ldots, y_{i_m} \in \mathfrak{n}^- \otimes A$.

Recall that $\mathfrak{sl}(1|1) \subset \mathfrak{gl}(1|1)$. Since $z \in Z(\mathfrak{sl}(1|1))$, we may need to go to the larger algebra $\mathfrak{gl}(1|1)$ to prove some results. Let $h' = E_{11}, h'' = E_{22}$ be a basis of the Cartan subalgebra of $\mathfrak{gl}(1|1)$, so that $z = h' + h'' \in \mathfrak{sl}(1|1)$. Let $a_n, b_n \in \mathbb{C}$ be defined by $h'_n \cdot w_0 = a_n w_0$ and $h''_n \cdot w_0 = b_n w_0$, so that $c_n = a_n + b_n$. Then for each $k \geq 0$, we can compute $H'_0 Y_{i_0} Y_{i_1} \cdots Y_{i_m} \cdot w_0 = (a_0 - k) Y_{i_0} Y_{i_1} \cdots Y_{i_k} \cdot w_0$ and $H''_0 Y_{i_0} Y_{i_1} \cdots Y_{i_k} \cdot w_0 = (b_0 + k) Y_{i_0} Y_{i_1} \cdots Y_{i_k} \cdot w_0$. Adding these together, we get $Z Y_{i_0} Y_{i_1} \cdots Y_{i_k} \cdot w_0 = (a_0 + b_0) Y_{i_0} Y_{i_1} \cdots Y_{i_k} \cdot w_0$. So for each $k \geq 0$, we see that $Y_{i_0} Y_{i_1} \cdots Y_{i_k} \cdot w_0$ is in a distinct weight space. Hence the condition that there exists an $m \geq 0$ such that $Y_{i_0} Y_{i_1} \cdots Y_{i_m} \cdot w_0 = 0$ for all $y_{i_0}, y_{i_1}, \ldots, y_{i_m} \in \mathfrak{n}^- \otimes A$ is equivalent to the support of the Weyl module being finite. Since finite dimensionality implies finite support, we must have this condition on the y_i's as part of the definition of the Weyl module. However, to make the definition more simple, we will start with the condition of $Y_0 Y_1 \cdots Y_m \cdot w_0 = 0$ for some $m \geq 0$ and show that this implies that $Y_{i_0} Y_{i_1} \cdots Y_{i_m} \cdot w_0 = 0$ for all $y_{i_0}, y_{i_1}, \ldots, y_{i_m} \in \mathfrak{n}^- \otimes A$.

The second half of this Chapter dicusses $\mathbb{Z} \times \mathbb{Z}$-graded $\mathfrak{sl}(1|1) \otimes \mathbb{C}[t, t^{-1}]$ modules.

Recall the definition of the Verma module $M(\Lambda)$ from Definition 3.1.1.

Definition 5.1.1. Let $m \in_{>0}$ and let $\mathbf{c} := (c_n)_{n \in \mathbb{Z}}$, where $c_n \in \mathbb{C}$. Let $\Lambda \in (\mathfrak{h} \otimes A)^*$

be such that $\Lambda(z \otimes t^n) = c_n$. Define $I(\Lambda, m) \subset M(\Lambda)$ to be the submodule generated by $Y_0 Y_1 \cdots Y_m \cdot w_0$. Then

$$W(\Lambda, m) := M(\Lambda)/I(\Lambda, m)\,. \tag{5.1}$$

$W(\Lambda, m)$ is called a Weyl module of $\mathfrak{sl}(1\,|\,1) \otimes A$.

More informally, $W(\Lambda, m)$ is the highest weight $\mathfrak{sl}(1|1) \otimes A$-module generated by w_0 and defined by the relations

$$\begin{aligned} (\mathfrak{n}^+ \otimes A) \cdot w_0 &= 0, \\ H_n \cdot w_0 &= \Lambda(h_n) w_0 \quad \forall\, h_n \in \mathfrak{h} \otimes A, \\ Y_0 Y_1 \cdots Y_m \cdot w_0 &= 0\ . \end{aligned} \tag{5.2}$$

Note that here we abuse notation by using w_0 as the generating vector of both $M(\Lambda)$ and $W(\Lambda, m)$.

Theorem 5.1.2. *Let $m \in \mathbb{Z}_{>0}$ and let $W(\Lambda, m)$ be as in Definition 5.1.1. Then*

1. *$W(\Lambda, m) \neq 0$ and only if $c_{k+m} = \alpha_1 c_{k+m-1} + \alpha_2 c_{k+m-2} + \cdots + \alpha_m c_k$ for all $k \in \mathbb{Z}_{>0}$ and for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$.*

2. *If $W(\Lambda, m) \neq 0$ then $\dim W(\Lambda, m)^{-k} = \binom{m}{k}$ for $0 \leq k \leq m$.*

3. *If $W'(\Lambda, m)$ is a $\mathfrak{g} \otimes A$-module generated by a nonzero vector w which satisfies the relations (5.2), then there exists a $\mathfrak{g} \otimes A$-submodule $V \subseteq W(\Lambda, m)$ such that $W'(\Lambda, m) \cong W(\Lambda, m)/V$.*

Remark 5.1.3. Since the roots of the characteristic polynomial of the recurrence relation $c_{k+m} = \alpha_1 c_{k+m-1} + \alpha_2 c_{k+m-2} + \cdots + \alpha_m c_k$ are not necessarily distinct, then

$W(\Lambda, m)$ is not necessarily a tensor product of evaluation modules. Hence, the modules $W(\Lambda, m)$ from Definition 4.1.1 are new. Their analogues for $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathrm{d}$ were studied in [Wil08], however they are not finite dimensional.

To prove the theorem, we use four key propositions, which are listed below.

Proposition 5.1.4. *Let $m \geq 0$. Then $Y_0 Y_1 \cdots Y_m \cdot w_0 = 0$ if and only if $Y_I \cdot w_0 = 0$ for any $I \in \mathbb{Z}^{m+1}$.*

Let M be the infinite matrix $M := [c_{i+j}]_{i,j \in \mathbb{Z}}$ and for $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, let $M_{I,J}$ be the matrix $M_{I,J} := \left[c_{i_s + j_l}\right]_{s,l=1,\ldots,n}$. Note that $\det M_{I,J}$ is an $n \times n$ minor of M.

Proposition 5.1.5. *Denote $z := h' + h''$ so that $Z_k \cdot w_0 = c_k w_0$. Then for $n \geq 1$ and for all $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$,*

$$X_I Y_J w_0 = (-1)^{\frac{n(n-1)}{2}} \det M_{I,J} w_0 \ .$$

In particular, $X_I Y_J w_0 = 0$ if and only if $\det M_{I,J} = 0$.

By the construction of $W(\Lambda, m)$ we assume that $Y_I \cdot w_0 = 0$ for every $I \in \mathbb{Z}^{m+1}$. Hence Proposition 5.1.5 tells us that every $(m+1) \times (m+1)$ minor of M is 0, meaning that the rank of M is at most m.

Proposition 5.1.6. *Let $m \in \mathbb{Z}_{\geq 0}$ and consider the sequence $\mathbf{c} = (c_n)_{n \in \mathbb{Z}}$. Then $\operatorname{rank} M \leq m$ if and only if the sequence $\mathbf{c}$ satisfies a linear recurrence relation with constant coefficients. In particular, the recurrence relation of least degree satisfied by the sequence $\mathbf{c}$ has minimal polynomial $p(t)$ where $\deg p(t) = \operatorname{rank} M$.*

Proposition 5.1.7. *The module $W(\Lambda, m)$ has finite dimension. In particular,* $\dim W(\Lambda, m) = 2^m$ *and* $\dim W(\Lambda, m)^k = \binom{m}{k}$ *for* $0 \leq k \leq m$.

For each of these propositions, there is a subsection further on in this chapter dedicated to the proof. The idea of the proof of Theorem 5.1.2 is as follows. If $W(\Lambda, m)$ is finite-dimensional, then it must have a finite support. To have finite support, there must exist a $k \in \mathbb{Z}_{>0}$ such that $W(\Lambda, m)^{\pm l} = 0$ for all $l \geq k$. Proposition 5.1.4 states that if the assumptions of Theorem 5.1.2 hold, then $W(\Lambda, m)^{\pm l} = 0$ for all $l \geq m$. Hence, $W(\Lambda, m)$ has finite support. Furthermore, Proposition 5.1.4 and 5.1.5 imply that $\operatorname{rank} M \leq m$. By Proposition 5.1.6, $\operatorname{rank} M = m$ is equivalent to the existence of a degree rank$M$ recursive relation with constant coefficients on the sequence $\mathbf{c} = \{c_i\}_{i \in \mathbb{Z}}$. Then we use Proposition 5.1.4 to show that each weight space of $W(\Lambda, m)$ is finite-dimensional. Since each weight space is finite-dimensional, and $W(\Lambda, m)$ has finite support, then $W(\Lambda, m)$ is finite dimensional. In short, we have the following implications:

$$\begin{array}{ccccccc} \text{Conditions (5.2)} & \Rightarrow & \text{Prop. 5.1.4} & \Rightarrow & \text{Prop. 5.1.5} & \Leftrightarrow & \text{Prop. 5.1.6} \\ & & \Downarrow & & & & \Downarrow \\ & & \text{Prop. 5.1.7} & & & & \text{Thm. 5.1.2.1} \\ & & \Downarrow & & & & \\ & & \text{Thm. 5.1.2.2} & & & & \end{array}$$

Each of these propositions will be proven in the subsections which follow.

Proof of Proposition 5.1.4

The proof of Proposition 5.1.4 requires a total order on the elements of $U(\mathfrak{sl}(1 \mid 1) \otimes A)$. To define such an order, we need the notion of the height of an element of $U(\mathfrak{sl}(1 \mid 1) \otimes A)$.

Definition 5.1.8. Consider $Y_I := Y_{i_0} \cdots Y_{i_m} \in U(\mathfrak{sl}(1 \mid 1) \otimes A)$, where $I = (i_0, \ldots, i_m) \in \mathbb{Z}^{m+1}$. Define the height of Y_I as $\mathrm{ht}(Y_I) = \sum_{k=0}^{m}(i_k - k)$.

Definition 5.1.9. For a fixed $m \in \mathbb{Z}_{>0}$ we define an order on the set $\{Y_I \mid I \in \mathbb{Z}^{m+1}\}$ as follows. Let $J = (j_0, \ldots, j_m), K = (k_0, \ldots, k_m) \in \mathbb{Z}^{m+1}$. We say $Y_J > Y_K$ if $\mathrm{ht}(Y_J) > \mathrm{ht}(Y_K)$ or if $\mathrm{ht}(Y_J) = \mathrm{ht}(Y_K)$ and there is an $s \geq 0$ such that $j_s > k_s$ and $j_t = k_t$ for all $s + 1 \leq t \leq m$.

Example 5.1.10. Let $J = (j_0, j_1, j_2, j_3, j_4, j_5, j_6, j_7) = (0, 1, 2, 3, 7, 8, 11)$ and $K = (k_0, k_1, k_2, k_3, k_4, k_5, k_6, k_7) = (0, 1, 2, 4, 6, 8, 11)$. Then $\mathrm{ht}(J) = \mathrm{ht}(K) = 11$ and $j_5 > k_5$, $j_6 = k_7$, and $j_7 = k_7$. So $Y_J > Y_K$.

We set up some notation using symmetric functions, which we use in the proof of Proposition 5.1.4. Let $e_k(z_1, \ldots, z_n)$ be the k^{th} elementary symmetric polynomial in variables $z_1, \ldots, z_n$, where $e_0 = 1$, and let $p_k(z_1, \ldots, z_n)$ be the k^{th} power-sum symmetric polynomial in variables $z_1, \ldots, z_n$. By Newton's formula from page 24 of [Mac98], we have $ne_n = \sum_{r=1}^{n}(-1)^{r-1}p_r e_{n-r}$ when $n \in \mathbb{Z}_{\geq 0}$. Recall that $H''_i = H'' \otimes t^i$, where $h'' = E_{22} \in \mathfrak{gl}(1|1)$. For $i, n \in \mathbb{Z}_{\geq 0}$ such that $i \geq n$ we define recursively in a similar way $Q_n(H''_1, \ldots, H''_i)$ by setting $Q_0(H''_1, \ldots, H''_i) = 1$ and

$$Q_n(H''_1, \ldots, H''_i) = \frac{1}{n}\sum_{r=1}^{n}(-1)^{r-1}H''_r Q_{n-r}(H''_1, \ldots, H''_i).$$

Proposition 5.1.11. *Let $m \geq 0$. Then $Y_0 Y_1 \cdots Y_m \cdot w_0 = 0$ if and only if $Y_I \cdot w_0 = 0$ for any $I \in \mathbb{Z}^{m+1}$.*

Proof. We split the proof into two steps, when $A = \mathbb{C}[t]$ and when $A = \mathbb{C}[t, t^{-1}]$. First we prove that $Y_0 Y_1 \cdots Y_m \cdot w_0 = 0$ implies $Y_I \cdot w_0 = 0$ for any $I \in \mathbb{Z}_{\geq 0}^{m+1}$, i.e., when $A = \mathbb{C}[t]$. Then we extend this to when $A = \mathbb{C}[t, t^{-1}]$ by proving that $Y_0 Y_1 \cdots Y_m \cdot w_0 = 0$ implies $Y_I \cdot w_0 = 0$ for any $I \in \mathbb{Z}^{m+1}$. The proofs require the use of two technical lemmas: Lemma 5.1.17 and Lemma 5.1.18. These lemmas and their proofs are found after the proof of this proposition.

Step 1: The set $\mathcal{S}_1 = \{Y_{j_0} Y_{j_1} \cdots Y_{j_m} \in U(\mathfrak{g} \otimes \mathbb{C}[t])\}$ is well-ordered, so we proceed by induction on the ordering. Since we assume that $Y_0 Y_1 \cdots Y_m \cdot w_0 = 0$, then acting by H_1'' results in

$$0 = Y_0 Y_1 \cdots Y_m w_0 = H_1'' Y_0 Y_1 \cdots Y_m w_0 = Y_0 Y_1 \cdots Y_{m+1} w_0 \ .$$

Since $\mathrm{ht}(Y_0 Y_1 \cdots Y_m) = 0$ and the only non-zero element of height 1 is $Y_0 Y_1 \cdots Y_{m+1}$, the base case holds.

Consider $Y_J := Y_{j_0} Y_{j_1} \cdots Y_{j_m} \in U(\mathfrak{sl}(1|1) \otimes \mathbb{C}[t])$. Let $s \geq 1$ be the least number such that $t := j_{m-s+1} - j_{m-s} > 1$. i.e., $Y_J := Y_{j_0} \cdots Y_{j_{m-s}} Y_{j_{m-s}+t} Y_{j_{m-s}+t+1} \cdots Y_{j_{m-s}+t+s-1}$. Suppose that $Y_K \cdot w_0 = 0$ for all $K \in \mathbb{Z}^{m+1}$ such that $Y_K < Y_J$. Consider

$$Y_{J'} = Y_{j_0} \cdots Y_{j_{m-s}} Y_{j_{m-s}+t-1} Y_{j_{m-s}+t} \cdots Y_{j_{m-s}+t+s-2} \ .$$

Then $\mathrm{ht}(Y_{J'}) = \mathrm{ht}(Y_J) - s$, so $Y_{J'} < Y_J$. By the inductive hypothesis $Y_{J'} \cdot w_0 = 0$. By Lemma 5.1.17, choosing $n = s$, we have

$$\begin{aligned} 0 &= Q_s(H_1'', \ldots, H_n'') Y_{j_{m-s}} Y_{j_{m-s}+t-1} Y_{j_{m-s}+t} \cdots Y_{j_{m-s}+t+s-2} \cdot w_0 \\ &= Y_{j_{m-s}} Y_{j_{m-s}+t} Y_{j_{m-s}+t+1} \cdots Y_{j_{m-s}+t+s-1} \cdot w_0 = Y_J \cdot w_0 \ . \end{aligned}$$

Step 2: Now we consider the case when $A = \mathbb{C}[t, t^{-1}]$. Case one showed that $Y_0 Y_1 \cdots Y_m w_0 = 0$ if and only if $Y_{i_0} Y_{i_1} \cdots Y_{i_m} w_0 = 0$ for all $i_0, \ldots, i_m \in \mathbb{Z}_{\geq 0}$. The set $\mathcal{S}_2 = \{Y_{j_0} Y_{j_1} \cdots Y_{j_m} \in U(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \,|\, Y_{j_0} Y_{j_1} \cdots Y_{j_m} < Y_0 Y_1 \cdots Y_m\}$ is well-ordered, so it suffices to show that $Y_0 Y_1 \cdots Y_m w = 0$ implies that $Y_{j_0} Y_{j_1} \cdots Y_{j_m} w_0 = 0$ for all $J = (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1}$ such that $Y_J < Y_0 Y_1 \cdots Y_m$. We will use induction on the ordering of $\mathcal{S}_2$. The only non-zero element with a height of -1 is $Y_{-1} Y_1 \cdots Y_m$, so to show that the base case holds, we show that $Y_{-1} Y_1 \cdots Y_m \cdot w_0 = 0$. Indeed,

$$0 = Y_0 Y_1 \cdots Y_m w_0 = H''_{-1} Y_0 Y_1 \cdots Y_m w_0 = Y_{-1} Y_1 \cdots Y_m w_0 \ .$$

Next consider $Y_J := Y_{j_0} Y_{j_1} \cdots Y_{j_m} w_0 \in U(\mathfrak{sl}(1 \,|\, 1) \otimes \mathbb{C}[t, t^{-1}])$. Let $s \geq 1$ be the least number such that $t := j_{m-s+1} - j_{m-s} > 1$.
i.e., $Y_J := Y_{j_0} \cdots Y_{j_{m-s-1}} Y_{j_{m-s}} Y_{j_{m-s}+t} Y_{j_{m-s}+t+1} \cdots Y_{j_{m-s}+t+s}$. Suppose that $Y_K \cdot w = 0$, where $Y_K := Y_{k_0} \cdots Y_{k_m}$, for all $Y_K > Y_J$. Consider

$$Y_{J'} := Y_{j_0} \cdots Y_{j_{m-s-1}} Y_{j_{m-s}} Y_{j_{m-s}+t+1} Y_{j_{m-s}+t+2} \cdots Y_{j_{m-s}+t+s} \cdot w_0 \ .$$

Since $\mathrm{ht}(Y_{J'}) = \mathrm{ht}(Y_J) + s$, then $Y_{J'} > Y_J$. By the inductive hypothesis $Y_{J'} \cdot w_0 = 0$. By Lemma 5.1.18, choosing $n = s$, we have

$$\begin{aligned} 0 &= Q_{-s}(H''_{-1}, \ldots, H''_{-s}) Y_{J'} \cdot w_0 \\ &= Y_{j_0} \cdots Y_{j_{s-1}} Y_{j_s} Y_{j_s+t} Y_{j_s+t+1} \cdots Y_{j_s+t+s-1} \cdot w_0 = Y_J \cdot w_0 \ . \end{aligned}$$

□

Two Technical Lemmas

In this subsection, we introduce technical notation to aid us in proving the Lemmas 5.1.17 and 5.1.18. This notation will not be used in other parts of the thesis. We will read from right to left in this subsection. For example, for $I \in \mathbb{Z}^{m+1}$, let $I[k]$ denote the k^{th} element of the tuple I when counting from the right. i.e., if $I = (i_1, \ldots, i_m)$, then $I[k] = i_{m+1-k}$.

For a sequence J, define $\tau_k(J)$ to be the sequence in which the 1 is added to the k^{th} entry of J when counting from the right (assuming J has at least k entries). i.e.,

$$\tau_k(J)[i] = \begin{cases} J[i] + 1, & i = k \\ J[i], & \text{otherwise} \end{cases}.$$

It follows that $\tau_k^{-1}(J)$ is the sequence in which 1 is subtracted from the k^{th} entry of J when counting from the right. Define $\sigma_k := \tau_k \tau_{k-1} \cdots \tau_1$. Define $\pi_{k,l}$ to be the $(l+1)$-cycle $\pi_{k,l} := (k-l \;\; k-l+1 \cdots k) \in S_{m+1}$ provided $l < k$. Then $\pi_{k,l}(J)$ is the $(m+1)$-tuple obtained by shifting the k^{th} entry of J (from the right) to the right by l places, while $\pi_{k,l}^{-1}(J)$ is the $(m+1)$-tuple obtained by shifting the $(k-l)^{\text{th}}$ entry of J (from the right) to the left by l places. We will use $Y_{\phi(J)}$ and $\phi(Y_J)$ interchangeably for $\phi \in \{\sigma, \tau\}$. Since $Y_i Y_j = -Y_j Y_i$, then $Y_{\pi_{k,l}(J)} = \text{sgn}(\pi_{k,l})(Y_J)$. Here are some examples to show how the operators act.

Example 5.1.12. Let $J = (j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8, j_9) \in \mathbb{Z}^9$. Since we read from the right, then $J[k] = j_{10-k}$ for all $1 \leq k \leq 9$. For example, $J[6] = j_4$. This example illustrates how the operators τ and σ act on J. By definition, τ_6 adds 1 to the sixth coordinate of J when reading from the right. i.e., it adds 1 to j_4, so

$\tau_6(J) = (j_1, j_2, j_3, j_4 + 1, j_5, j_6, j_7, j_8, j_9)$. The operator τ_6^2 adds 1 twice to $J[6]$. i.e., it adds 2 to j_4, so $\tau_6^2(J) = (j_1, j_2, j_3, j_4 + 2, j_5, j_6, j_7, j_8, j_9)$. The operator σ_6 adds 1 to the first six coordinates of J, when reading from the right. This is written as $\sigma_6(J) = \tau_1\tau_2\tau_3\tau_4\tau_5\tau_6(J) = (j_1, j_2, j_3, j_4 + 1, j_5 + 1, j_6 + 1, j_7 + 1, j_8 + 1, j_9 + 1)$. Recalling that we read from the right, the 4-cycle $\pi_{6,3} = (3\,4\,5\,6)$ moves j_4 to the right by 3 places, which forces j_5, j_6, and j_7 to move left by one place. Explicitly, $\pi_{6,3}(J) = (j_1, j_2, j_3, j_5, j_6, j_7, j_4, j_8, j_9)$.

Example 5.1.13. This example illustrates how the operators act on a tuple J that is in a special form. Let $J = (j_1, j_2, j_3, j_3 + t - 1, j_3 + t, j_3 + t + 1, j_3 + t + 2, j_3 + t + 3, j_3 + t + 4) \in \mathbb{Z}^9$. Then when reading from the right, the first 6 coordinates of J differ by 1. The 4-cycle $\pi_{6,3} = (3\,4\,5\,6)$ permutes the coordinates as follows: $\pi_{6,3}(J) = (j_1, j_2, j_3, j_3 + t, j_3 + t + 1, j_3 + t + 2, j_3 + t - 1, j_3 + t + 3, j_3 + t + 4)$. Recall the notation $J[3] = j_3 + t + 2$, $J[4] = j_3 + t + 1$, $J[5] = j_3 + t$, and $J[6] = j_3 + t - 1$. Then, since we read from the right, the tuple $\pi_{6,3}(J)$ can also be interpreted as taking J, subtracting 3 from the third coordinate, and adding 1 to the fourth, fifth, and sixth coordinates. i.e., $\pi_{6,3}(J) = \tau_6\tau_5\tau_4\tau_3^{-3}(J)$. Since $Y_iY_j = -Y_jY_i$, then $\pi_{6,3}(Y_J) = (-1)^3\tau_6\tau_5\tau_4\tau_3^{-3}(Y_J) = \operatorname{sgn}(\pi_{6,3})\tau_6\tau_5\tau_4\tau_3^{-3}(Y_J)$.

Below are two examples illustrating how these operators act on J when combined and when J is in a special form. In certain cases we may rewrite π using τ and σ. For example, let $k = 6, n = 5$, and $r = 3$. Then σ_3 adds 1 to the first three coordinates when reading from the right, then τ_6^3 adds 3 to the 6th coordinate when reading from the right, and $\pi_{6,5}$ shifts the 6th entry (from the right) to the right by 5 places.

$$\begin{aligned}
&\pi_{6,5}\tau_6^3\sigma_3(J) \qquad (5.3)\\
&= \pi_{6,5}\tau_6^3\sigma_3(j_1, j_2, j_3, j_3+t-1, j_3+t, j_3+t+1, j_3+t+2, j_3+t+3, j_3+t+4)\\
&= \pi_{6,5}\tau_6^3(j_1, j_2, j_3, j_3+t-1, j_3+t, j_3+t+1, j_3+t+3, j_3+t+4, j_3+t+5)\\
&= \pi_{6,5}(j_1, j_2, j_3, j_3+t+2, j_3+t, j_3+t+1, j_3+t+3, j_3+t+4, j_3+t+5)\\
&= (j_1, j_2, j_3, j_3+t, j_3+t+1, j_3+t+3, j_3+t+4, j_3+t+5, j_3+t+2)\\
&= \tau_1^{-4}\sigma_3\sigma_6(J).
\end{aligned}$$

For an example of remark 3 below, let $k = 2, n = 5$, and $r = 3$. Then

$$\begin{aligned}
&\pi_{2,1}\tau_2^3\sigma_3(J)\\
&= \pi_{2,1}\tau_2^3\sigma_3(j_1, j_2, j_3, j_3+t-1, j_3+t, j_3+t+1, j_3+t+2, j_3+t+3, j_3+t+4)\\
&= \pi_{2,1}\tau_2^3(j_1, j_2, j_3, j_3+t-1, j_3+t, j_3+t+1, j_3+t+3, j_3+t+4, j_3+t+5)\\
&= \pi_{2,1}(j_1, j_2, j_3, j_3+t-1, j_3+t, j_3+t+1, j_3+t+3, j_3+t+7, j_3+t+5)\\
&= (j_1, j_2, j_3, j_3+t-1, j_3+t, j_3+t+1, j_3+t+3, j_3+t+5, j_3+t+7)\\
&= \tau_1^1\sigma_3\sigma_2(J)
\end{aligned}$$

Remark 5.1.14. Let $J = (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1}$ be such that $j_i < j_{i+1}$ for all $0 \leq i \leq m-1$, and such that there exists an $s \geq 1$ and $t \geq 2$ such that

$$J[i] = \begin{cases} j_{m-s} + t + s - i - 1, & 1 \leq i \leq s \\ j_{m+1-i}, & s+1 \leq i \leq m+1 \end{cases}.$$

Here are some observations about $\pi_{k,l}(J)$ when $k \leq s$, recalling that $J[i+1] = J[i] - 1$ for $1 \leq i \leq s-1$:

1. $\pi_{k,l}$ shifts $J[k]$ to the right by l places and $J[k-l], \ldots, J[k-1]$ to the left by one place. Write $J' = \pi_{k,l}(J)$. Then $J'[k-l] = J[k] = j_{m-s} + t + s - k - 1 = J[k-l] - l = \tau_{k-l}^{-l}(J[k-l])$. And for $0 \leq i \leq l-1$, $J'[k-i] = J[k-i-1] = j_{m-s} + t + s - (k-i-1) - 1 = J[k-i] + 1 = \tau_{k-i}(J[k-i])$. Hence $\pi_{k,l}(Y_J) = (-1)^l \tau_k \cdots \tau_{k-l+1} \tau_{k-l}^{-l}(Y_J) = \operatorname{sgn}(\pi_{k,l}) \tau_k \cdots \tau_{k-l+1} \tau_{k-l}^{-l}(Y_J)$. This is illustrated in equation (5.3) of Example 5.1.13, as $\pi_{6,3}(Y_J) = \operatorname{sgn}(\pi_{6,3}) \tau_6 \tau_5 \tau_4 \tau_3^{-3}(Y_J)$.

2. It follows that if $k \geq n + 2 - r$, then $\pi_{k,k-1} \tau_k^r \sigma_{n+1-r}(J)$ adds 1 to $J[1], \ldots, J[n+1-r]$, then adds r to $J[k]$, and then shifts $J[k]$ over to the right by $k-1$ places, putting it in the first position. This in turn shifts $J[1], \ldots, J[k-1]$ over to the left by 1 place each. So $J'[1] = J[k] + r = j_{m-s} + t + s - k - 1 + r = J[1] + r - k + 1 = \tau_1^{r-k+1}(J[1])$. For $2 \leq i \leq n + 2 - r$, $J'[i] = J[i-1] + 1 = j_{m-s} + t + s - (i-1) - 1 + 1 = J[i] + 2 = \tau_i^2(J[i])$. And for $n+3-r \leq i \leq k$, $J'[i] = J[i-1] = J[i]+1 = \tau_i(J[i])$. Hence $\pi_{k,k-1} \tau_k^r \sigma_{n+1-r}(Y_J) = (-1)^{k-1} \tau_1^{r-k-1} \sigma_{n+2-r} \sigma_k(Y_J)$. This is illustrated in Example 5.1.13.

3. If $k \leq n + 1 - r \leq s$, then $\pi_{k,k-1} \tau_k^r \sigma_{n+1-r}(J)$ adds 1 to $J[1], \ldots, J[n+1-r]$ (including to $J[k]$), then adds r to $J[k]$, and then shifts $J[k]$ over to the right by $k-1$ places, putting it in the first position. This in turn shifts $J[1], \ldots, J[k-1]$ over to the left by 1 place each. So $J'[1] = J[k] + r + 1 = j_{m-s} + t + s - k - 1 + r + 1 = J[1] + r - k + 2 = \tau_i^{r-k+2}(J[1])$. For $2 \leq i \leq k$, $J'[i] = J[i-1] + 1 = j_{m-s} + t + s - (i-1) - 1 + 1 = J[i] + 2 = \tau_i^2(J[i])$. And for $k+1 \leq i \leq n+1-r$, $J'[i] = J[i-1] = J[i] + 1 = \tau_i(J[i])$. Hence $\pi_{k,k-1} \tau_k^r \sigma_{n+1-r}(Y_J) = (-1)^{k-1} \tau_1^{r-k} \sigma_{n+1-r} \sigma_k$.

4. In particular, $\pi_{k,k-1}\tau_k^{n+1}(Y_J) = (-1)^{k-1}\tau_1^{n+1-k}\sigma_k(Y_J)$.

5. It also follows that $\pi_{n+1,r-1}\tau_{n+1}^r\sigma_{n+1-r}$ adds 1 to $J[1], \ldots, J[n+1-r]$, then adds r to $J[n+1]$, and then shifts $J[n+1]$ over to the right by $r-1$ places (i.e., to the $(n+2-r)^{\text{th}}$ place.) This in turn shifts $J[n+2-r], \ldots, J[n]$ over to the left by 1 each. So $J'[n+2-r] = J[n+1] + r = J[n+2-r] + 1$. For $1 \leq i \leq n+1-r$, $J'[i] = J[i] + 1$, and for $n+3-r \leq i \leq n+1$, $J'[i] = J[i-1] = J[i] + 1$. Hence $\pi_{n+1,r-1}\tau_{n+1}^r\sigma_{n+1-r}Y_J = (-1)^{r-1}\sigma_{n+1}Y_J$.

Remark 5.1.15. Let $J = (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1}$ be such that $j_i < j_{i+1}$ for all $0 \leq i \leq m-1$, and such that there exists an $1 \leq s \leq m$ and $t \geq 2$ such that

$$J[i] = \begin{cases} j_{m-s} + t + s - i + 1, & 1 \leq i \leq s \\ j_{m+1-i}, & s+1 \leq i \leq m+1 \end{cases}.$$

Recall that $\pi_{k,l} := (k-l \;\; k-l+1 \cdots k) \in S_{m+1}$ and $\pi_{k_l}(J)$ is the $(m+1)$-tuple obtained by shifting the k^{th} entry of J (from the right) to the right by l places.

1. Recall from Remark 5.1.14 that $\pi_{k,l}(Y_J) = (-1)^l\tau_k \cdots \tau_{k-l+1}\tau_{k-l}^{-l}(Y_J) = \text{sgn}(\pi_{k,l})\tau_k \cdots \tau_{k-l+1}\tau_{k-l}^{-l}(Y_J)$. It follows that if $k \leq s-n+r-1$, then $\pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}$ subtracts 1 from $J[s-n+r], \ldots, J[s]$, then subtracts r from $J[k]$, and then shifts $J[k]$ over to the right by $k-1$ places, putting it in the first position. This in turn shifts $J[1], \ldots, J[k-1]$ over to the left by 1 place each. Let $J' := \pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)$. We need to consider four cases.

   - For $i = 1$, $J'[1] = J[k] - r = j_{m-s} + t + s - k + 1 - r = J[1] - r - k + 1 = \tau_1^{-r-k+1}(J[1])$.

- For $2 \leq i \leq k$, $J'[i] = J[i-1] = J[i] + 1 = \tau_i(J[i])$.
- For $k+1 \leq i \leq s-n+r-1$ and $s+1 \leq i \leq m+1$, $J'[i] = J[i]$.
- And for $s-n+r \leq i \leq s$, $J'[i] = J[i] - 1 = \tau_i^{-1}(J[i])$.

Combining this we get

$$\begin{aligned}\pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(Y_J) &= \operatorname{sgn}(\pi_{k,k-1})\tau_s^{-1}\cdots\tau_{s-n+r}^{-1}\tau_k\cdots\tau_2\tau_1^{-r-k+1} \\ &= \operatorname{sgn}(\pi_{k,k-1})\sigma_s^{-1}\sigma_{s-n+r-1}\sigma_k\tau_1^{-r-k}Y_J \ .\end{aligned}$$

2. It also follows that if $s-n+r \leq k \leq s$, then $\pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}$ subtracts 1 from $J[s-n+r], \ldots, J[s]$ (including from $J[k]$), then subtracts r from $J[k]$, and then shifts $J[k]$ over to the right by $k-1$ places, putting it in the first position. This in turn shifts $J[1], \ldots, J[k-1]$ over to the left by 1 place each. Let $J' := \pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)$. We need to consider five cases.

   - For $i = 1$, $J'[1] = J[k]-r-1 = j_{m-s}+t+s-k-r = J[1]-k-r = \tau_1^{-r-k}(J[1])$.
   - For $2 \leq i \leq s-n+r$, $J'[i] = J[i-1] = J[i] + 1 = \tau_i(J[i])$.
   - For $s-n+r+1 \leq i \leq k$, $J'[i] = J[i-1]-1 = j_{m-s}+t+s-(i-1)+1-1 = J[i]$.
   - For $k+1 \leq i \leq s$, $J'[i] = J[i] - 1 = \tau_i^{-1}(J[i])$.
   - Finally, when $s+1 \leq i \leq m+1$, then $J'[i] = J[i]$.

   Combining this we get

$$\pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(Y_J) = \tau_s^{-1}\cdots\tau_k^{-1}\tau_{s-n+r}\cdots\tau_2\tau_1^{-k-r}$$

$$= \operatorname{sgn}(\pi_{k,k-1})\sigma_s^{-1}\sigma_k\sigma_{s-n+r}\tau_1^{-r-k-1}Y_J \ .$$

Remark 5.1.16. Let $J = (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1}$ be such that $j_i < j_{i+1}$ for all $0 \leq i \leq m-1$, and such that there exists an $1 \leq s \leq m$ and $t \geq 2$ such that

$$J[i] = \begin{cases} j_{m-s} + t + s - i + 1, & 1 \leq i \leq s \\ j_{m+1-i}, & s+1 \leq i \leq m+1 \end{cases} .$$

Recall that $\pi_{k,l} := (k-l \ \ k-l+1 \cdots k) \in S_{m+1}$ and $\pi_{k,l}^{-1}(J)$ is the $(m+1)$-tuple obtained by shifting the $(k-l)^{\text{th}}$ entry of J (from the right) to the left by l places. Then $\pi_{s-n+r-1,r-1}^{-1}(J) = (s-n+r-1 \ \ s-n+r-2 \cdots s-n)(J)$ is the $(m+1)$-tuple obtained by shifting the $s-n^{\text{th}}$ entry of J (from the right) to the left by $r-1$ places.

We observe that $\pi_{s-n+r-1,r-1}^{-1}\tau_{s-n}^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(Y_J) = (-1)^{r-1}\sigma_s^{-1}\sigma_{s-n}(Y_J)$. Notice that $J[i+1] = J[i] - 1$ for $1 \leq i \leq s-1$.

1. We look at how the composition $\tau_{s-n}^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}$ acts on J. The operators σ_s^{-1} and $\sigma_{s-(n+1-r)}$ add 1 to the first $s-n-1+r$ coordinates of J and subtract 1 from the first s coordinates of J, respectively. Hence the composition $\sigma_s^{-1}\sigma_{s-(n+1-r)}$ subtracts 1 from $J[s-n+r], J[s-n+r], \ldots, J[s]$. Then τ_{s-n}^{-r} subtracts r from $J[s-n]$. Notice that $s-n \leq s-n-1+r$. So we get

$$\tau_{s-n}^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}J[i] = \begin{cases} j_{m-s}+t+s-i+1, & 1 \le i \le s-n-1 \\ j_{m-s}+t+n+1-r, & i = s-n \\ j_{m-s}+t+s-i+1, & s-n+1 \le i \le s-n-1+r \\ j_{m-s}+t+s-i, & s-n+r \le i \le s \\ j_{m+1-i}, & s+1 \le i \le m+1 \end{cases}.$$

2. Next, $\pi_{s-n+r-1,r-1}^{-1}$ shifts the $(s-n)^{\text{th}}$ entry of $\tau_{s-n}^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}J$ (from the right) to the left by $r-1$ places. So $j_{m-s}+t+n+1-r$ moves to the $s-n+r-1^{th}$ place.

$$\begin{aligned} &\pi_{s-n+r-1,r-1}^{-1}\tau_{s-n}^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}J[i] \\ &= \begin{cases} j_{m-s}+t+s-i+1, & 1 \le i \le s-n-1 \\ j_{m-s}+t+s-i, & s-n \le i \le s-n-2+r \\ j_{m-s}+t+n+1-r, & i = s-n+r-1 \\ j_{m-s}+t+s-i, & s-n+r \le i \le s \\ j_{m+1-i}, & s+1 \le i \le m+1 \end{cases} \\ &= \begin{cases} j_{m-s}+t+s-i+1, & 1 \le i \le s-n-1+r \\ j_{m-s}+t+s-i, & s-n \le i \le s \\ j_{m+1-i}, & s+1 \le i \le m+1 \end{cases}. \end{aligned}$$

So you can see that, up to sign, $\pi_{s-n+r-1,r-1}^{-1}\tau_{s-n}^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}$ subtracts 1 from

$J[s-n], \ldots, J[s]$, and acts trivially on the other components of J, which is exactly how $\sigma_s^{-1}\sigma_{s-n}$ acts on J. Recall that $\pi^{-1}_{s-n+r-1,r-1}$ is an r-cycle. Skew-symmetry then tells us that $\pi^{-1}_{s-n+r-1,r-1}\tau^{-r}_{s-n}\sigma_s^{-1}\sigma_{s-(n+1-r)}(Y_J)$
$= \operatorname{sgn}(\pi^{-1}_{s-n+r-1,r-1})\sigma_s^{-1}\sigma_{s-n}(Y_J) = (-1)^{r-1}\sigma_s^{-1}\sigma_{s-n}(Y_J)$.

Lemma 5.1.17. *Let $J = (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1}$ be such that $j_i < j_{i+1}$ for all $0 \leq i \leq m-1$, and such that there exists an $s \geq 1$ and $t \geq 2$ such that*

$$J[i] = \begin{cases} j_{m-s} + t + s - i - 1, & 1 \leq i \leq s \\ j_{m+1-i}, & s+1 \leq i \leq m+1 \end{cases} .$$

i.e., $(J) = (j_0, \cdots, j_{m-s}, j_{m-s} + t - 1, j_{m-s} + t, j_{m-s} + t + 1, \ldots, j_{m-s} + t + s - 2)$. Further suppose that $Y_K \cdot w_0 = 0$ for all $K \in \mathbb{Z}^{m+1}$ such that $Y_K < Y_{\sigma_n(J)}$. Then for all $1 \leq n \leq s$,

$$Q_n(H_1'', \ldots, H_n'')Y_J \cdot w_0 = \sigma_n(Y_J) \cdot w_0 \ . \tag{5.4}$$

In particular,

$$Q_s(H_1'', \ldots, H_n'')Y_J \cdot w_0 = \sigma_s(Y_J) \cdot w_0,$$

where $\sigma_s(J) = (j_0, \cdots, j_{m-s}, j_{m-s} + t, j_{m-s} + t + 1, \ldots, j_{m-s} + t + s - 1)$.

Proof. Let $J \in \mathbb{Z}^{m+1}$ be as in the statement of the lemma. Note that

$$\operatorname{ht}(Y_J) = \sum_{i=s+1}^{m+1} (j_{m+1-i} - (m+1-i)) + (j_{m-s} + t - m + s - 1)s,$$

and for any $1 \leq n \leq s$,

$$\operatorname{ht}(Y_{\sigma_n(J)}) = \sum_{i=s+1}^{m+1} (j_{m+1-i} - (m+1-i)) + (j_{m-s} + t - m + s - 1)(n-s)$$

$$+ (j_{m-s} + t - m + s)n$$
$$= \mathrm{ht}Y_J + n \ .$$

We proceed by induction on n. When $n = 1$,

$$Q_1(H_1'')Y_J \cdot w_0 = H_1''Y_J \cdot w_0 = \sum_{k=2}^{m+1} Y_{\tau_k(J)} \cdot w_0 + Y_{\tau_1(J)} \cdot w_0 \ .$$

Note that when $2 \leq k \leq m+1$, then $\mathrm{ht}(Y_{\tau_k(J)}) = \mathrm{ht}Y_J + 1 = \mathrm{ht}(Y_{\sigma_1(J)})$. However, $Y_{\sigma_1(J)}[1] = j_{m-s} + t + s - 1$ and $Y_{\tau_k(J)}[1] = j_{m-s} + t + s - 2$. Hence $Y_{\tau_k(J)} < Y_{\sigma_1(J)}$ by Definition 5.1.9. So $Y_{\tau_k(J)} \cdot w_0 = 0$ for all $2 \leq k \leq m+1$. Hence $Q_1(H_1'')Y_J \cdot w_0 = Y_{\tau_1(J)} \cdot w_0 = Y_{\sigma_1(J)} \cdot w_0$.

Now suppose n is such that $1 \leq n+1 \leq s$ and all $i \leq n$ satisfy the lemma. By the inductive hypothesis,

$$H_r''Q_{n+1-r}(H_1'', \ldots, H_{n+1-r}'')Y_J \cdot w_0 = H_r''Y_{\sigma_{n+1-r}(J)} \cdot w_0 = \sum_{k=1}^{m+1} Y_{\tau_k^r\sigma_{n+1-r}(J)} \cdot w_0 \ .$$

for each $1 \leq r \leq n+1$. Like in the base case, when $n+2 \leq k \leq m+1$ then $\mathrm{ht}(\tau_k^r\sigma_{n+1-r}(Y_J)) = \mathrm{ht}(J) + n + 1 - r + r = \mathrm{ht}(\sigma_{n+1}(Y_J))$. In this case, since $r \geq 1$, then $n - r + 2 \leq n + 1$, so σ_{n+1} adds 1 to $J[n-r+2]$, and we get $\sigma_{n+1}(J)[n+2-r] = j_{m-s} + t + s - n + r - 2$. Together, $\tau_k^r\sigma_{n+1-r}$ add 1 to the first $n+1-r$ components of J and r to the k^{th} component of J. Since $k \geq n+2$, $\tau_k^r\sigma_{n+1-r}$ acts trivially on $J[n+2-r]$ and $\tau_k^r\sigma_{n+1-r}(J)[n+2-r] = j_{m-s} + t + s - n + r - 3 < \sigma_{n+1}(J)[n+2-r]$. Furthermore, both σ_{n+1} and σ_{n+1-r} add 1 to the first $n+1-r$ components of J, so $\sigma_{n+1}(J)[i] = \sigma_{n+1-r}(J)[i]$ for all $1 \leq i \leq n+1-r$. Hence $Y_{\sigma_{n+1}(J)}$ is the greater

element by Definition 5.1.9. So $Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 = 0$ for all $n+2 \leq k \leq m+1$ and

$$H_r'' Q_{n+1-r}(H_1'', \ldots, H_{n+1-r}'') Y_J \cdot w_0 = \sum_{k=1}^{n+1} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 \ .$$

Using Newton's formula, we get

$$\begin{aligned}(n+1) Q_{n+1}(H_1'', \ldots, H_{n+1}'') Y_J \cdot w_0 &= \sum_{r=1}^{n+1} (-1)^{r-1} H_r'' Y_{\sigma_{n+1-r}(J)} \cdot w_0 \\ &= \sum_{r=1}^{n+1} (-1)^{r-1} \sum_{k=1}^{n+1} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 \ .\end{aligned}$$

Recall from Remark 5.1.14.5 that for $1 \leq r \leq n+1$, if you apply $\pi_{n+1,r-1}$ to $\tau_{n+1}^r \sigma_{n+1-r}(Y_J)$, it is equal to $(-1)^{r-1} \sigma_{n+1}(Y_J)$, where the $(-1)^{r-1}$ accounts for the minus signs coming from rearranging the Y_i's. So

$$\begin{aligned}&(n+1) Q_{n+1}(H_1'', \ldots, H_{n+1}'') Y_J \cdot w_0 \\ &= \sum_{r=1}^{n+1} (-1)^{r-1} \sum_{k=1}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 + \sum_{r=1}^{n+1} (-1)^{r-1} Y_{\tau_{n+1}^r \sigma_{n+1-r}(J)} \cdot w_0 \\ &= \sum_{r=1}^{n+1} (-1)^{r-1} \sum_{k=1}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 + \sum_{r=1}^{n+1} (-1)^{r-1} (-1)^{r-1} Y_{\sigma_{n+1}(J)} \cdot w_0 \\ &= \sum_{r=1}^{n+1} (-1)^{r-1} \sum_{k=1}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 + \sum_{r=1}^{n+1} Y_{\sigma_{n+1}(J)} \cdot w_0 \\ &= \sum_{r=1}^{n+1} (-1)^{r-1} \sum_{k=1}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 + (n+1) Y_{\sigma_{n+1}(J)} \cdot w_0 \ .\end{aligned}$$

To prove the result, it remains to show that

$$\sum_{r=1}^{n+1} (-1)^{r-1} \sum_{k=1}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 = 0 \ . \tag{5.5}$$

To see more clearly what is going on, we will permute the components of $Y_{\tau_k^r \sigma_{n+1-r}(J)}$ so that the k^{th} component is moved to the first position. To do this, we use the operator $\pi_{k,k-1}$, recalling that $\pi_{k,k-1}$ shifts the k^{th} component of J (when counting from the right) to the right by $k-1$ places. When doing so, we split the sums up into $1 \leq k \leq n+1-r$ and $k \geq n+2-r$. Doing so, we get

$$\begin{aligned}
&\sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=1}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 \\
&= \sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=1}^{n+1-r} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 + \sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=n+2-r}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 \\
&= \sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=1}^{n+1-r} \pi_{k,k-1} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 + \sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=n+2-r}^{n} \pi_{k,k-1} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 \ .
\end{aligned}$$

By Remark 5.1.14, when $1 \leq k \leq n+1-r$, then $\pi_{k,k-1}\tau_k^r \sigma_{n+1-r}(Y_J)$ $= (-1)^{k-1}\tau_1^{r-k}\sigma_{n+1-r}\sigma_k(Y_J)$, and when $n+2-r \leq k$, then $\pi_{k,k-1}\tau_k^r \sigma_{n+1-r}(Y_J) = (-1)^{k-1}\tau_1^{r-k-1}\sigma_{n+2-r}\sigma_k(Y_J)$. So

$$\begin{aligned}
\sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=1}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 &= \sum_{r=1}^{n+1}\sum_{k=1}^{n+1-r} (-1)^{r+k}\tau_1^{r-k}\sigma_{n+1-r}\sigma_k(Y_J) \cdot w_0 \\
&+ \sum_{r=1}^{n+1}\sum_{k=n+2-r}^{n} (-1)^{r+k}\tau_1^{r-k-1}\sigma_{n+2-r}\sigma_k(Y_J) \cdot w_0 \ .
\end{aligned}$$

Switching the order of summation in the second sum, we get

$$\begin{aligned}
\sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=1}^{n} Y_{\tau_k^r \sigma_{n+1-r}(J)} \cdot w_0 &= \sum_{r=1}^{n+1}\sum_{k=1}^{n+1-r} (-1)^{r+k}\tau_1^{r-k}\sigma_{n+1-r}\sigma_k(Y_J) \cdot w_0 \\
&+ \sum_{k=1}^{n}\sum_{r=n+2-k}^{n+1} (-1)^{r+k}\tau_1^{r-k-1}\sigma_{n+2-r}\sigma_k(Y_J) \cdot w_0 \ .
\end{aligned}$$

Next we change variables, replacing r by k' and k by r'. (When the new variables are named this way, it is clear that the two sums cancel.) If we let $k' = n + 2 - r$ in the second sum, and then $r' = n + 1 - k$ in the second sum, we get

$$\begin{aligned}
&\sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=1}^{n} Y_{\tau_k^r\sigma_{n+1-r}(J)}\cdot w_0 \\
&= \sum_{r=1}^{n+1}\sum_{k=1}^{n+1-r}(-1)^{r+k}\tau_1^{r-k}\sigma_{n+1-r}\sigma_k(Y_J)\cdot w_0 + \sum_{k=1}^{n}\sum_{k'=1}^{k}(-1)^{n-k'+k}\tau_1^{n+1-k'-k}\sigma_{k'}\sigma_k(Y_J)\cdot w_0 \\
&= \sum_{r=1}^{n+1}\sum_{k=1}^{n+1-r}(-1)^{r+k}\tau_1^{r-k}\sigma_{n+1-r}\sigma_k(Y_J)\cdot w_0 \\
&+ \sum_{r'=1}^{n}\sum_{k'=1}^{n+1-r'}(-1)^{r'+k'+1}\tau_1^{-k'+r'}\sigma_{k'}\sigma_{n+1-r'}(Y_J)\cdot w_0 = 0\ .
\end{aligned}$$

□

We may denote elements of the universal enveloping algebra as $Y_J := Y_{j_1}\cdots Y_{j_n} \in U(\mathfrak{gl}(1\,|1))$ where $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$. Lemma 5.1.18 shows that a similar pattern holds when we start decreasing the entries of J. For $i, n \in \mathbb{Z}_{\geq 0}$ such that $i \geq n$ define recursively $Q_{-n}(H''_{-1}, \ldots, H''_{-i})$ by $Q_0(H''_{-1}, \ldots, H''_{-i}) = 1$ and

$$Q_{-n}(H''_{-1}, \ldots, H''_{-i}) := \frac{1}{n}\sum_{r=1}^{n}(-1)^{r-1}H''_{-r}Q_{-(n-r)}\left(H''_{-1}, \ldots, H''_{-(n-r)}\right)\ .$$

In the following Lemma, we read the entries of the tuple J from right to left. For example, if $J = (1, 2, 3, 4)$ then $J[1] = 4$.

Lemma 5.1.18. *Let $J = (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1}$ be such that $j_i < j_{i+1}$ for all*

$0 \leq i \leq m-1$, *and such that there exists an* $1 \leq s \leq m$ *and* $t \geq 2$ *such that*

$$J[i] = \begin{cases} j_{m-s} + t + s - i + 1, & 1 \leq i \leq s \\ j_{m+1-i}, & s+1 \leq i \leq m+1 \end{cases}.$$

Further suppose that $Y_K \cdot w_0 = 0$ *for all* $K \in \mathbb{Z}^{m+1}$ *such that* $Y_K > Y_{\sigma_s^{-1}\sigma_{s-n}(J)}$. *Then for all* $1 \leq n \leq s$,

$$Q_{-n}(H''_{-1}, \ldots, H''_{-n}) Y_J \cdot w_0 = \sigma_s^{-1}\sigma_{s-n}(Y_J) \cdot w_0 \ . \tag{5.6}$$

The proof of this is similar to the proof of Lemma 5.1.17. This lemma says that if $Y_J \cdot w_0$ is acted on by $Q_n(H''_{-1}, \ldots, H''_{-n})$, where n is an integer between 1 and s, then 1 is subtracted from the first n terms on the sequence, starting at the term $Y_{j_{m-s}+t+1}$. The following example illustrates the lemma.

Example 5.1.19. Let $J \in \mathbb{Z}^9$ be such that $j_i < j_{i+1}$ for all $1 \leq i \leq m+1$, and there exists $t \geq 2$ such that

$$J[i] = \begin{cases} j_{m-s} + t + s - i + 1, & 1 \leq i \leq 4 \\ j_{m+1-i}, & 5 \leq i \leq 9 \end{cases}.$$

Then $J = (j_0, j_1, j_2, j_3, j_4, j_4+t+1, j_4+t+2, j_4+t+3, j_4+t+4)$ and $Y_J = Y_{j_0}Y_{j_1}Y_{j_2}Y_{j_3}Y_{j_4}Y_{j_4+t+1}Y_{j_4+t+2}Y_{j_4+t+3}Y_{j_4+t+4}$ and we have the following equalities

- $Q_{-1}(H''_{-1})Y_J \cdot w_0 = Y_{j_0}Y_{j_1}Y_{j_2}Y_{j_3}Y_{j_4}Y_{j_4+t}Y_{j_4+t+2}Y_{j_4+t+3}Y_{j_4+t+4} \cdot w_0$
- $Q_{-2}(H''_{-1}, H''_{-2})Y_J \cdot w_0 = Y_{j_0}Y_{j_1}Y_{j_2}Y_{j_3}Y_{j_4}Y_{j_4+t}Y_{j_4+t+1}Y_{j_4+t+3}Y_{j_4+t+4} \cdot w_0$
- $Q_{-3}(H''_{-1}, H''_{-2}, H''_{-3})Y_J \cdot w_0 = Y_{j_0}Y_{j_1}Y_{j_2}Y_{j_3}Y_{j_4}Y_{j_4+t}Y_{j_4+t+1}Y_{j_4+t+2}Y_{j_4+t+4} \cdot w_0$

- $Q_{-4}(H''_{-1}, H''_{-2}, H''_{-3}, H''_{-4})Y_J \cdot w_0 = Y_{j_0}Y_{j_1}Y_{j_2}Y_{j_3}Y_{j_4}Y_{j_4+t}Y_{j_4+t+1}Y_{j_4+t+2}Y_{j_4+t+3} \cdot w_0$

  .

Proof. Let $J \in \mathbb{Z}^{m+1}$ be as in the statement of the lemma. Note that

$$\mathrm{ht}(Y_J) = \sum_{i=s+1}^{m+1} (j_{m+1-i} - (m+1-i)) + (j_{m-s} + t + s - m)s,$$

and for any $1 \leq l \leq s$, $\mathrm{ht}(Y_{\sigma_s^{-1}\sigma_{s-l}(J)}) = \sum_{i=s+1}^{m+1}(j_{m+1-i} - (m+1-i)) + (j_{m-s} + t + s - m)(s-l) + (j_{m-s} + t + s - m - 1)l = \mathrm{ht}Y_J - l$.

We proceed by induction on n. When $n = 1$,

$$Q_{-1}(H''_{-1})Y_J \cdot w_0 = H''_{-1}Y_J \cdot w_0 = \sum_{k=1}^{s-1} Y_{\tau_k^{-1}(J)} \cdot w_0 + Y_{\tau_s^{-1}(J)} \cdot w_0 + \sum_{k=s+1}^{m+1} Y_{\tau_k^{-1}(J)} \cdot w_0 \ .$$

Notice that $\mathrm{ht}(Y_{\tau_k^{-1}(J)}) = \mathrm{ht}Y_J - 1 = \mathrm{ht}(Y_{\sigma_s^{-1}\sigma_{s-1}(J)})$. However, when $1 \leq k \leq s-1$, then $\tau_k^{-1}(J)[k+1] = (J)[k+1] = j_{m-s}+t+s-k$ and $\tau_k^{-1}(J)[k] = j_{m-s}+t+s-k+1-1 = \tau_k^{-1}(J)[k+1]$. So a factor of Y_i^2 appears in $Y_{\tau_k^{-1}(J)}$, meaning $Y_{\tau_k^{-1}(J)} \cdot w_0 = 0$. And when $s+1 \leq k \leq m+1$, then $\tau_k^{-1}(J)[s] = j_{m-s} + t + 1 > j_{m-s} + t = \sigma_s^{-1}\sigma_{s-1}(J)[s]$, and $\tau_k^{-1}(J)[i] = \sigma_s^{-1}\sigma_{s-1}(J)[i]$ for all $1 \leq i \leq s-1$. By Definition 5.1.9, $Y_{\tau_k^{-1}(J)} > Y_{\sigma_s^{-1}\sigma_{s-1}(J)}$, so $Y_{\tau_k^{-1}(J)} \cdot w_0 = 0$. Hence

$$Q_{-1}(H''_{-1})Y_J \cdot w_0 = Y_{\tau_s^{-1}(J)} \cdot w_0 = \sigma_s^{-1}\sigma_{s-1}(Y_J) \cdot w_0 \ .$$

Suppose n is such that $1 \leq n+1 \leq s$ and satisfies the lemma. By the inductive hypothesis,

$$H''_{-r}Q_{n+1-r}(H''_1, \ldots, H''_{n+1-r})Y_J \cdot w_0 = H''_{-r}Y_{\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0$$

$$= \sum_{k=1}^{m+1} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 \ .$$

for each $1 \leq r \leq n+1$.

Like in the base case, when $s+1 \leq k \leq m+1$ then $\mathrm{ht}(\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(Y_J)) = \mathrm{ht}(J) - (n+1) = \mathrm{ht}(\sigma_s^{-1}\sigma_{s-(n+1)}(Y_J))$. Since $k \geq s+1$ and $r \geq 1$, then $Y_{\sigma_s^{-1}(J)}[1] = j_{m-s} + t + s - 1$ and $Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)}[1] = Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_r(J)}[1] = j_{m-s} + t + s$. Hence $Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)}$ is the greater element by Definition 5.1.9. So $Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 = 0$ for all $s+1 \leq k \leq m+1$ and

$$H''_{-r}Q_{-(n+1-r)}(H''_{-1}, \ldots, H''_{-(n+1-r)})Y_J \cdot w_0 = \sum_{k=1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 \ .$$

Using Newton's formula, we get

$$\begin{aligned}(n+1)Q_{-(n+1)}(H''_{-1}, \ldots, H''_{-(n+1)})Y_J \cdot w_0 &= \sum_{r=1}^{n+1}(-1)^{r-1}H''_{-r}Y_{\sigma_{-(n+1-r)}(J)} \cdot w_0 \qquad (5.7)\\ &= \sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 \ .\end{aligned}$$

For each $1 \leq r \leq n+1$, let us consider the case when $1 \leq k \leq s-(n+1)$. The operators $\sigma_s^{-1}\sigma_{s-(n+1-r)}$ together act trivially on the first $s-(n+1-r)$ entries of J, when reading from the right. Since $1 \leq k \leq s-(n+1)$, then $k \leq k+r \leq s-(n+1-r)$. So $\sigma_s^{-1}\sigma_{s-(n+1-r)}$ acts trivially on $J[k]$ and $J[k+r]$. Hence $\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)[k+r] = (J)[k+r] = j_{m-s} + t + s - k - r + 1$ and $\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)[k] = J[k] - r = j_{m-s} + t + s - k + 1 - r$. So a factor of Y_i^2 appears in $Y_{\tau_k^{-1}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)}$, meaning

$Y_{\tau_k^{-1}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 = 0$. This reduces equation (5.7) to

$$
\begin{aligned}
&(n+1)Q_{-(n+1)}(H''_{-1}, \ldots, H''_{-(n+1)})Y_J \cdot w_0 \\
&= \sum_{r=1}^{n+1}(-1)^{r-1} \sum_{k=s-n}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0. \\
&= \sum_{r=1}^{n+1}(-1)^{r-1} \sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 + \sum_{r=1}^{n+1}(-1)^{r-1} Y_{\tau_{s-n}^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 \\
&= \sum_{r=1}^{n+1}(-1)^{r-1} \sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 + \sum_{r=1}^{n+1}(-1)^{r-1} \pi_{s-n,r-1}^{-1}\sigma_s^{-1}\sigma_{s-n}(Y_J) \cdot w_0 \\
&= \sum_{r=1}^{n+1}(-1)^{r-1} \sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 + \sum_{r=1}^{n+1}(-1)^{r-1}(-1)^{r-1}\sigma_s^{-1}\sigma_{s-n}(Y_J) \cdot w_0 \\
&= \sum_{r=1}^{n+1}(-1)^{r-1} \sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 + (n+1)\sigma_s^{-1}\sigma_{s-n}(Y_J) \cdot w_0 \ .
\end{aligned}
$$

To prove the lemma, it remains to show that

$$
\sum_{r=1}^{n+1}(-1)^{r-1} \sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 = 0 \ . \tag{5.8}
$$

To see more clearly what is going on, we will permute the components of $Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)}$ so that the k^{th} component is moved to the first position. To do this, we use the operator $\pi_{k,k-1}$, recalling that $\pi_{k,k-1}$ shifts the k^{th} component of J (when counting from the right) to the right by $k-1$ places. So we have

$$
\begin{aligned}
&\sum_{r=1}^{n+1}(-1)^{r-1} \sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)} \cdot w_0 \\
&= \sum_{r=1}^{n+1}(-1)^{r-1} \sum_{k=s-n+1}^{s} \pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(Y_J) \cdot w_0 \ .
\end{aligned}
$$

By Remark 5.1.15, when $s-n+r \leq k \leq s$, then $\pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-n+r-1}Y_J = (-1)^{k-1}\sigma_s^{-1}\sigma_{k-1}\sigma_{s-n+r-1}\tau_1^{-r-k-1}Y_J$. And when $s-n+1 \leq k \leq s-n-1+r$, then $\pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}Y_J = (-1)^{k-1}\sigma_s^{-1}\sigma_k\sigma_{s-(n+1-r)}\tau_1^{-r-k}Y_J$. Splitting the sum into two, we get

$$\begin{aligned}
&\sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)}\cdot w_0 \\
&= \sum_{r=1}^{n}(-1)^{r-1}\sum_{k=s-n+r}^{s} \pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(Y_J)\cdot w_0 \\
&+ \sum_{r=2}^{n+1}(-1)^{r-1}\sum_{k=s-n+1}^{s-n+r-1} \pi_{k,k-1}\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(Y_J)\cdot w_0 \\
&= \sum_{r=1}^{n}\sum_{k=s-n+r}^{s}(-1)^{r-1}(-1)^{k-1}\sigma_s^{-1}\sigma_k\sigma_{s-n+r}\tau_1^{-r-k-1}(Y_J)\cdot w_0 \\
&+ \sum_{r=2}^{n+1}\sum_{k=s-n+1}^{s-n-1+r}(-1)^{r-1}(-1)^{k-1}\sigma_s^{-1}\sigma_k\sigma_{s-n-1+r}\tau_1^{-r-k}(Y_J)\cdot w_0\ .
\end{aligned}$$

Switch the order of summation in the second sum to get

$$\begin{aligned}
&\sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)}\cdot w_0 \\
&= \sum_{r=1}^{n}\sum_{k=s-n+r}^{s}(-1)^{r+k}\sigma_s^{-1}\sigma_k\sigma_{s-n+r}\tau_1^{-r-k-1}(Y_J)\cdot w_0 \\
&+ \sum_{k=s-n+1}^{s}\sum_{r=-s+n+1+k}^{n+1}(-1)^{r+k}\sigma_s^{-1}\sigma_k\sigma_{s-n-1+r}\tau_1^{-r-k}(Y_J)\cdot w_0\ .
\end{aligned}$$

Next we change variables, replacing r by k' and k by r'. (When the new variables are named this way, it is clear that the two sums cancel.) If we let $k' = r+s-n-1$ in the second sum, and then $r' = k-s+n$ in the second sum, we get

$$\begin{aligned}
&\sum_{r=1}^{n+1}(-1)^{r-1}\sum_{k=s-n+1}^{s} Y_{\tau_k^{-r}\sigma_s^{-1}\sigma_{s-(n+1-r)}(J)}\cdot w_0\\
&=\sum_{r=1}^{n}\sum_{k=s-n+r}^{s}(-1)^{r+k}\sigma_s^{-1}\sigma_k\sigma_{s-n+r}\tau_1^{-r-k-1}(Y_J)\cdot w_0\\
&+\sum_{k=s-n+1}^{s}\sum_{k'=k}^{s}(-1)^{k'-s+n+1+k}\sigma_s^{-1}\sigma_k\sigma_{k'}\tau_1^{s-n-1-k'-k}(Y_J)\cdot w_0\\
&=\sum_{r=1}^{n}\sum_{k=s-n+r}^{s}(-1)^{r+k}\sigma_s^{-1}\sigma_k\sigma_{s-n+r}\tau_1^{-r-k-1}(Y_J)\cdot w_0\\
&+\sum_{r'=1}^{n}\sum_{k'=s-n+r'}^{s}(-1)^{r'+k'+1}\sigma_s^{-1}\sigma_{s-n+r'}\sigma_{k'}\tau_1^{-r'-k'-1}(Y_J)\cdot w_0=0\ .
\end{aligned}$$

Hence equation (5.8) holds. □

Proof of Proposition 5.1.5

Let M be the infinite matrix $M := [c_{i+j}]_{i,j\in\mathbb{Z}}$ and for $I=(i_1,\dots,i_n), J=(j_1,\dots,j_n)\in\mathbb{Z}^n$, let $M_{I,J}$ be the matrix $M_{I,J} := \left[c_{i_s+j_l}\right]_{s,l=1,\dots,n}$. Note that $\det M_{I,J}$ is an $n\times n$ minor of M.

Proposition 5.1.20. *Denote $z := h' + h''$ so that $Z_k\cdot w_0 = c_k w_0$. Then for $n\geq 1$ and for all $I=(i_1,\dots,i_n), J=(j_1,\dots,j_n)\in\mathbb{Z}^n$,*

$$X_I Y_J w_0 = (-1)^{\frac{n(n-1)}{2}}\det M_{I,J} w_0\ .$$

In particular, $X_I Y_J w_0 = 0$ if and only if $\det M_{I,J}=0$.

Proof. First note that $XY = -YX + [X, Y]$. Hence, the Leibniz rule gives

$$[X, Y_1 Y_2 \cdots Y_k] = (-1)^k Y_1 Y_2 \cdots Y_k X + \sum_{s=1}^{k} (-1)^{s-1} Y_1 \cdots Y_{s-1} [X, Y_s] Y_{s+1} \cdots Y_k$$

for all $k \geq 1$. We will proceed by induction. When $n = 1$, then

$$X_{i_1} Y_{j_i} w = Y_{j_1} X_{i_1} w + [X_{i_1}, Y_{j_i}] w = Z_{i_1+j_i} w = c_{i_1+j_i} w = (-1)^{\frac{1}{2}1(1-1)} \det [c_{i_1+j_i}] \, w \ .$$

Now suppose the result holds for some $n \geq 1$. Let $i_1, \ldots, i_{n+1}; j_1, \ldots, j_{n+1} \in \mathbb{Z}$. Then

$$\begin{aligned}
&X_{i_1} \cdots X_{i_n} X_{i_{n+1}} Y_{j_1} \cdots Y_{j_n} Y_{j_{n+1}} w_0 \\
&= \sum_{s=1}^{n+1} (-1)^{s-1} X_{i_1} \cdots X_{i_n} Y_{j_1} \cdots Y_{j_{s-1}} [X_{i_{n+1}}, Y_{j_s}] Y_{j_{s+1}} \cdots Y_{j_{n+1}} w_0 \\
&= \sum_{s=1}^{n+1} (-1)^{s-1} X_{i_1} \cdots X_{i_n} Y_{j_1} \cdots Y_{j_{s-1}} Z_{i_{n+1}+j_s} Y_{j_{s+1}} \cdots Y_{j_{n+1}} w_0 \\
&= \sum_{s=1}^{n+1} (-1)^{s-1} X_{i_1} \cdots X_{i_n} Y_{j_1} \cdots \widehat{Y_{j_s}} \cdots Y_{j_{n+1}} Z_{i_{n+1}+j_s} w_0 \ \text{ (since } Z \text{ is central)} \\
&= \sum_{s=1}^{n+1} (-1)^{s-1} c_{i_{n+1}+j_s} X_{i_1} \cdots X_{i_n} Y_{j_1} \cdots \widehat{Y_{j_s}} \cdots Y_{j_{n+1}} w_0 \\
&= \sum_{s=1}^{n+1} (-1)^{s-1} c_{i_{n+1}+j_s} (-1)^{\frac{1}{2}n(n-1)} |c_{i_k+j_l}|_{k=1,\ldots,n, l=1,\ldots,\hat{s},\ldots n+1} w_0 \text{ (by the inductive hypothesis)} \\
&= \sum_{s=1}^{n+1} (-1)^{s-1+\frac{1}{2}n(n-1)+2n+2} c_{i_{n+1}+j_s} |c_{i_k+j_l}|_{k=1,\ldots,n, l=1,\ldots,\hat{s},\ldots n+1} w_0 \\
&= (-1)^{\frac{1}{2}n(n+1)} \sum_{s=1}^{n+1} (-1)^{n+1+s} c_{i_{n+1}+j_s} |c_{i_k+j_l}|_{k=1,\ldots,n, l=1,\ldots,\hat{s},\ldots n+1} w_0 \ .
\end{aligned}$$

Since $|c_{i_k+j_l}|_{k=1,\ldots,n, l=1,\ldots,\hat{s},\ldots n+1}$ is the determinant of the submatrix obtained by removing the $(n+1)^{\text{th}}$ row and the s^{th} column, by using the Laplace expansion, we

get

$$X_{i_1} \cdots X_{i_n} X_{i_{n+1}} Y_{j_1} \cdots Y_{j_n} Y_{j_{n+1}} w_0 = (-1)^{\frac{1}{2}n(n+1)} \det \left[c_{i_k+j_l}\right]_{k,l=1,\ldots,n-1} w_0 \ .$$

□

Proof of Proposition 5.1.6

Proposition 5.1.6, appearing below as Proposition 5.1.21, describes the relationship between the rank of the matrix $M = [c_{i+j}]_{i,j\in\mathbb{Z}}$ and the recurrence relation on the sequence $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$. This subsection restates the proposition and provides a proof.

Proposition 5.1.21. *Let $m \in \mathbb{Z}_{\geq 0}$ and consider the sequence $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$. Then $\operatorname{rank} M \leq m$ if and only if the sequence $\mathbf{c}$ satisfies an order m linear recurrence relation with constant coefficients. In particular, the recurrence relation of least degree satisfied by the sequence $\mathbf{c}$ has minimal polynomial $p(t)$ where $\deg p(t) = \operatorname{rank} M$.*

Proof. Suppose $\operatorname{rank} M = k \leq m$. Then there exists a non-zero $k \times k$ minor of M, say

$$\begin{vmatrix} c_{j_1} & c_{j_2} & \cdots & c_{j_k} \\ \vdots & & \ddots & \vdots \\ c_{j_1+k-1} & c_{j_2+k-1} & \cdots & c_{j_k+k-1} \end{vmatrix},$$

and any $(k+1) \times (k+1)$ minor of M is zero. Hence for any $n \in \mathbb{Z}$,

$$0 = \begin{vmatrix} c_n & c_{j_1} & c_{j_2} & \cdots & c_{j_k} \\ c_{n+1} & c_{j_1+1} & c_{j_2+1} & \cdots & c_{j_k+1} \\ \vdots & & \ddots & & \vdots \\ c_{n+k} & c_{j_1+k} & c_{j_2+k} & \cdots & c_{j_k+k} \end{vmatrix}.$$

By using Laplace expansion on the first column, we get

$$0 = \begin{vmatrix} c_n & c_{j_1} & c_{j_2} & \cdots & c_{j_k} \\ c_{n+1} & c_{j_1+1} & c_{j_2+1} & \cdots & c_{j_k+1} \\ \vdots & & \ddots & & \vdots \\ c_{n+k} & c_{j_1+k} & c_{j_2+k} & \cdots & c_{j_k+k} \end{vmatrix} = \sum_{i=0}^{k} c_{n+i} C_{i+1,1},$$

where $C_{s,t}$ is the cofactor obtained by deleting row s and column t. Hence $0 = \sum_{i=0}^{k} c_{n+i} C_{i+1,1}$ is equivalent to

$$c_{n+k} = -\sum_{i=0}^{k-1} \left(\frac{C_{i+1,1}}{C_{k+1,1}} \right) c_{n+i} = \sum_{i=0}^{k-1} \alpha_i c_{n+i},$$

where $\alpha_i \in \mathbb{C}$ and do not depend on n. Consider the matrix

$$B = \begin{bmatrix} c_n & c_{j_1} & c_{j_2} & \cdots & c_{j_k} \\ c_{n+1} & c_{j_1+1} & c_{j_2+1} & \cdots & c_{j_k+1} \\ \vdots & & \ddots & & \vdots \\ c_{n+k} & c_{j_1+k} & c_{j_2+k} & \cdots & c_{j_k+k} \end{bmatrix}.$$

Since $C_{k+1,1} \neq 0$, then the vectors

$$\begin{bmatrix} c_{j_1} \\ c_{j_2} \\ \vdots \\ c_{j_k} \end{bmatrix}, \begin{bmatrix} c_{j_1+1} \\ c_{j_2+1} \\ \vdots \\ c_{j_k+1} \end{bmatrix}, \ldots, \begin{bmatrix} c_{j_1+k-2} \\ c_{j_2+k-2} \\ \vdots \\ c_{j_k+k-2} \end{bmatrix}, \begin{bmatrix} c_{j_1+k-1} \\ c_{j_2+k-1} \\ \vdots \\ c_{j_k+k-1} \end{bmatrix} \tag{5.9}$$

are linearly independent. Now suppose the c_i's satisfy a recurrence relation of order less than k. i.e., Suppose there exist $\beta_i \in \mathbb{C}$ not all zero such that $c_{n+k-1} = \sum_{i=1}^{k-2} \beta_i c_{n+i}$ for all $n \in \mathbb{Z}$. Then

$$\beta_0 \begin{bmatrix} c_{j_1} \\ c_{j_2} \\ \vdots \\ c_{j_k} \end{bmatrix} + \beta_1 \begin{bmatrix} c_{j_1+1} \\ c_{j_2+1} \\ \vdots \\ c_{j_k+1} \end{bmatrix} + \cdots + \beta_{k-2} \begin{bmatrix} c_{j_1+k-2} \\ c_{j_2+k-2} \\ \vdots \\ c_{j_k+k-2} \end{bmatrix} - \begin{bmatrix} c_{j_1+k-1} \\ c_{j_2+k-1} \\ \vdots \\ c_{j_k+k-1} \end{bmatrix} = 0$$

which contradicts the linear independence of the vectors in (5.9). Hence, the sequence $\mathbf{c}$ satisfies a degree k homogeneous linear recurrence relation with constant coefficients and does not satisfy a homogeneous linear recurrence relation of lesser order.

Now suppose $c_{k+m} = \alpha_1 c_{k+m-1} + \alpha_2 c_{k+m-2} + \cdots \alpha_m c_k$ for all $k \geq 0$. Recall that $M_{I,J} := [c_{i+j}]_{i \in I, j \in J}$ where $I, J \in \mathbb{Z}^{m+1}$. Let M be the infinite matrix $M := [c_{i+j}]_{i,j \in \mathbb{Z}}$. Then $\det M_{I,J}$ is an $(m+1) \times (m+1)$ minor of M. Let v_l be the l^{th} column of M.

Since $c_{k+m} = \alpha_m c_k + \alpha_{m-1} c_{k+1} + \cdots + \alpha_2 c_{k+m-2} + \alpha_1 c_{k+m-1}$, then for all $k \geq 0$

$$v_{m+k} = \begin{bmatrix} c_{m+k} \\ c_{m+k+1} \\ c_{m+k+2} \\ \vdots \end{bmatrix} = \alpha_m \begin{bmatrix} c_k \\ c_{k+1} \\ c_{k+2} \\ \vdots \end{bmatrix} + \alpha_{m-1} \begin{bmatrix} c_{k+1} \\ c_{k+2} \\ c_{k+3} \\ \vdots \end{bmatrix} + \cdots + \alpha_2 \begin{bmatrix} c_{m+k-2} \\ c_{m+k-1} \\ c_{m+k} \\ \vdots \end{bmatrix} + \alpha_1 \begin{bmatrix} c_{m+k-1} \\ c_{m+k} \\ c_{m+k+1} \\ \vdots \end{bmatrix}$$

$$= \alpha_m v_k + \alpha_{m-1} v_{k+1} + \cdots + \alpha_2 v_{m+k-2} + \alpha_1 v_{m+k-1} \ .$$

In particular, $v_m = \alpha_m v_0 + \alpha_{m-1} v_1 + \cdots + \alpha_2 v_{m-2} + \alpha_1 v_{m-1}$. If we assume that there exists an $L \geq 0$, such that for all $0 \leq l \leq L$ there exist $\beta_{l,0}, \beta_{l,1}, \ldots, \beta_{l,m-1}$ such that $v_{m+l} = \beta_{l,0} v_0 + \beta_{l,1} v_1 + \cdots + \beta_{l,m-1} v_{m-1}$, then

$$\begin{aligned} v_{m+L+1} &= \alpha_m v_{L+1} + \alpha_{m-1} v_{L+2} + \cdots + \alpha_2 v_{m+L-1} + \alpha_1 v_{m+L} \\ &= \alpha_m (\beta_{L+m-1,0} v_0 + \beta_{L+m-1,1} v_1 + \cdots + \beta_{L+m-1,m-1} v_{m-1}) \\ &\quad + \cdots + \alpha_1 (\beta_{L,0} v_0 + \beta_{L,1} v_1 + \cdots + \beta_{L,m-1} v_{m-1}) \\ &\in \operatorname{span}\{v_0, v_1, \ldots, v_{m-1}\} \ . \end{aligned}$$

Hence $v_l \in \operatorname{span}\{v_0, v_1, \ldots, v_{m-1}\}$. Moreover, if $\{v_0, v_1, \ldots, v_{m-1}\}$ were not linearly independent, then we would have a k-degree linear recurrence relation on the c_i's, which is not what we assumed. So $\dim \operatorname{span}\{v_0, v_1, \ldots, v_{m-1}\} = m$, and we can find vectors $v_{j_0}, v_{j_1}, \ldots, v_{j_m}$ such that they are linearly dependent. Now let $i := (i_0, i_1, \ldots, i_m)$ and

define

$$C_\infty := \left\{ v_l := \begin{bmatrix} c_l \\ c_{l+1} \\ c_{l+2} \\ \vdots \end{bmatrix} \mid l \in \mathbb{Z}_{\geq 0} \right\}, C_m := \left\{ v_l := \begin{bmatrix} c_l \\ c_{l+1} \\ c_{l+2} \\ \vdots \\ c_{l+m} \end{bmatrix} \mid l \in \mathbb{Z}_{\geq 0} \right\}.$$

Define the linear map

$$\begin{gathered} \Phi_i : C_\infty \to C_m \\ \begin{bmatrix} c_s \\ c_{s+1} \\ c_{s+2} \\ \vdots \\ \vdots \end{bmatrix} \mapsto \begin{bmatrix} c_{s+i_0} \\ c_{s+i_1} \\ c_{s+i_2} \\ \vdots \\ c_{s+i_m} \end{bmatrix}. \end{gathered}$$

Since the set $\{v_{j_0}, v_{j_1}, \ldots, v_{j_m}\}$ is linearly dependent in C_∞ then the vectors in the image

$$\{\Phi_i(v_{j_0}), \Phi_i(v_{j_1}), \ldots, \Phi_i(v_{j_m})\} = \left\{ \begin{bmatrix} c_{j_0+i_0} \\ c_{j_0+i_1} \\ \vdots \\ c_{j_0+i_m} \end{bmatrix}, \begin{bmatrix} c_{j_1+i_0} \\ c_{j_1+i_1} \\ \vdots \\ c_{j_1+i_m} \end{bmatrix}, \ldots, \begin{bmatrix} c_{j_m+i_0} \\ c_{j_m+i_1} \\ \vdots \\ c_{j_m+i_m} \end{bmatrix} \right\}$$

are linearly dependent. These are precisely the columns of M_{m+1}, hence $\det M_{I,J} = 0$ and $\text{rank} M \leq m$. □

Proof of Proposition 5.1.7

Proposition 5.1.7, restated below as Proposition 5.1.22, gives the dimension of each $\mathbb{Z}$-graded component of $W(\Lambda, m)$. The proof is found after the statement and a basis of $W(\Lambda, m)$ can be found in the proof.

Proposition 5.1.22. *The module $W(\Lambda, m)$ is finite-dimensional. In particular $\dim W(\Lambda, m) = 2^m$ and $\dim W(\Lambda, m)^{-k} = \binom{m}{k}$ when $0 \leq k \leq m$.*

Proof. First we prove that $W(\Lambda, m)$ is finite-dimensional. Let $n \geq m$. First we claim that $Y_n \cdot w_0 \in \operatorname{span}\{Y_0 \cdot w_0, \ldots, Y_{m-1} \cdot w_0\}$. Indeed, by Proposition 5.1.4, $Y_0 \cdots Y_{m-1} Y_n \cdot w_0 = 0$. Then, by Lemma 5.1.25, for all $i_0, \ldots, i_{m-1} \in \mathbb{Z}$

$$\begin{aligned} 0 = Y_0 \cdots Y_{m-1} Y_n \cdot w_0 &= X_{i_{m-1}} \cdots X_{i_0} Y_0 \cdots Y_{m-1} Y_n \cdot w_0 \\ &= \sum_{k=0}^{m} (-1)^{\frac{m(m+1)}{2}+k} \det\left([c_{i_s+t}]_{s\in S, t\in T\setminus\{k\}}\right) Y_k \cdot w_0 \\ &\quad + (-1)^{\frac{m(m+1)}{2}+n} \left(\det[c_{i_s+t}]_{s\in S, t\in T\setminus\{n\}}\right) Y_n \cdot w_0, \end{aligned}$$

where $S := \{0, \ldots, m-1\}$ and $T := \{0, \ldots, m-1, n\}$. Since $W(\Lambda, m)$ is a highest weight module, then each weight space $W(\Lambda, m)^{-k}$, where $0 \leq k \leq m$, is spanned by $\{Y_{j_1} \cdots Y_{j_k} \cdot w_0 \,|\, j_1, \ldots, j_k \in \mathbb{Z}\} = \{Y_{j_1} \cdots Y_{j_k} \cdot w_0 \,|\, j_1, \ldots, j_k \in \mathbb{Z}, 0 \leq j_1, \ldots, j_k \leq m-1\}$. Hence $\dim W(\mathbf{c}, m)^{m-2k} < \infty$. Since $\operatorname{supp} W(\Lambda, m) < \infty$, then $\dim W(\Lambda, m) < \infty$.

Next, we prove that $\dim W(\Lambda, m) = 2^m$. By Lemma 5.1.26, every $Y_I \cdot w_0 := Y_{i_1} Y_{i_2} \cdots Y_{i_k} w_0 \in W(\Lambda, m)^{-k}$ is a linear combination of elements in $\mathcal{B}_{-k} = \{Y_{j_1} \cdots Y_{j_k} w_0 \,|\, 0 \leq j_1 < \cdots < j_k \leq m-1 \}$. When $k \geq m+1$, then $\mathcal{B}_{-k} = \emptyset$. We take $\mathcal{B}_0 := \{w_0\}$. When $0 \leq k \leq m$, then $|\mathcal{B}_{-k}| = \binom{m}{k}$. It remains to show that

the set $\mathcal{B}_{-k}$ is linearly independent for each $0 \leq k \leq m$.

We can write the set $\mathcal{B}_{-k}$ in the following way, which will allow us to prove linear independence. For $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, let $Y_I := Y_{i_1} \cdots Y_{i_n}$. Fix $0 \leq l \leq m$. Let $K \subseteq \{0, 1, \ldots, m-1\}$ such that $|K| = l$. Let $I_K = (i_1, \ldots, i_{m-l}) \in \mathbb{Z}^{m-l}$ be such that $0 \leq i_1 < \cdots < i_{m-l} \leq m-1$ and $i_s \notin K$ for all s. Then we can write $\mathcal{B}_{-(m-l)} = \{Y_{I_K} \cdot w_0 \,|\, K \subseteq \{0, 1, \ldots, m-1\}, |K| = l\}$.

Since $\mathcal{B}_{-m} = \{Y_0 \cdots Y_{m-1} w_0\}$, this set is clearly linearly independent. Suppose the set $\mathcal{B}_{-(m-l)}$ is linearly independent for some $l \geq 0$. Let $u := \sum_K \lambda_K Y_{I_K} \cdot w_0 = 0$, where the sum is over all possible subsets $K \subseteq \{0, 1, \ldots, m-1\}$ of size l and $\lambda_K \in \mathbb{C}$. For each such K, let $J_K = (j_1, \ldots, j_{m-l}) \in \mathbb{Z}^{m-l}$ be such that $j_1 < \cdots < j_{m-l}$ and $j_s \notin K$ for all s. Then

$$0 = Y_{J_K} \cdot u = \lambda_K Y_{J_K} Y_{I_K} \cdot w_0 = (-1)^\alpha \lambda_K Y_0 Y_1 \cdots Y_{m-1} \cdot w_0 \ .$$

Since $Y_0 \cdots Y_{m-1} w_0$ is a basis vector of $W(\Lambda, m)^{-m}$ and we assumed that $W(\Lambda, m) \neq 0$, then $\lambda_K = 0$. Hence the set $\mathcal{B}_{-(m-l)}$ is linearly independent. □

Example 5.1.23. The weight diagram of $W(\Lambda, 3)$ is illustrated in Figure 5.1. There are four weight spaces. Each weight space is represented by a column of nodes, which are the basis vectors of the weight space. The arrows show how the elements of $U(\mathfrak{g} \otimes A)$ act on the weight spaces; X_n acts by moving a vector in $W(\Lambda, m)^i$ to $W(\Lambda, m)^{i+1}$, Y_n acts by moving a vector in $W(\Lambda, m)^i$ to $W(\Lambda, m)^{i-1}$, and Z_n acts by scalar multiplication by c_n. Note that $Y_n \cdot (Y_0 Y_1 Y_2 w_0) = 0$ and $X_n \cdot w_0 = 0$ for all $n \in \mathbb{Z}$.

Corollary 5.1.24. *Let $\mathcal{B}_0 := \{w_0\}$ and for $1 \leq k \leq m$, the set $\mathcal{B}_{-k}$ is a basis for*

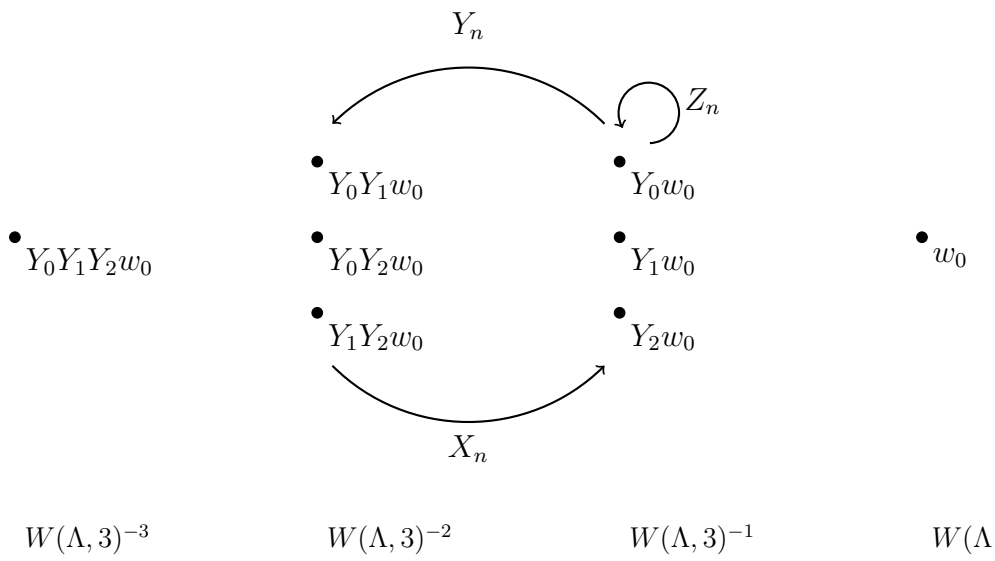


Figure 5.1: Weight diagram of $W(\Lambda, 3)$

$W(m,\Lambda)^{-k}$. *Furthermore, when the rank of* $[c_{i+j}]_{i,j\in\mathbb{Z}}$ *is* $k \leq m$, *then* $(Y \otimes t^{m-k}p(t)) \cdot w_0 = 0$, *where* $p(t)$ *is the minimal characteristic polynomial of the recurrence relation from Proposition 5.1.6. This is equivalent to saying that if* $g \cdot w_0 \in W(m,\Lambda)$, *then there exists a unique* $g_0 \in \mathfrak{g} \otimes \left(A\big/(t^{m-k}p(t))\right) = \mathfrak{g} \otimes \left(\mathbb{C}[t]\big/(t^{m-k}p(t))\right)$ *such that* $g \cdot w_0 = g_0 \cdot w_0$.

Proof. The first assertion follows directly from the proof of Proposition 5.1.7. Since $Y_m \cdot w_0 \in W(m,\Lambda)^{-1}$, then $Y_m \cdot w_0 \in \operatorname{span}\{Y_j w_0 \,|\, 0 \leq j \leq m-1\}$. i.e., there exist $\lambda_j \in \mathbb{C}$, not all 0, such that $Y_m \cdot w_0 = \sum_{j=0}^{m-1} \lambda_j Y_j \cdot w_0$. If we act by X_n for any $n \in \mathbb{Z}$, we get

$$c_{n+m}w_0 = \left(\sum_{j=0}^{m-1} \lambda_j c_{n+j}\right) w_0 \ .$$

Since we assume $W(m,\Lambda) \neq 0$, then $w_0 \neq 0$ and so

$$c_{n+m} = \sum_{j=0}^{m-1} \lambda_j c_{n+j} \ . \tag{5.10}$$

By Proposition 5.1.6 there is a degree $k \leq m$ linear recurrence relation on the c_i's with characteristic polynomial $p(t) = t^k - \sum_{j=0}^{k-1} \alpha_{k-j} t^j$. So we can write $c_{n+m} = \sum_{j=1}^{k} \alpha_j c_{n+m-j}$. Comparing to equation (5.10), we deduce that $\lambda_j = \alpha_{k-m+j}$ for $m-k \leq j \leq m-1$, and $\lambda_j = 0$ otherwise. So

$$(Y \otimes t^m) \cdot w_0 = \sum_{j=m-k}^{m-1} \alpha_{k-m+j} (Y \otimes t^j) \cdot w_0 \ .$$

This is the same as $(Y \otimes t^{m-k} p(t)) \cdot w_0 = 0$. □

Lemma 5.1.25. *Let $m \geq 1$. Let $S := \{0, \ldots, m-1\}$, $T := \{0, \ldots, m\}$, $I := (i_0, \ldots, i_{m-1}) \in \mathbb{Z}^m$, and $J := (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1}$. Then*

$$X_I Y_J \cdot w_0 = \sum_{k=0}^{m} (-1)^{\frac{m(m+1)}{2}+k} \det\left([c_{i_s+j_t}]_{s \in S, t \in T \setminus \{k\}} \right) Y_{j_k} \cdot w_0 \ .$$

Proof. We proceed by induction. Let $z := [x, y] = h' + h'' \in Z(\mathfrak{g})$. Also note that $[x, y] = xy + yx$, since $x, y \in \mathfrak{g}_{\overline{1}}$. In the case when $m = 1$, we have

$$\begin{aligned} X_{i_0} Y_{j_0} Y_{j_1} w_0 &= -Y_{j_0} X_{i_0} Y_{j_1} w_0 + [X_{i_0}, Y_{j_0}] Y_{j_1} w_0 \\ &= -Y_{j_0} [X_{i_0}, Y_{j_1}] w_0 + Y_{j_0} Y_{j_1} X_{i_0} w + Z_{i_0+j_0} Y_{j_1} w_0 \\ &= (-1)^{0+1} c_{i_0+j_1} Y_{j_0} w_0 + (-1)^{1+1} c_{i_0+j_0} Y_{j_1} w_0. \end{aligned}$$

Let $m - 1 \geq 1$ and suppose that for all $I' := (i_0, \ldots, i_{m-2}) \in \mathbb{Z}^{m-1}$, and $J' := (j_0, \ldots, j_{m-1}) \in \mathbb{Z}^m$

$$X_{I'} Y_{J'} \cdot w_0 = \sum_{k=0}^{m-1} (-1)^{\frac{m(m+1)}{2}+k} \det\left([c_{i_s+j_t}]_{s \in S, t \in T \setminus \{k\}} \right) Y_{j_k} \cdot w_0 \tag{5.11}$$

where $S := \{0, \ldots, m-2\}$ and $T := \{0, \ldots, m-1\}$. Let $I = (i_0, \ldots, i_{m-1}) \in \mathbb{Z}^m$ and $J := (j_0, \ldots, j_m) \in \mathbb{Z}^{m+1}$. Then

$$\begin{aligned}
&X_I Y_J \cdot w_0 = X_{I'} X_{i_{m-1}} Y_J \cdot w_0 \\
&= \sum_{l=0}^{m} (-1)^l X_{I'} Y_{j_0} \cdots [X_{i_{m-1}}, Y_{j_l}] \cdots Y_{j_m} \cdot w_0 \\
&= \sum_{l=0}^{m} (-1)^l X_{I'} Y_{j_0} \cdots \widehat{Y_{j_l}} \cdots Y_{j_m} Z_{i_{m-1}+j_l} \cdot w_0 \text{ since } z_{i_{m-1}+j_l} \in Z(\mathfrak{g} \otimes A) \text{ for all } l \geq 0 \\
&= \sum_{l=0}^{m} (-1)^l c_{i_{m-1}+j_l} X_{I'} Y_{j_0} \cdots \widehat{Y_{j_l}} \cdots Y_{j_m} \cdot w_0 \ .
\end{aligned}$$

Since for all $k \geq 0$ $X_{I'} Y_{j_0} \cdots \widehat{Y_{j_l}} \cdots Y_{j_m} \cdot w_0$ is of the form the left side of equation (5.11), by the induction

$$\begin{aligned}
&c_{i_{m-1}+j_l} X_{I'} Y_{j_0} \cdots \widehat{Y_{j_l}} \cdots Y_{j_m} \cdot w_0 \\
&= c_{i_{m-1}+j_l} \sum_{\substack{k=0 \\ k \neq l}}^{m} (-1)^{\frac{m(m+1)}{2}+k} \det\left([c_{i_s+j_t}]_{s \in S, t \in T \setminus \{k,l\}} \right) Y_k \cdot w_0.
\end{aligned}$$

So

$$\begin{aligned}
X_I Y_J \cdot w_0 &= \sum_{l=0}^{m} (-1)^l c_{i_{m-1}+j_l} X_{I'} Y_{j_0} \cdots \widehat{Y_{j_l}} \cdots Y_{j_m} \cdot w_0 \\
&= \sum_{l=1}^{m} \sum_{k=0}^{l-1} c_{i_{m-1}+j_l} (-1)^{\frac{m(m+1)}{2}+k+l} \det\left([c_{i_s+j_t}]_{s \in S, t \in T \setminus \{k,l\}} \right) Y_{j_k} \cdot w_0 \\
&+ \sum_{l=0}^{m} \sum_{k=l+1}^{m} c_{i_{m-1}+j_l} (-1)^{\frac{m(m+1)}{2}+k+l+1} \det\left([c_{i_s+j_t}]_{s \in S, t \in T \setminus \{l,k\}} \right) Y_{j_k} \cdot w_0 \ .
\end{aligned}$$

Since these are two finite sums, we may switch the order of summation. So

$$\begin{aligned}
X_I Y_J \cdot w_0 &= \sum_{k=0}^{m-1} \sum_{l=k+1}^{m} c_{i_{m-1}+j_l} (-1)^{\frac{(m-1)m}{2}+k+l} \det\left([c_{i_s+j_t}]_{s\in S, t\in T\setminus\{k,l\}}\right) Y_{j_k} \cdot w_0 \\
&+ \sum_{k=1}^{m} \sum_{l=0}^{k-1} c_{i_{m-1}+j_l} (-1)^{\frac{(m-1)m}{2}+k+l+1} \det\left([c_{i_s+j_t}]_{s\in S, t\in T\setminus\{l,k\}}\right) Y_{j_k} \cdot w_0 \\
&= \sum_{k=0}^{m-1} (-1)^{m-1} (-1)^{\frac{(m-1)m}{2}+k} \sum_{l=k+1}^{m} (-1)^{m+l} c_{i_{m-1}+j_l} \det\left([c_{i_s+j_t}]_{s\in S, t\in T\setminus\{k,l\}}\right) Y_{j_k} \cdot w_0 \\
&+ \sum_{k=1}^{m} (-1)^{m} (-1)^{\frac{(m-1)m}{2}+k} \sum_{l=0}^{k-1} c_{i_{m-1}+j_l} (-1)^{m+l+1} \det\left([c_{i_s+j_t}]_{s\in S, t\in T\setminus\{l,k\}}\right) Y_{j_k} \cdot w_0 \\
&= \sum_{k=0}^{m} (-1)^{\frac{m(m+1)}{2}+k} \det\left([c_{i_s+j_t}]_{s\in S\cup\{i_{m-1}\}, t\in T\setminus\{k\}}\right) Y_{j_k} \cdot w_0,
\end{aligned}$$

where the last equality is obtained by expanding the determinant of the matrix $[c_{i_s+j_t}]_{s\in S\cup\{i_{m-1}\}, t\in T\setminus\{k\}}$ along the m^{th} row. □

Lemma 5.1.26. *Let $l \geq 2$ and $m \geq l$.*

Let $Y_{i_s} w_0 = \lambda_{s,0} Y_0 w_0 + \lambda_{s,1} Y_1 w_0 + \cdots + \lambda_{s,m-1} Y_{m-1} w_0$ for $s = 0, 1, \ldots, m-1$. Let M denote the $m \times m$ matrix

$$M = [\lambda_{k,j}]_{k,j=0}^{m-1} \ .$$

Let $K = (k_0, \ldots, k_{l-1})$, where $0 \leq k_0 < k_1 < \cdots < k_{l-1} \leq m$, and let $J = (j_0, \ldots, j_{l-1})$ where $0 \leq j_0 < j_1 < \cdots < j_{l-1} \leq m-1$. Let $\det({}_lM_{K,J})$ denote the $l \times l$ minor of M indexed by rows $k_1, k_2, \ldots, k_l$ and columns $j_1, j_2, \ldots, j_l$ are deleted. For $J \in J_l = \{J = (j_0, \ldots, j_{l-1}) \,|\, 0 \leq j_0 < j_1 < \cdots < j_{l-1} \leq m-1\}$, let $Y_J := Y_{j_0} Y_{j_1} \cdots Y_{j_{l-1}}$. Then for all $i_0, i_1, \ldots, i_{l-1} \in \mathbb{Z}_{\geq 0}$,

$$Y_{i_0} Y_{i_1} \cdots Y_{i_{l-1}} w = \sum_{J\in J_l} \det({}_lM_{K,J}) Y_J w_0$$

where $K = (0, 1, \ldots, l-1)$.

Proof. We use induction on l. The case when $l = 2$ is as follows: Consider $y_{i_s} w$ for $s = 0, 1$ and $M = [\lambda_{k,j}]_{k,j=0}^{m-1}$. Let $\det({}_2M_{K,J})$ denote the 2×2 minor of M indexed by $K = \{0, 1\}$ and $J = \{j_0, j_1\}$. Let $i_0, i_1 \in \mathbb{Z}_{\geq 0}$. Then

$$Y_{i_0} Y_{i_1} w = Y_{i_0} \left(\sum_{n_1=0}^{m-1} \lambda_{1,n_1} Y_{n_1} w \right) = - \sum_{n_1=0}^{m-1} \lambda_{1,n_1} Y_{n_1} Y_{i_0} w_0 = - \sum_{n_1=0}^{m-1} \sum_{n_0=0}^{m-1} \lambda_{0,n_0} \lambda_{1,n_1} Y_{n_1} Y_{n_0} w_0$$
$$= \sum_{n_1=0}^{m-1} \sum_{n_0=0}^{m-1} \lambda_{0,n_0} \lambda_{1,n_1} Y_{n_0} Y_{n_1} w_0 \ .$$

Now we change variables so that in the sum above all the terms are in the order $y_{j_0} y_{j_1} w$ where $0 \leq j_0 < j_1 \leq m-1$. We split the sum into two parts: $n_0 < n_1$ and $n_0 > n_1$. Then

$$Y_{i_0} Y_{i_1} w = \sum_{n_1=0}^{m-1} \sum_{n_0=0}^{n_1-1} \lambda_{0,n_0} \lambda_{1,n_1} Y_{n_0} Y_{n_1} w_0 + \sum_{n_1=0}^{m-2} \sum_{n_0=n_1+1}^{m-1} \lambda_{0,n_0} \lambda_{1,n_1} Y_{n_0} Y_{n_1} w_0$$
$$= \sum_{n_1=1}^{m-1} \sum_{n_0=0}^{n_1-1} \lambda_{0,n_0} \lambda_{1,n_1} Y_{n_0} Y_{n_1} w_0 - \sum_{n_1=0}^{m-2} \sum_{n_0=n_1+1}^{m-1} \lambda_{0,n_0} \lambda_{1,n_1} Y_{n_1} Y_{n_0} w_0 \ .$$

In the first sum, let $j_0 = n_0$, $j_1 = n_1$ and in the second sum, let $j_0 = n_1$ and $j_1 = n_0$. Then

$$Y_{i_0} Y_{i_1} w = \sum_{j_1=1}^{m-1} \sum_{j_0=0}^{j_1-1} \lambda_{0,j_0} \lambda_{1,j_1} Y_{j_0} Y_{j_1} w_0 - \sum_{j_0=0}^{m-2} \sum_{j_1=j_0+1}^{m-1} \lambda_{0,j_1} \lambda_{1,j_0} Y_{j_0} Y_{j_1} w_0 \ .$$

In the second sum, changing the order of summation gives

$$Y_{i_0} Y_{i_1} w = \sum_{j_1=1}^{m-1} \sum_{j_0=0}^{j_1-1} \lambda_{0,j_0} \lambda_{1,j_1} Y_{j_0} Y_{j_1} w_0 - \sum_{j_1=1}^{m-1} \sum_{j_0=0}^{j_1-1} \lambda_{0,j_1} \lambda_{1,j_0} Y_{j_0} Y_{j_1} w_0$$

$$= \sum_{j_1=1}^{m-1} \sum_{j_0=0}^{j_1-1} (\lambda_{0,j_0}\lambda_{1,j_1} - \lambda_{0,j_1}\lambda_{1,j_0}) Y_{j_0} Y_{j_1} w_0$$
$$= \sum_{J \in J_2, K=(0,1)} \det\left({}_2M_{J,K}\right) Y_J w_0 \ .$$

Now assume this holds for some $l \geq 2$. Let $K = (0, 1, ..., l-1)$ and $K' = (0, 1, \ldots, l)$. Let $i_0, i_1, \ldots, i_l \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned} Y_{i_0} Y_{i_1} \cdots Y_{i_{l-1}} Y_{i_l} w &= (-1)^l Y_{i_l} Y_{i_0} Y_{i_1} \cdots Y_{i_{l-1}} w_0 \\ &= (-1)^l Y_{i_l} \sum_{J \in J_l} \det\left({}_lM_{J,K}\right) Y_J w_0 \text{ (by induction)} \\ &= \sum_{J \in J_l} \det\left({}_lM_{J,K}\right) Y_J Y_{i_l} w_0 \ . \end{aligned}$$

By the first part of the proof of Proposition 5.1.7 $Y_{i_l} w_0 = \sum_{j=0}^{m-1} \lambda_{l,j} Y_j w_0$ for some $\lambda_{l,j} \in \mathbb{C}$. So

$$Y_{i_0} Y_{i_1} \cdots Y_{i_{l-1}} Y_{i_l} w_0 = \sum_{J \in J_l} \sum_{j=0}^{m-1} \lambda_{l,j} \det\left({}_lM_{J,K}\right) Y_J Y_j w_0 \ .$$

To obtain the $(l+1)\times(l+1)$ minor $\det({}_{l+1}M_{K',J'})$, from $\det\left({}_lM_{J,K}\right)$, we need to use rows $0, 1, \ldots, l$ and columns $j'_0, j'_1, \ldots, j'_l$ from the matrix M, where $J' = (j'_0, j'_1, \ldots, j'_l) \in J_{l+1}$. Each l-tuple J can be made into an $(l+1)$-tuple by adjoining one more element. Since in this context all elements in an $(l+1)$-tuple are distinct, there are $m - l$ elements to choose from. Moreover, there are $\binom{m}{l}$ many l-tuples, which gives a total of $\binom{m}{l}(m-l)$ distinct pairs (J, j_l) such that $J = (j_0, j_1, \ldots, j_{l-1}) \in J_l$, $0 \leq j_l \leq m-1$, and $j_l \neq j_i$ for $i = 0, 1, \ldots, l-1$. In other words, there are $\binom{m}{l}(m-l)$ many distinct terms $\det({}_lM_{K,J'})\lambda_{l,j_l} y'_J w$, where $J' \in J_{l+1}$; or $\binom{m}{l}(m-l)$ many distinct pairs (J, j_l) such

that $(j_0, j_1, \ldots, j_{l-1}) = J \in J_l$ and $0 \leq j_l \leq m-1$ and $j_l \neq j_t$ for $t = 0, 1, \ldots, l-1$. Moreover, each $(l+1)$-tuple can be formed in this way from $l+1$ distinct l-tuples. (i.e., the $(l+1)$-tuple $(j'_0, j'_1, \ldots, j'_l)$ can be formed from $(j'_0, \ldots, \widehat{j'_s}, \ldots, j'_l)$ where $0 \leq s \leq l$). Consider the set $\{J, j_l\} = \{j_0, j_1, \ldots, j_l\}$, which comes from the pair (J, j_l). When we look at all of the sets $\{j_0, \ldots, j_{l-1}, j_l\}$ which come from all $\binom{m}{l}(m-l)$ distinct pairs (J, j_l), then we get $\binom{m}{l}(m-l)\dfrac{1}{l+1} = \binom{m}{l+1}$ distinct sets. There are exactly $\binom{m}{l+1}$ many distinct $(l+1)$-tuples. Hence the sum

$$\sum_{J \in J_l} \sum_{j=0}^{m-1} \lambda_{l,j} \det\left({}_lM_{J,K}\right) Y_J Y_j w$$

can be rewritten as

$$\begin{aligned}
&\sum_{J' \in J_{l+1}} \sum_{s=0}^{l} \det\left({}_lM_{K,(j'_0,\ldots,\widehat{j'_s},\ldots,j'_l)}\right) \lambda_{l,j_s} Y_{j'_0} \cdots \widehat{Y_{j'_s}} \cdots Y_{j'_l} Y_{j'_s} w \\
&= \sum_{J' \in J_{l+1}} \sum_{s=0}^{l} (-1)^{l-s} \det\left({}_lM_{K,(j'_0,\ldots,\widehat{j'_s},\ldots,j'_l)}\right) \lambda_{l,j_s} Y_{j'_0} \cdots Y_{j'_s} \cdots Y_{j'_l} w \\
&= \sum_{J' \in J_{l+1}} \left(\sum_{s=0}^{l} (-1)^{(l+1)+(s+1)} \det\left({}_lM_{K,(j'_0,\ldots,\widehat{j'_s},\ldots,j'_l)}\right) \lambda_{l,j_s} \right) Y_{J'} w \\
&= \sum_{J \in J_l} \left(\sum_{j_l=0}^{m-1} (-1)^{(l+1)+(s+1)} \lambda_{l,j_l} \det\left({}_lM_{K,J}\right) \right) Y_J Y_{j_l} w \\
&= \sum_{J' \in J_{l+1}} \det\left({}_{l+1}M_{K,J}\right) Y_{J'} w \ .
\end{aligned}$$

□

In Proposition 4.1 of [CFR21], Calixto, Futorny, and Rocha show that for a basic classical Lie superalgebra $\mathfrak{g}$ such that $\mathfrak{g}_{\bar{0}}$ is semisimple, the annihilator $\mathrm{Ann}_A V$ of a simple bounded weight $g \otimes A$-module V is radical. An irreducible Weyl module of

$\mathfrak{gl}(1|1) \otimes \mathbb{C}[t, t^{-1}]$ is an example of such a module. In the following proposition, we explicitly write the annihilator of the loop Weyl module.

Corollary 5.1.27. *Let $\mathfrak{g} = \mathfrak{sl}(1|1) \otimes A$ and let $W(\Lambda, m)$ be a non-zero simple Weyl module. Then $\mathrm{Ann}_A W(\Lambda, m) = (p(t))$, where $p(t)$ is the characteristic equation of the recurrence relation on $\mathbf{c} = \{c_i\}_{i \in \mathbb{Z}}$.*

Proof. Let w_0 be the generator of $W(\Lambda, m)$. We first show that $\mathrm{Ann}_{\mathbb{C}[t,t^{-1}]} w_0 = (p(t))$. Recall that $(\mathfrak{n}^+ \otimes A) \cdot w_0 = 0$. Furthermore, since $p(t)$ is the characteristic polynomial of the recurrence relation $c_{k+m} = \alpha_1 c_{k+m-1} + \alpha_2 c_{k+m-2} + \cdots + \alpha_m c_k$ for all $k \in \mathbb{Z}$, then $p(t) = t^m - \sum_{i=1}^m \alpha_i t^{m-i}$. Recall from Definition 5.1.1 that $z \otimes t^n \cdot w_0 = c_n w_0$ for any $n \in \mathbb{Z}$. So for any $k \in \mathbb{Z}$,

$$\begin{aligned} Z \otimes t^k p(t) \cdot w_0 &= Z \otimes \left(t^{m+k} - \sum_{i=1}^m \alpha_i t^{m+k-i} \right) \cdot w_0 \\ &= Z \otimes t^{m+k} \cdot w_0 - \sum_{i=1}^m \alpha_i \left(Z \otimes t^{m+k-i} \cdot w_0 \right) \\ &= (c_{m+k} - (\alpha_1 c_{m+j-1} + \alpha_2 c_{m+k-2} + \cdots + \alpha_m c_k)) \cdot w_0 = 0. \end{aligned}$$

Then for any $q(t) = \sum_k \beta_k t^k \in A$,

$$\begin{aligned} (Z \otimes q(t)p(t)) \cdot w_0 &= (z \otimes \left(\sum_k \beta_k t^k \right) (t^m - \alpha_1 t^{m-1} - \alpha_2 t^{m-2} - \cdots - \alpha_m)) \cdot w_0 \\ &= \left(Z \otimes \left(\sum_k \beta_k t^k \right) (t^m - \alpha_1 t^{m-1} - \alpha_2 t^{m-2} - \cdots - \alpha_m) \right) \cdot w_0 \\ &= \sum_k \beta_k \left(z \otimes t^k p(t) \right) \cdot w_0 = 0 \ . \end{aligned}$$

Finally, suppose $(Y \otimes p(t)) \cdot w_0 \neq 0$. Since $W(\Lambda, m)$ is simple, then there exists

$n \in \mathbb{Z}$ such that $(X \otimes t^n)(Y \otimes p(t)) \cdot w_0 = cw_0$ for some $c \in \mathbb{C}^*$. Fix such an n. Then

$$(X \otimes t^n)(Y \otimes p(t)) \cdot w_0 = (-YX + [X, Y]) \otimes t^n p(t)) \cdot w_0 = (Z \otimes t^n p(t)) \cdot w_0 = 0 \ .$$

This contradicts the fact that $W(\Lambda, m)$ is simple. Hence $(Y \otimes p(t)) \cdot w_0 = 0$, and $p(t) \subseteq \mathrm{Ann}_{\mathbb{C}[t,t^{-1}]} w_0$.

Now let $q(t) := \sum_{i=0}^{n} \beta_i t^i \in \mathrm{Ann}_{\mathbb{C}[t,t^{-1}]} w_0$, where $\beta_n \neq 0$. Then

$$0 = (Z \otimes q(t)) \cdot w_0 = (\beta_0 c_0 + \beta_1 c_1 + \cdots \beta_n c_n) \cdot w_0 \ .$$

This gives us the recurrence relation $c_n = \dfrac{-\beta_0}{\beta_n} c_0 - \dfrac{\beta_1}{\beta_n} c_1 - \cdots - \dfrac{\beta_{n-1}}{\beta_n} c_{n-1}$. Since $W \neq 0$, by part 1. of Theorem 5.1.2, there exists a degree m recurrence relation on the c_i's. So $n \geq m$ and $q(t) \in (p(t))$. Hence $(p(t)) = \mathrm{Ann}_{\mathbb{C}[t,t^{-1}]} w_0$.

Since W is a weight module, every non-zero element of $W(\Lambda, m)$ can be written as a sum of homogeneous elements. Each such homogeneous element can be written as a linear combination of the basis vectors of its corresponding weight space, which are of the form $(Y \otimes t^{j_0})(Y \otimes t^{j_1}) \cdots (Y \otimes t^{j_n}) \cdot w_0$ where $j_0, \ldots, j_n \in \mathbb{Z}$. We can see that

$$\begin{aligned}
&(X \otimes p(t))(Y \otimes t^{j_0})(Y \otimes t^{j_1}) \cdots (Y \otimes t^{j_n}) \cdot w_0 \\
&= (X \otimes t^m)(Y \otimes t^{j_1}) \cdots (Y \otimes t^{j_n}) \cdot w_0 - \sum_{i=0}^{m-1} \alpha_i (X \otimes t^i)(Y \otimes t^{j_1}) \cdots (Y \otimes t^{j_n}) \cdot w_0 \\
&= \sum_{s=0}^{n} (-1)^s \left(Y_{j_0} \cdots [X_m, Y_{j_s}] \cdots Y_{j_n} \cdot w - \sum_{i=0}^{m-1} \alpha_i Y_{j_0} \cdots [X_i, Y_{j_s}] \cdots Y_{j_n} \cdot w_0 \right) \\
&= \sum_{s=0}^{n} (-1)^s \left(Y_{j_0} \cdots \widehat{Y_{j_s}} \cdots Y_{j_n} Z_{m+j_s} \cdot w - \sum_{i=0}^{m-1} \alpha_i Y_{j_0} \cdots \widehat{Y_{j_s}} \cdots Y_{j_n} Z_{i+j_s} \cdot w_0 \right)
\end{aligned}$$

$$= \sum_{s=0}^{n} (-1)^s \left(c_{m+j_s} - \sum_{i=0}^{m-1} \alpha_i c_{i+j_s} \right) Y_{j_0} \cdots \widehat{Y_{j_s}} \cdots Y_{j_n} Z_{m+j_s} \cdot w_0 = 0 \ .$$

Furthermore, $(Y \otimes p(t))(Y \otimes t^{j_0})(Y \otimes t^{j_1}) \cdots (Y \otimes t^{j_n}) \cdot w_0 = (Y \otimes t^{j_0})(Y \otimes t^{j_1}) \cdots (Y \otimes t^{j_n})(Y \otimes p(t)) \cdot w_0 = 0$, as shown above. Finally, since $Z \in Z(\mathfrak{g})$, then $(Z \otimes p(t))(Y \otimes t^{j_0})(Y \otimes t^{j_1}) \cdots (Y \otimes t^{j_n}) \cdot w_0 = 0$.

□

Reducibility and Submodules of $W(\Lambda, m)$

This subsection discusses the reducibility of the $\mathfrak{sl}(1 \,|\, 1) \otimes A$-module $W(\Lambda, m)$. When $W(\Lambda, m)$ is reducible we find all of its nontrivial submodules.

Let $M = [c_{i+j}]_{i,j \in \mathbb{Z}}$ and $\mathbf{c} = \{c_i \,|\, i \in \mathbb{Z}\}$. By Proposition 5.1.6, there is a $k \leq m$ such that $\mathbf{c}$ satisfies a linear recurrence relation of degree k and does not satisfy any linear recurrence relations of lesser degree. Let $p(t)$ be the minimal characteristic polynomial of this recurrence relation.

Proposition 5.1.28. *Suppose that the sequence* $\mathbf{c}$ *associated to the module* $W(\Lambda, m)$ *satisfies an order* $k \leq m$ *linear recurrence relation with constant coefficients, and does not satisfy any linear recurrence relations of lesser degree. Then* $W(\Lambda, m)$ *is simple if and only if* $k = m$*. If* $k < m$*, then* $W(\Lambda, m)$ *has the following filtration:*

$$0 = W_{m-k} \subset W_{m-k-1} \subset \cdots \subset W_1 \subset W_0 = W(\Lambda, m)$$

where $W_i / W_{i+1} \cong \binom{m-k}{i} W(\Lambda, k)[-i]$.

Proof. Let $s : m - k$ and let $I \subseteq [s] = \{1, \ldots, s\}$. Label the highest weight vector as

$w_{[s]} := w_0$ and for nonempty subsets $I \subseteq [s]$ let

$$w_{I^c} = \prod_{i \in I} (Y \otimes t^{i-1} p(t)) \cdot w_0 \in W(\Lambda, m)^{-|I|} \ .$$

Since $[x \otimes t^i, y \otimes t^j] = z \otimes t^{i+j}$ and $Z \otimes t^{i+j}$ is in the centre of $U(\mathfrak{sl}(1 \,|\, 1) \otimes A)$ and $(Z \otimes p(t)) \cdot w_0 = 0$, then it is clear that $(\mathfrak{n}^+ \otimes A) \cdot w_{I^c} = 0$ for all such sets I. We only consider vectors made up by elements of the form $Y \otimes t^n p(t)$ where $n \leq s-1$, since by Corollary 5.1.24 we are working with the algebra $\mathfrak{g} \otimes \left(\mathbb{C}[t] \big/ (t^s p(t)) \right)$.

Suppose $k < m$. By Proposition 5.1.6 and since $\deg p(t) = k$, then

$w_{I^c} \in \operatorname{span} \mathcal{B}_{-|I|} = \{ Y_{j_1} \cdots Y_{j_n} \cdot w_0 \,|\, 0 \leq j_1 < \cdots < j_{|I|} \leq m-1 \ \}$.

i.e., $0 \neq w_{I^c} \in W(m, \Lambda)^{-|I|}$. Hence each subset of I produces a unique singular vector in $W(m, \Lambda)^{-|I|}$ which generates a nontrivial submodule of $W(m, \Lambda)$.

Proposition 5.1.4 states that $W(\Lambda, m)^{-(m+1)} = 0$. Hence if $w_{I^c} \in W(\Lambda, m)^{-|I|}$ then $Y_0 Y_1 \cdots Y_{m-l} \cdot w_{I^c} \in W(\Lambda, m)^{-(m+1)} = 0$. Therefore w_{I^c} generates a unique $\mathfrak{sl}(1 \,|\, 1) \otimes A$ module $X_{I^c} \cong W(\Lambda, m - |I|)[-|I|]$. There are $\binom{m-k}{|I|}$ such submodules. Furthermore, if $I, J \subseteq [s]$, then $X_{I^c} \cap X_{J^c} = X_{I^c \cap J^c} \cong W(\Lambda, m - |I \cup J|)[-|I \cup J|]$ by construction.

To show that these are the only non-trivial submodules, suppose there is a non-zero vector $v = \prod_q (Y \otimes q(t)) \cdot w_0 \in W(m, \Lambda)^{-l}$ for some $1 \leq l \leq m$, where v is written in terms of basis vectors $\mathcal{B}_{-l}$, such that $(\mathfrak{n}^+ \otimes A) \cdot v = 0$. Then each $q(t)$ is a characteristic polynomial for a recursion satisfied by $\mathbf{c}$, so $p(t)$ divides $q(t)$. Hence v is a linear combination of vectors of the form w_{I^c} for some $I \subseteq [s]$ of size l.

Now suppose $k = m$. For the sake of contradiction, assume that $W(\Lambda, m)$ is not simple. Then there exists a non-zero vector $v \in W(\Lambda, m)^{-l}$ for some $1 \leq l \leq m$ such

that $(\mathfrak{n}^+ \otimes A) \cdot v = 0$. By Corollary 5.1.24, v is of the form

$$v = \prod_{i=1}^{l} (Y \otimes q_i(t)) \cdot w_0,$$

where $Y \otimes q_i(t)$ is a non-zero linear combination of vectors in the set $\{Y \otimes t^j \,|\, 0 \leq j \leq m-1\}$. i.e., $\deg q_i(t) \leq m-1$. Since $(\mathfrak{n}^+ \otimes A) \cdot v = 0$, then the sequence $\mathbf{c}$ satisfies each polynomial $q_i(t)$ implying that $\deg q_i(t) \geq m$, which is a contradiction. Hence $W(\Lambda, m)$ is simple.

In the other direction, we know that rank$M \leq m$ by Lemma 5.1.6. By above $W(\Lambda, m)$ is simple only if rank$M = m$. □

Remark 5.1.29. By Proposition 5.1.28, $W_1 \cong \mathbb{C}^{m-k} \otimes W(\Lambda, m-1)[-1]$ is a proper submodule of $W(\Lambda, m)$. In the proof of Proposition 5.1.28, we explicitly wrote down $m-k$ linearly independent singular vectors of degree -1 which generate W_1. These vectors were $w_{[m-k]\setminus\{i\}} := (Y \otimes t^{i-1}p(t)) \cdot w_0$ where $i \in [m-k]$. Let

$$X_{[m-k]\setminus\{i\}} := U(\mathfrak{g} \otimes A) \cdot w_0$$

be the $U(\mathfrak{g} \otimes A)$-submodule of $W(\Lambda, m)$ generated by $w_{[m-k]\setminus\{i\}}$. Then

$$W_1 = \sum_{i=1}^{m-k} X_{[m-k]\setminus\{i\}} \ .$$

Now consider the set $\mathcal{B} = \{Y \otimes t^{i-1}p(t) \,|\, 1 \leq i \leq m-k\} \subset U(\mathfrak{g} \otimes A)$. The linear independence of $\mathcal{B}$ follows from the linear independence of the polynomials $\{t^{i-1}p(t) \,|\, 1 \leq i \leq m-k\}$. Hence $\text{span}_{\mathbb{C}}\mathcal{B} \cong \mathbb{C}^{m-k}$ as vector spaces. With this choice

of basis for $\mathbb{C}^{m-k}$,

$$W_i = \bigwedge^i(\mathbb{C}^{m-k}) \otimes W(\Lambda, m-i)[-i] \ .$$

Furthermore, if $I, J \subseteq [m-k]$ then this choice of basis allows us to see that $X_I \cap X_J = X_{I \cap J}$ and we may construct a lattice of submodules of $W(\Lambda, m)$. This is illustrated in the following example.

Example 5.1.30. Consider the Weyl module $W(\Lambda, 5)$, where $\text{rank}[c_{i+j}]_{i,j\in\mathbb{Z}} = 2$. Then $W(\Lambda, 5)$ has the following submodules:

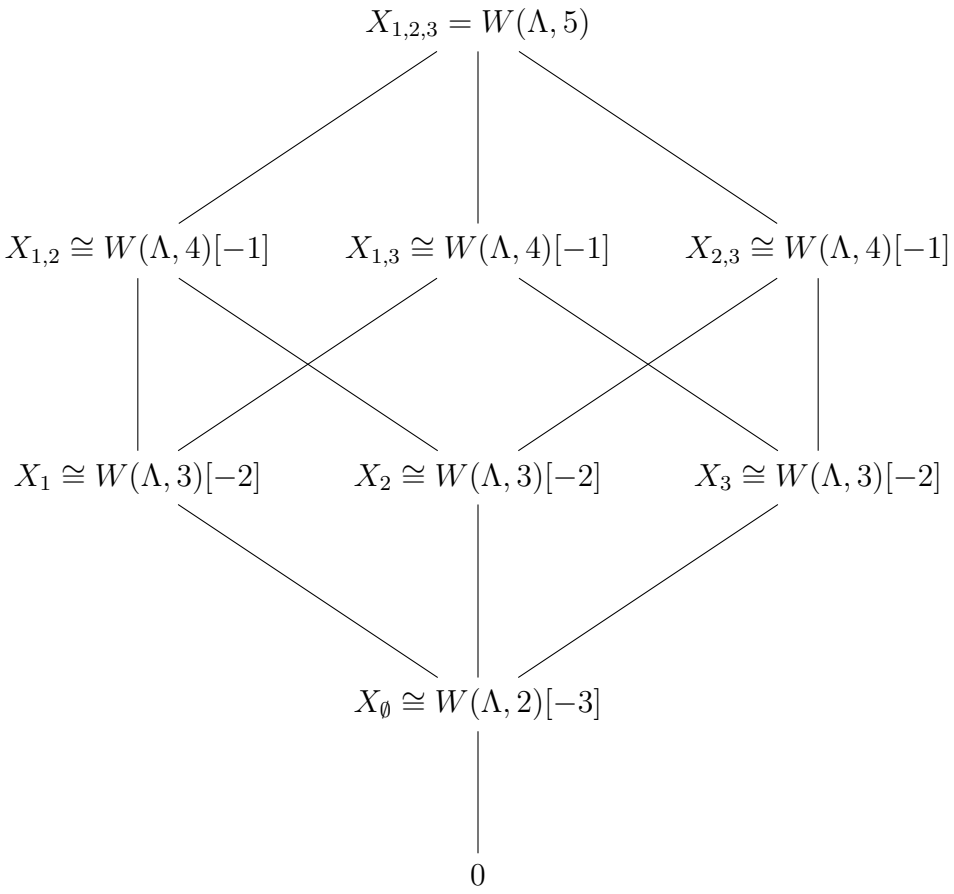


Figure 5.2: Lattice of submodules of $W(\Lambda, 5)$, when $\text{rank}[c_{i+j}]_{i,j\in\mathbb{Z}} = 2$.

5.2 $\mathbb{Z} \times \mathbb{Z}$ graded modules of the loop algebra $\mathfrak{sl}(1|1) \otimes \mathbb{C}[t, t^{-1}]$

Let $\mathfrak{g} = \mathfrak{sl}(1|1)$ with the standard basis $\{x, y, z\}$ so that

$$[x, y] = z, \text{ and } [x, x] = [y, y] = 0.$$

The Lie superalgebra $\mathfrak{g}$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \; y \in \mathfrak{n}^-, z \in \mathfrak{h}, x \in \mathfrak{n}^+.$$

Recall that for any Lie superalgebra $\mathfrak{l}$, the loop Lie superalgebra is defined as $\overline{\mathfrak{l}} = \mathfrak{l} \otimes \mathbb{C}[t, t^{-1}]$. The decomposition (2.1) also applies to Lie superalgebras, and defines a decomposition of the loop algebra $\overline{\mathfrak{g}}$ given by

$$\overline{\mathfrak{g}} = \overline{\mathfrak{n}}^- \oplus \overline{\mathfrak{h}} \oplus \overline{\mathfrak{n}}^+ \; .$$

Note that this gradation is compatible with the Lie bracket, so $\overline{\mathfrak{g}}$ is well-defined as a $\mathbb{Z} \times \mathbb{Z}$-graded algebra. We have studied two types of modules, which will be discussed below. From now on, we use the notation $g_k := g \otimes t^k \in \overline{\mathfrak{g}}$ for $g \in \mathfrak{g}$, $k \in \mathbb{Z}$.

5.2.1 Finite Dimensional $\mathbb{Z}$-graded $\overline{\mathfrak{g}}$-modules

In the previous section, we considered the $\mathbb{Z}$-grading on $\overline{\mathfrak{g}}$ and studied finite dimensional $\mathbb{Z}$-graded $\overline{\mathfrak{g}}$-modules. Let M be such a module. Since M has finite support, there is a

largest $\mu_0 \in \text{supp}M$ such that there exists $0 \neq w_0 \in M_{\mu_0}$ with the properties

$$(\mathfrak{n}^+ \otimes \mathbb{C}[t, t^{-1}]) \cdot w_0 = 0, \tag{5.12}$$

$$Z_n \cdot w_0 = a_n w_0,\ a_n \in \mathbb{C} \tag{5.13}$$

$$Y_I \cdot w_0 = 0 \text{ for all } I \in \mathbb{Z}^{m+1}. \tag{5.14}$$

We showed that $Y_I \cdot w_0 = 0$ for all $I \in \mathbb{Z}^{m+1}$ is equivalent to $Y_0 Y_1 \cdots Y_m \cdot w_0 = 0$. This gives us a $\overline{\mathfrak{g}}$-module with generators and relations. Using these generators and relations, we proved each weight space of M is finite-dimensional. In fact, we showed that $\dim M = 2^m$ and $\dim M^{-k} = \binom{m}{k}$ for $0 \leq k \leq m$.

5.2.2 $\mathbb{Z} \times \mathbb{Z}$-graded $\overline{\mathfrak{g}}$-modules with finite $\mathbb{Z}$-support

Let M be a $\mathbb{Z} \times \mathbb{Z}$ graded $\overline{\mathfrak{g}}$-module. Let $M^{(k,l)}$ denote the (k, l)th graded component of M, where k is the grading in the horizontal direction, and l, also referred to as the "level", is the grading in the vertical direction. We define the $\mathbb{Z}$-support of M in the horizontal direction by the set $\{k \in \mathbb{Z} \mid M^{(k,l)} \neq 0 \text{ for some } l \in \mathbb{Z}\}$. We consider the $\mathbb{Z} \times \mathbb{Z}$ graded $\overline{\mathfrak{g}}$-modules M with finite $\mathbb{Z}$-support in the horizontal direction, meaning that there exists $N \in \mathbb{Z}_{\geq 0}$ such that $M^{(\pm k,l)} = 0$ for all $l \in \mathbb{Z}$ and for all $k \geq N$. We will describe a basis, and derive an explicit formula for the character of such a module. Let $\mathfrak{b}$ denote the Borel subalgebra of $\mathfrak{g}$. For any $\mathbb{Z}$-graded $\overline{\mathfrak{b}}$-module L define

$$M(L) := \text{Ind}_{\overline{\mathfrak{b}}}^{\overline{\mathfrak{g}}} L = U(\overline{\mathfrak{g}}) \otimes_{U(\overline{\mathfrak{b}})} L\ .$$

This definition makes sense since L is also a $U(\mathfrak{b})$-module. By the Poincaré-Birkhoff-Witt theorem, as $U(\overline{\mathfrak{n}}^-)$-modules, $M(L) \cong U(\overline{\mathfrak{n}}^-) \otimes L$. Hence $M(L)$ admits

a $\mathbb{Z} \times \mathbb{Z}$ gradation via

$$\deg(Y_{i_1} \cdots Y_{i_k} \cdot w) = (k, \deg w + (i_1 + \cdots + i_k)), \tag{5.15}$$

where $y_i \in \overline{\mathfrak{n}}^-$ and $w \in L$. Let $I(L)$ be the unique maximal submodule of $M(L)$ such that $I(L) \cap L = 0$. ($I(L)$ can be taken as the sum of all submodules of $M(L)$ which intersect L trivially). Note that we do not consider maximal submodules I of $M(L)$ such that $I \cap L \neq 0$, since there is no such unique maximal submodule. We define

$$V(L) := M(L)/I(L) \ .$$

The following lemma states precisely when $V(L)$ is irreducible.

Lemma 5.2.1. *The following statements are equivalent:*

(i) $V(L)$ is an irreducible $\overline{\mathfrak{g}}$-module.

(ii) L is an irreducible $\overline{\mathfrak{b}}$-module.

(iii) L is an irreducible $\overline{\mathfrak{h}}$-module.

Proof. We know that $V(L) := M(L)/I(L)$ is irreducible if and only if $I(L)$ is a maximal $\overline{\mathfrak{g}}$-submodule of $M(L)$.

(i) implies (iii):

If $I(L)$ is a maximal submodule, then since $I(L) \cap L = 0$, there is no proper non-zero $\overline{\mathfrak{h}}$- submodule $W \subset L$ such that $I(L) \subseteq U(\overline{\mathfrak{n}}^-) \cdot W$.

(iii) implies (i)

Claim: $I(L)$ is the unique maximal $\overline{\mathfrak{g}}$-submodule of $M(L)$. Indeed, suppose there

exists a submodule $Y(L) \subseteq M(L)$ such that $Y(L) \cap L \neq 0$. Since L is an irreducible $\overline{\mathfrak{h}}$-module, then $L \subseteq Y(L)$, which implies that $Y(L) = M(L)$. Hence $I(L)$ is the unique maximal $\overline{\mathfrak{g}}$-submodule of $M(L)$, and $V(L)$ is the unique irreducible quotient.

(iii) implies (ii):

Suppose L is an irreducible $\overline{\mathfrak{h}}$-module. Then L is also an irreducible $U(\overline{\mathfrak{h}})$-module. So for any $v, w \in L$, there exists $Z \in U(\overline{\mathfrak{h}})$ such that $Z \cdot v = w$. Now suppose there exists a non-zero $U(\overline{\mathfrak{b}})$-submodule of L, say W. Let $w \in W$. Then by the above, for every $v \in (L\ W) \cup \{0\}$, there exists $Z \in U(\overline{\mathfrak{h}})$ such that $Z \cdot w = v$. So $v \in W$, which implies that $v = 0$. Hence $W = L$, and L is an irreducible $U(\overline{\mathfrak{b}})$-module.

(ii) implies (iii):

Suppose L is an irreducible $\mathbb{Z}$-graded $\overline{\mathfrak{b}}$-module with respect to the grading $\deg(X \otimes t^n) = 1$ and $\deg(Z \otimes t^n) = 0$. We can write $L = \bigoplus_{i \in \mathbb{Z}} L_i$.

Claim: For all $v \in L$, $X \otimes t^n \cdot v = 0$.

Proof of claim: Suppose not. i.e., suppose there exists $k \in \mathbb{Z}$ and $v \in L_k$ such that $0 \neq X \otimes t^n \cdot v \in L_{k+1}$. Then $L^k = \bigoplus_{i \geq k} L_i$ is a $\overline{\mathfrak{b}}$-submodule of L. This contradicts the irreducibility of L as a $\overline{\mathfrak{b}}$-module. Let $W \subset L$ be a proper $\overline{\mathfrak{h}}$-submodule. Let $w \in W$ and $v \in (L \setminus W) \cup \{0\}$. Since L is an irreducible $\overline{\mathfrak{b}}$-module, then there exists $Z \in \overline{\mathfrak{b}}$ such that $Z \cdot w = v$. In particular, $Z = Z_n$ for some $n \in \mathbb{Z}$. Since W is a proper $\overline{\mathfrak{h}}$-submodule, then $v = w = 0$. □

5.2.3 Irreducible $\mathbb{Z}$-graded $\overline{\mathfrak{h}}$-modules

In order to study irreducible $\overline{\mathfrak{g}}$-modules $V(L)$, the previous lemma showed that must choose L to be an irreducible $\overline{\mathfrak{h}}$-module. Furthermore, for $V(L)$ to have finite-dimensional graded components, we must choose L to have finite-dimensional graded

components. Hence, let L be as in Proposition 3.2.1 and suppose that $L_0 \neq 0$. By Proposition 3.2.1, we know that there exists an $r > 0$ such that

$$L = \bigoplus_{i \in \mathbb{Z}} L_i$$

where $\dim L_{ri} = 1$ for every $i \in \mathbb{Z}$. As in the proof of the proposition, if we choose a non-zero basis vector v_0 of L_0, then there exists $X \in \overline{\mathfrak{h}}^r$ and such that $v_{ir} := X^i \cdot v_0 \in L_{ir}$ is non-zero for all $i \in \mathbb{Z}_{>0}$. Similarly, there exists $Y \in \overline{\mathfrak{h}}^r$ such that $v_{-ir} := Y^i \cdot v_0 \in L_{-ir}$ is non-zero for all $i \in \mathbb{Z}_{>0}$. Hence we may fix a basis of L to be $\{v_{ri} \,|\, i \in \mathbb{Z}\}$, where $L_{ri} = \mathrm{span}_{\mathbb{C}}\{v_{ri}\}$.

For every $i \in \mathbb{Z}$, since L is a $\mathbb{Z}$-graded $\overline{\mathfrak{h}}$-module, write $z_i \cdot v_k =: a_i v_{i+k} \in L_{i+k}$, where $a_i \in \mathbb{C}$, $k \in r\mathbb{Z}$. Since $L_i = 0$ when $i \notin (r)$, then $a_i = 0$ if $r \nmid i$. Hence L defines a sequence $\{a_i\}_{i \in \mathbb{Z}}$, where $a_i : \mathbb{Z} \to \mathbb{C}$ is the product of q_r and some other function $g : \mathbb{Z} \to \mathbb{C}$.

5.2.4 General $\mathbb{Z}$-graded $\overline{\mathfrak{h}}$-modules

Definition 5.2.2. Let $\mathbf{a} := \{a_i\}_{i \in \mathbb{Z}}$ be a sequence with $a_i \in \mathbb{C}$ and define an $\overline{\mathfrak{h}}$-module $L(\mathbf{a}) = \mathrm{span}\{w_n \,|\, n \in \mathbb{Z}\}$ via

$$z_i \cdot w_n = a_i w_{n+i} \ .$$

We can consider $L(\mathbf{a})$ as a $\overline{\mathfrak{b}}$-module by letting $x_i \cdot w_n = 0$ for all $i, n \in \mathbb{Z}$. Then $L(\mathbf{a})$ is a $U(\overline{\mathfrak{b}})$-module as well, so we can define the $U(\overline{\mathfrak{g}})$-module

$$M(\mathbf{a}) := \mathrm{Ind}_{\overline{\mathfrak{b}}}^{\overline{\mathfrak{g}}} L(a) = U(\overline{\mathfrak{g}}) \otimes_{U(\overline{\mathfrak{b}})} L(\mathbf{a}) \ . \tag{5.16}$$

As before, $M(\mathbf{a}) \cong U(\overline{\mathfrak{n}}^-) \otimes L(\mathbf{a})$ as $U(\overline{\mathfrak{n}}^-)$-modules, so $M(\mathbf{a})$ admits a $\mathbb{Z} \times \mathbb{Z}$ gradation. Let $I(\mathbf{a})$ be the unique maximal submodule of $M(\mathbf{a})$ such that $I(\mathbf{a}) \cap L(\mathbf{a}) = 0$ and define

$$V(\mathbf{a}) := M(\mathbf{a})/I(\mathbf{a}) \ , \tag{5.17}$$

which inherits the $\mathbb{Z} \times \mathbb{Z}$ gradation of $M(\mathbf{a})$. Let $V(\mathbf{a})^{(k,l)}$ denote the (k,l) graded component of $V(\mathbf{a})$. Since $L(\mathbf{a})$ is a $\mathbb{Z}$-graded $\overline{\mathfrak{b}}$-module, by Lemma 5.2.1, $V(\mathbf{a})$ is irreducible if and only if $L(\mathbf{a})$ is an irreducible.

Proposition 5.2.3. *Let $V(\mathbf{a})$ be as in (5.17) and let $A = [a_{i+j}]_{i,j\in\mathbb{Z}}$. The following are equivalent:*

1. *The set $\{k \in \mathbb{Z} \mid V(\mathbf{a})^{(k,l)} \neq 0\}$ is finite.*
2. *$V(\mathbf{a})$ has finite dimensional graded subcomponents.*
3. *The rank of A is finite.*
4. *The sequence $\mathbf{a} = \{a_n\}$ is exponential-polynomial.*

Lemma 5.2.4. *Let $m \geq 1$. Let $S := \{0, \ldots, m-1\}$ and $T := \{0, \ldots, m\}$. Then*

$$\begin{aligned} &X_{i_{m-1}} \cdots X_{i_0} Y_{j_0} \cdots Y_{j_m} \cdot w_l \\ &= \sum_{k=0}^{m} (-1)^{m+k} \det\left([a_{i_s+j_t}]_{s\in S, t\in T\setminus\{k\}}\right) Y_{j_k} \cdot w_{l+i_0+\cdots+i_{m-1}+j_0+\cdots+\widehat{j_k}+\cdots+j_m}. \end{aligned}$$

Proof. (Of Lemma 5.2.4) We proceed by induction on m. In the case when $m = 1$,

$$\begin{aligned} X_{i_0} Y_{j_0} Y_{j_1} w_l &= -a_{i_0+j_1} Y_{j_0} w_{l+i0+j1} + a_{i_0+j_0} Y_{j_1} w_{l+i_0+j_0} \\ &= \sum_{k=0}^{1} (-1)^{k+1} \det\left([a_{i_s+j_t}]_{s\in\{0\}, t\in\{0,1\}\setminus\{k\}}\right) Y_{j_k} \cdot w_{l+i_0+j_0+\cdots+\widehat{j_k}+\cdots+j_1} \ . \end{aligned}$$

Now assume that this result holds for some $m-1 \geq 1$. Let $S = \{1, \ldots, m\}$ and $T = \{0, 1, \ldots, m\}$. Since $X_i Y_j = -Y_j X_i + Z_{i+j}$, we have

$$
\begin{aligned}
&X_{i_{m-1}} \cdots X_{i_0} Y_{j_0} \cdots Y_{j_m} \cdot w_l = \sum_{n=0}^{m} (-1)^n X_{i_{m-1}} \cdots X_{i_0} Y_{j_0} \cdots Z_{i_0+j_n} \cdots Y_{j_m} \cdot w_l \\
&= \sum_{n=0}^{m} (-1)^n a_{i_0+j_n} X_{i_{m-1}} \cdots X_{i_0} Y_{j_0} \cdots \widehat{Y_{j_n}} \cdots Y_{j_m} \cdot w_{l+i_0+j_n} \\
&= \sum_{n=0}^{m} (-1)^n a_{i_0+j_n} \sum_{k=0, k\neq n}^{m} (-1)^{m-1+k} \det\left([a_{i_s+j_t}]_{s\in S, t\in T\setminus\{k,n\}}\right) Y_{j_k} w_{l+i_0+\cdots+i_{m-1}+j_0+\cdots+\widehat{j_k}+\cdots j_m} \\
&= \sum_{k=0}^{m} (-1)^{m-1+k} a_{i_0+j_n} \sum_{n=0}^{k-1} (-1)^n \det\left([a_{i_s+j_t}]_{s\in S, t\in T\setminus\{k,n\}}\right) Y_{j_k} w_{l+i_0+\cdots+i_{m-1}+j_0+\cdots+\widehat{j_k}+\cdots j_m} \\
&+ \sum_{k=0}^{m} (-1)^{m+k} a_{i_0+j_n} \sum_{n=k+1}^{m} (-1)^{n-1} \det\left([a_{i_s+j_t}]_{s\in S, t\in T\setminus\{k,n\}}\right) Y_{j_k} w_{l+i_0+\cdots+i_{m-1}+j_0+\cdots+\widehat{j_k}+\cdots j_m}.
\end{aligned}
$$

For each $0 \leq k \leq m$,

$$
\begin{aligned}
&\sum_{n=0}^{k-1} (-1)^n \det\left([a_{i_s+j_t}]_{s\in S, t\in T\setminus\{k,n\}}\right) Y_{j_k} w_{l+i_0+\cdots+i_{m-1}+j_0+\cdots+\widehat{j_k}+\cdots j_m} \\
&+ \sum_{n=k+1}^{m} (-1)^{n-1} \det\left([a_{i_s+j_t}]_{s\in S, t\in T\setminus\{k,n\}}\right) Y_{j_k} w_{l+i_0+\cdots+i_{m-1}+j_0+\cdots+\widehat{j_k}+\cdots j_m}
\end{aligned}
$$

is the Laplace expansion of $[a_{i_s+j_t}]_{s\in S, t\in T\setminus\{k\}}$ along the first row. Hence the result follows. □

Proof. (Of Proposition 5.2.3) We will prove this in the following order: 2. $\Leftrightarrow$ 1. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4.

2. implies 1.

Fix an $l \in \mathbb{Z}$. If $V(\mathbf{a})$ has finite-dimensional graded subcomponents, then in particular, $n_l := \dim V(\mathbf{a})^{(-1,l)} \in \mathbb{Z}_{\geq 0}$. Hence for all $i_0, i_1, \ldots, i_{n_l} \in \mathbb{Z}$, we can find $c_0, c_1, \ldots, c_{n_l} \in$

$\mathbb{C}$ such that $c_{n_l} \neq 0$ and $c_0 Y_{i_0} w_{l-i_0} + c_1 Y_{i_1} w_{l-i_1} + \cdots + c_{n_l} Y_{i_{n_l}} w_{l-i_{n_l}} = 0$. Acting by $Y_{i_0} Y_{i_1} \cdots Y_{i_{n_l-1}}$, we get $Y_{i_0} Y_{i_1} \cdots Y_{i_{n_l}} \cdot w_{l-i_0-\cdots-i_{n_l}} = 0$, (since $Y_i^2 = 0$ for all $i \in \mathbb{Z}$). Let $N+1 = \max\{\dim V(\mathbf{a})^{(-1,l)} \mid l \in \mathbb{Z}\}$. Then $Y_{i_0} Y_{i_1} \cdots Y_{i_{N+1}} \cdot w_l = 0$ for all $i_k, l \in \mathbb{Z}$. i.e., $V(\mathbf{a})^{(-(N+1),l)} = 0$ for all $l \in \mathbb{Z}$. Therefore, we have the containment
$\{k \in \mathbb{Z} \mid V(\mathbf{a})^{(k,l)} \neq 0\} \subseteq \{0\} \cup [N]$.

1. implies 2.

Suppose there exists an $N \in \mathbb{Z}_{>0}$ such that $V(\mathbf{a})^{(-(N+1),l)} = 0$ for all $l \in \mathbb{Z}$. This means that for all $l, n \in \mathbb{Z}$, $Y_0 \cdots Y_{N-1} Y_n w_{l-n-(0+1+\cdots+(N-1))} = 0$. Let $S = \{0, \ldots, N-2\}$ and $T = S \cup \{n\}$. By Lemma 5.2.4, for all $i_0, \ldots, i_{N-1} \in \mathbb{Z}$, we get

$$\begin{aligned} 0 &= X_{i_{N-1}} \cdots X_{i_0} Y_0 \cdots Y_{N-1} Y_n w_{l-n-(0+1+\cdots+(N-1))} \\ &= \sum_{k=0}^{N-1} (-1)^{N+k} \det\left([a_{i_s+j_t}]_{s\in S, t\in T\setminus\{k\}} Y_k \cdot w_{l-k}\right) + \left([a_{i_s+t}]_{s\in S, t\in T\setminus\{n\}} Y_n \cdot w_{l-n}\right). \end{aligned}$$

Hence $Y_n w_{l-n} \in \operatorname{span}\{Y_i w_{l-i} \mid i \in 0, 1, \ldots, N-1\}$.
i.e., $\bigoplus_{l\in\mathbb{Z}} V(\mathbf{a})^{(-1,l)} = \operatorname{span}_{\mathbb{C}}\{Y_i w_{l-i} \mid i \in \{0, 1, \ldots, N-1\}, l \in \mathbb{Z}\}$. Since $Y_i^2 = 0$, it follows that $V(\mathbf{a})^{(-k,l)}$ is spanned by $\{Y_{j_1} \cdots Y_{j_k} \cdot w_{l-(j_1+\cdots+j_k)} \mid 0 \leq j_1 < \cdots < j_k \leq N-1\}$.

1. implies 3.

Suppose that $V(\mathbf{a})$ has finite support. Above we showed that in this case, $V(\mathbf{a})$ is spanned by elements of the form $Y_{j_1} \cdots Y_{j_k} \cdot w_{l-(j_1+\cdots+j_k)}$, where $k > 0$ and $j_1, \ldots, j_k, l \in \mathbb{Z}$. Moreover, there exists a minimal $N \in \mathbb{Z}_{>0}$ such that $Y_{j_1} \cdots Y_{j_{N+1}} \cdot w_{l-(j_1+\cdots+j_{N+1})} = 0$ for all $j_1, \ldots, j_K, l \in \mathbb{Z}$. By Lemma 5.2.4, for any $i_1, \ldots, i_K, j_1, \ldots, j_{N+1}, n \in \mathbb{Z}$, we

have

$$\begin{aligned} 0 &= X_{i_1} \cdots X_{i_{N+1}} Y_{j_1} \cdots Y_{j_{N+1}} \cdots w_{l-(j_1+\cdots+j_{N+1})} \\ &= (-1)^{\frac{N(N+1)}{2}} \det \left[a_{i_s+j_t}\right]_{s,t=1,\ldots,N+1} w_{l+i+1+\cdots+i_{N+1}} \ . \end{aligned}$$

Hence, every minor determinant of A of order greater than or equal to $N+1$ is zero, which implies that A has finite rank.

3. implies 1.

Suppose that $\mathrm{rk}A = N < \infty$. We claim that $V(\mathbf{a})^{(-(N+1),l)} = 0$ for all $l \in \mathbb{Z}$. Indeed, recall that $V(\mathbf{a}) = M(\mathbf{a})/I(\mathbf{a})$ where $I(\mathbf{a}) = \sum I$ such that $I \subset V(\mathbf{a})$ is a submodule of $V(\mathbf{a})$ and $I(\mathbf{a}) \cap L(\mathbf{a}) = 0$. Hence any element of the form $Y_{j_0} \cdots Y_{j_k} \cdot w_{l-(j_1+\cdots+j_k)}$ does not generate a submodule of $V(\mathbf{a})$. Now suppose there exist $j_0, \ldots, j_N, l \in \mathbb{Z}$ such that $Y_{j_0} \cdots Y_{j_N} \cdot w_{l-(j_1+\cdots+j_N)} \neq 0$. Then $0 \neq X_{i_N} \cdots X_{i_0} Y_{j_0} \cdots Y_{j_N} \cdot w_{l-(j_1+\cdots+j_N)} = A' w_{l+i_0+\cdots+i_N}$,where A' is a $(N+1) \times (N+1)$ minor of A. Since $\mathrm{rk}A = N$, this is a contradiction.

3. implies 4.

If $\mathrm{rk}A = N$ for some $N \in \mathbb{Z}_{\geq 0}$, then there is a linear recurrence relation of order N on the a_k's. Namely, $a_{k+N} = \alpha_1 a_{k+N-1} + \cdots + \alpha_N a_k$ where $\alpha_i \in \mathbb{C}$. Let $p(t) = t^N - \alpha_1 t^{N-1} - \cdots - \alpha_N$ be the characteristic polynomial of this recurrence relation, and let $\lambda_1, \ldots, \lambda_s$ be the roots of p with multiplicities m_i, respectively. Then the solution of this recurrence relation is

$$a_k = \sum_{i=1}^{s} g_i(k) \lambda_i^k,$$

where $g_i \in \mathbb{C}[k]$ is a polynomial of degree $m_i - 1$.

4. implies 3.

Suppose the sequence $\mathbf{a}$ is exponential polynomial. Then $\mathbf{a}$ is of the form $a_k = \sum_{i=1}^{s} g_i(k)\lambda_i^k$, for $k \in \mathbb{Z}$, which implies that a_k is a solution of the homogeneous linear recurrence relation $a_{k+N} = \alpha_1 a_{k+N-1} + \cdots + \alpha_N a_k$ where $\alpha_i \in \mathbb{C}$, for some $N \in \mathbb{Z}_{\geq 0}$. Following the proof of Proposition 5.1.6, this implies that every $\det \left[a_{i_s+j_t}\right]_{s,t=0,1,\ldots,N} = 0$ for all $i_0, \ldots, i_N, j_0, \ldots, j_N \in \mathbb{Z}$. i.e, $\operatorname{rank} A \leq N$. □

Corollary 5.2.5. *Let $A = \left[a_{i+j}\right]_{i,j\in\mathbb{Z}}$. If $\operatorname{rank} A = N$, then $V^{(-k,l)} = 0$ for all $k > N$ and all $l \in \mathbb{Z}$.*

Proof. Let m be such that $V^{(-k,l)} = 0$ for all $k > m$ and all $l \in \mathbb{Z}$. From the proof of 3. implies 1., we know that $m \leq N$. Claim: $m \geq N$. Indeed, if $k \leq N$ and $0 = Y_{j_0} \cdots Y_{j_{k-1}} w_{l-(j_1+\cdots+j_{k-1})}$ for all $j_0, \ldots, j_{k-1}, l \in \mathbb{Z}$, then

$$0 = X_{i_{k-1}} \cdots X_{i_0} Y_{j_0} \cdots Y_{j_{k-1}} w_{l-(j_1+\cdots+j_{k-1})} = \det\left(\left[a_{i_s+j_t}\right]_{s,t=0,\ldots,k-1}\right) w_{w_{l+(i_1+\cdots+i_{k-1})}}$$

for all $i_0, \ldots, i_{k-1}, j_0, \ldots, j_{k-1}, l \in \mathbb{Z}$. Then all $k \times k$ minors of A are zero. Since $\operatorname{rank} A = N$ this a contradiction. □

Proposition 5.2.6. *If $\{a_n\}_{n\in\mathbb{Z}}$ is exponential-polynomial of degree r, then*

1. $$L(\mathbf{a}) = \bigoplus_{j=0}^{r-1} L(\mathbf{a})\langle j\rangle,$$

   where $L := L(\mathbf{a})\langle 0\rangle$ is an irreducible $\overline{\mathfrak{h}}$-module as in Lemma 5.2.1 and $L(\mathbf{a})\langle j\rangle \cong L$ for $j = 0, \ldots, r-1$.

2. $$V(\mathbf{a}) = \bigoplus_{j=0}^{r-1} V(\mathbf{a})\langle j\rangle,$$

where $V(\mathbf{a})\langle 0\rangle = V(L)$.

Proof. 1. For $j \in \{0, \ldots, r-1\}$, let $L(\mathbf{a})\langle j\rangle := \mathrm{span}\{w_{rn+j} \,|\, n \in \mathbb{Z}\}$. Since $L(\mathbf{a}) = \mathrm{span}\{w_n \,|\, n \in \mathbb{Z}\}$, then $L(\mathbf{a}) = \bigoplus_{j=0}^{r-1} L(\mathbf{a})\langle j\rangle$. Fix $j \in \{0, \ldots, r-1\}$. We will show that $L(\mathbf{a})\langle j\rangle \cong L$.

Since $\{a_n\}$ is exponential-polynomial of degree r, then it can be written as

$$a_n = q_r(n) \cdot \sum_{i \in I} g_i(n)(\lambda_i)^n,$$

where I is a finite set. Let $R := \{n \in \mathbb{Z} \,|\, g_i(n) = 0 \text{ for some } i \in I\} \subset \mathbb{Z}$ be the set of integer roots of each g_i. Since I is finite, then R must be finite. Hence we can find $n_1, \ldots, n_k, m_1, \ldots, m_k \in r\mathbb{Z} \setminus (R \cap r\mathbb{Z})$ such that $n_1 + \cdots + n_k = r$ and $m_1 + \cdots + m_l = -r$. Let $\xi := Z_{n_1} \cdots Z_{n_k}$ and $\gamma := Z_{m_1} \cdots Z_{m_l} \in U(\overline{\mathfrak{h}})$. For all $s, t \in \mathbb{Z}$ with $s > t$,

$$\xi^{s-t} \cdot w_{rt+j} = (a_{n_1} \cdots a_{n_k})^{s-t} w_{rs+j} \neq 0$$
$$\gamma^{s-t} \cdot w_{rs+j} = (a_{m_1} \cdots a_{m_l})^{s-t} w_{rt+j} \neq 0 \ .$$

Moreover, $q_r(n) = 0$ when $n \neq 0 \mod r$. So for $i \neq j$, there is no element $\zeta \in U(\overline{\mathfrak{h}})$ such that $\zeta \cdot w_{rm+j} \in L_i$ for some $m \in \mathbb{Z}$.

Therefore, for each $j = 0, \ldots, r-1$, $L(\mathbf{a})\langle j\rangle$ is an irreducible $\mathbb{Z}$-graded $\overline{\mathfrak{h}}$-module. Hence by Lemma 5.2.1, $L(\mathbf{a})\langle j\rangle \cong L$ for each $j = 0, \ldots, r-1$.

2. We will show that $V(\overline{a}) = (U(\overline{\mathfrak{g}}) \otimes L(\overline{a}))/I(\overline{a}) = \bigoplus_{j=0}^{r-1}(U(\overline{\mathfrak{g}}) \otimes L(\mathbf{a})\langle j\rangle)/I\langle j\rangle$.

Let $I(\mathbf{a}) \subset U(\overline{\mathfrak{g}}) \otimes_{U(\overline{b})} L(\mathbf{a}) =: M(\mathbf{a})$ be the maximal proper submodule such that $I(\mathbf{a}) \cap L(\mathbf{a}) = 0$. It is clear that such a maximal submodule exists, since

we may take the sum of all submodules $I' \subset M(\mathbf{a})$ such that $I' \cap L(\mathbf{a}) = 0$. For $0 \leq j \leq r-1$, let $I\langle j\rangle \subset U(\overline{\mathfrak{g}}) \otimes_{U(\overline{\mathfrak{b}})} L(\mathbf{a})\langle j\rangle =: M(\mathbf{a})\langle j\rangle$ be the maximal proper submodule of $M(\mathbf{a})\langle j\rangle$. Since $U(\overline{\mathfrak{g}})$ is graded by $\deg y_i = -1$, $\deg x_i = 1$, and $\deg h_i = 0$ for $i \in \mathbb{Z}$, then $M(\mathbf{a})\langle j\rangle$ and $I\langle j\rangle$ inherit this grading.

We claim that $I(\mathbf{a}) = \bigoplus_{j=0}^{r-1} I\langle j\rangle$.

Indeed, for all $0 \leq j \leq r-1$, since $L(\mathbf{a})[j]$ is an irreducible $\overline{\mathfrak{h}}$-module, and $I\langle j\rangle$ is a proper submodule of $M(\mathbf{a})\langle j\rangle$, then $I\langle j\rangle \cap L(\mathbf{a})\langle j\rangle = 0$. So $I\langle j\rangle \cap L(\mathbf{a}) = 0$ for all $0 \leq j \leq r-1$. Since we took $I(\mathbf{a})$ to be the sum of all submodules of $U(\overline{\mathfrak{g}}) \otimes L(\mathfrak{a})$ which intersect $L(\mathbf{a})$ trivially, then $\bigoplus_{j=0}^{r-1} I\langle j\rangle \subseteq I(\mathbf{a})$.

For the other direction, we proceed by induction on the grading of $I(\mathbf{a})$.

Base case: Let $v \in I(\mathbf{a})$ such that $\deg v = -1$. Then we may write $v = \sum_{j=0}^{r-1} v_j$ where $v_j \in U(\overline{\mathfrak{g}}) \otimes L(\mathbf{a})[j]$ are homogeneous. (The case where v_j are not homogeneous is implied from the homogeneous case). Since $\deg v = -1$, then $v_j \in I\langle j\rangle$. Since otherwise, by the maximality of $I\langle j\rangle$, there exists $X_n \in U(\overline{\mathfrak{g}})$ such that $0 \neq X_n v_j \in L(\mathbf{a})\langle j\rangle$, which contradicts $v_j \in I(\mathbf{a})$.

Inductive step: Let $v = \sum_{j=0}^{r-1} v_j \in I(\mathbf{a})$ be of degree $-k$, where $v_j \in M(\mathbf{a})\langle j\rangle$ is homogeneous. Then $v_j \in I\langle j\rangle$. Since if not, there is $X_n \in U(\overline{\mathfrak{g}})$ such that $v_j' := X_n v_j \notin I\langle j\rangle$. However, $\deg v_j' = -(k-1)$, so $v_j' \in I\langle j\rangle$ by the inductive hypothesis. Thus $I(\mathbf{a}) = \bigoplus_{j=0}^{r-1} I\langle j\rangle$.

By definition and the claim above,

$$V(\mathbf{a}) = (U(\overline{\mathfrak{g}}) \otimes L(\mathbf{a}))/I(\mathbf{a}) \tag{5.18}$$

$$= \left(U(\bar{\mathfrak{g}}) \otimes \bigoplus_{j=0}^{r-1} L(\mathbf{a})\langle j\rangle \right) / I(\mathbf{a}) \tag{5.19}$$

$$= \left(\bigoplus_{j=0}^{r-1} U(\overline{\mathfrak{g}}) \otimes L(\mathbf{a})\langle j \rangle \right) / I(\mathbf{a}) \text{ (since it is a finite sum)} \tag{5.20}$$

$$= \left(\bigoplus_{j=0}^{r-1} M(\mathbf{a})\langle j \rangle \right) / \left(\bigoplus_{j=0}^{r-1} I\langle j \rangle \right) \tag{5.21}$$

$$\cong \bigoplus_{j=0}^{r-1} (M(\mathbf{a})\langle j \rangle / I\langle j \rangle) \tag{5.22}$$

$$= \bigoplus_{j=0}^{r-1} V(\mathbf{a})\langle j \rangle \ . \tag{5.23}$$

□

The module $V(\mathbf{a})$ is $\mathbb{Z} \times \mathbb{Z}$ graded. Let $V(\mathbf{a})^{(k,n)}$ be the (k,n) graded component of $V(\mathbf{a})$, where $k \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$. Let $V(\mathbf{a})\langle j \rangle^{(k,n)}$ denote the (k,n) graded component of $V(\mathbf{a})\langle j \rangle$, where $k \in \mathbb{Z}_{\geq 0}$ and $n \in r\mathbb{Z}$.

Proposition 5.2.7. *Let $\mathbf{a} = \{a_n\}_{n \in \mathbb{Z}}$ be exponential-polynomial of degree r and let $A = [a_{i+j}]_{i,j \in \mathbb{Z}}$. Suppose* $\operatorname{rank} A = N$. *Then*

1. *$V(\mathbf{a})^{(-k,l)}$ has basis $\{Y_{i_1} \cdots Y_{i_k} \cdot w_{l-(i_1+\cdots+i_k)} \,|\, 0 \leq i_1 < \cdots < i_k \leq N-1\}$, so* $\dim V(\mathbf{a})^{(-k,l)} = \binom{N}{k}$, *and*

2. *$V(\mathbf{a})\langle j \rangle^{(-k,l)}$ has basis $\{Y_{i_1} \cdots Y_{i_k} \cdot w_{l-(i_1+\cdots+i_k)} \,|\, 0 \leq i_1 < \cdots < i_k \leq N-1$ and $l - (i_1 + \cdots + i_k) = j \mod r\}$.*

Proof. 1. By Corollary 5.2.5 and from the proof of Proposition 5.2.3 we know that for each $l \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$, $V(\mathbf{a})^{(-k,l)}$ is spanned by the set $\mathcal{B}_{k,l} = \{Y_{j_1} \cdots Y_{j_k} \cdot w_{l-(j_1+\cdots+j_k)} \,|\, 0 \leq j_1 < \cdots < j_k \leq N-1\}$. We also assume that the set $\{w_l \,|\, l \in \mathbb{Z}\}$ is a basis for $L = \bigoplus_{l \in \mathbb{Z}} V(\mathbf{a})^{(0,l)}$. It remains to show that the for each $k \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}$, the set $\mathcal{B}_{k,l}$ is linearly independent. We will induct on

k.

For the case when $k = 1$, suppose there exist $\beta_0, \beta_1, \cdots, \beta_{N-1} \in \mathbb{C}$ such that

$$\sum_{l \in \mathbb{Z}} \beta_0 Y_0 w_l + \beta_1 Y_1 w_{l-1} + \cdots + \beta_{N-1} Y_{N-1} w_{l-(N-1)} = 0.$$

Then, acting by X_i for any $i \in \mathbb{Z}$, we have

$$\beta_0 a_i w_{l+i} + \beta_1 a_{i+1} w_{l+i} + \cdots + \beta_{N-1} a_{N-1} w_{l+i} = 0.$$

This implies that $\beta_0 a_i + \beta_1 a_{i+1} + \cdots + \beta_{N-1} a_{N-1} = 0$ for all $i \in \mathbb{Z}$. If not all of the β_j are zero, then this implies that $\mathrm{rk} A \leq N - 1$, which is a contradiction to that assumption that $\mathrm{rk} A = N$. Hence the set $\mathcal{B}_{1,l}$ is linearly independent.

Now suppose the set $\mathcal{B}_{k,l}$ is linearly independent for some $k \in \mathbb{Z}_{\geq 0}$. Let $J_{k+1} := \{\mathbf{j} = (j_1, \cdots, j_{k+1}) \in \mathbb{Z}^{k+1} \,|\, 0 \leq j_1 < \cdots < j_{k+1} \leq N - 1\}$. For $\mathbf{j} \in J_{k+1}$, define $Y_{\mathbf{j}} := Y_{j_1} \cdots Y_{j_{k+1}}$. Suppose there exist $\beta_{\mathbf{j}} \in \mathbb{C}$, such that

$$0 = \sum_{\mathbf{j} \in J_{k+1}} \beta_{\mathbf{j}} Y_{\mathbf{j}} w_{l-(j_1+\cdots+j_k)} \ .$$

Then for any $i \in \mathbb{Z}$,

$$\begin{aligned} 0 &= X_i \cdot \left(\sum_{\mathbf{j} \in J_{k+1}} \beta_{\mathbf{j}} Y_{j_1} \cdots Y_{j_{k+1}} \cdot w_{l-(j_1+\cdots+j_{k+1})} \right) \\ &= \sum_{\mathbf{j} \in J_{k+1}} \beta_{\mathbf{j}} X_i Y_{j_1} \cdots Y_{j_{k+1}} \cdot w_{l-(j_1+\cdots+j_{k+1})} \\ &= \sum_{\mathbf{j} \in J_{k+1}} \beta_{\mathbf{j}} \sum_{s=1}^{k+1} (-1)^{s+1} a_{i+j_s} Y_{j_1} \cdots \widehat{Y_{j_k}} \cdots Y_{j_{k+1}} \cdot w_{l-(j_1+\cdots+\widehat{j_s}+\cdots+j_{k+1})}. \end{aligned}$$

From each element $\mathbf{j} = (j_1, \ldots, j_{k+1})$ of J_{k+1}, we can obtain $k+1$ distinct elements of J_k by removing one of the j_i's. This is what we are doing in the sum above. Hence, the sum above is composed of $(k+1)\binom{N-1}{k+1} = \binom{N}{k}(N-k)$ distinct summands of the form

$$\beta_{\mathbf{j}} a_{i+j_s} Y_{j_1} \cdots \widehat{Y_{j_s}} \cdots Y_{j_{k+1}} \cdot w_{l-(j_1+\cdots+\widehat{j_s}+\cdots+j_{k+1})} \ .$$

Moreover, each element of J_{k+1} can be formed in an analogous way by taking an element $\mathbf{j}' = (j'_1, \ldots, j'_k) \in J_k$ and inserting j'_{k+1} such that $j'_{k+1} \notin \{j'_1, \ldots, j'_k\}$. There are $N-k$ possible choices of j'_{k+1}. A total of $\binom{N}{k}(N-k)$ elements of J_{k+1} are created. We may describe these elements. Let s be such that $j'_{s-1} < j'_{k+1} < j'_s$, or $s = k+1$ if $j'_k < j'_{k+1}$. Write $\mathbf{j} = (j_1, \ldots, j_{k+1}) \in J_{k+1}$ by setting $j_1 := j'_1, \ldots, j_{s-1} := j'_{s-1}, j_s := j'_{k+1}, j_{s+1} = j'_s, j_{s+2} = j'_{s+1}, \ldots, j_{k+1} = j'_k$. To each of these new elements of $\mathbf{j} \in J_{k+1}$, we may associate precisely one element of $V(\mathbf{a})^{(-k,l)}$ of the form

$$a_{i+j_s} Y_{j'_1} \cdots Y_{j'_k} w_{l+i-(j'_1+\cdots+j'_k)} = a_{i+j_s} Y_{\mathbf{j}'} w_{l+i-(j'_1+\cdots+j'_k)} \ .$$

For $\mathbf{j} = (j_1, \ldots, j_{k+1}) \in J_{k+1}$ and $\mathbf{j}' = (j'_1, \ldots, j'_k) \in J_k$, write $\beta_{\mathbf{j}} = \beta_{j_1,\ldots,j_{k+1}}$, $\beta_{\mathbf{j}'} := \beta j'_1, \ldots, j'_k$, $\beta_{\mathbf{j}',j_s} := \beta j'_1, \ldots, j_s, \ldots j'_k$. Then there is a one-to-one correspondence between the sets

$$\left\{ \beta_{\mathbf{j}} a_{i+j_s} Y_{j_1} \cdots \widehat{Y_{j_s}} \cdots Y_{j_{k+1}} \cdot w_{l-(j_1+\cdots+\widehat{j_s}+\cdots+j_{k+1})} \;\middle|\; \begin{smallmatrix} (j_1,\ldots,j_{k+1}) \in J_{k+1}, \\ j_s \in \{j_1,\ldots,j_{k+1}\} \end{smallmatrix} \right\}$$

and

$$\left\{\beta_{\mathbf{j}',j_s} a_{i+j_s} Y_{j'_1} \cdots Y_{j'_k} \cdot w_{l-(j'_1+\cdots+j'_k)} \,|\, (j'_1,\ldots,j'_k) \in J_k,\, j_s \notin \{j'_1,\ldots,j'_k\}\right\} \ .$$

Hence

$$\begin{aligned} 0 &= \sum_{\mathbf{j}\in J_{k+1}} \sum_{s=1}^{k+1} (-1)^{s+1} \beta_{\mathbf{j}} a_{i+j_s} Y_{j_1} \cdots \widehat{Y_{j_k}} \cdots Y_{j_{k+1}} \cdot w_{l-(j_1+\cdots+\widehat{j_s}+\cdots+j_{k+1})} \\ &= \sum_{\mathbf{j}'\in J_k} \left(\sum_{s\notin\{j'_1,\ldots,j'_k\}} (-1)^{s+1} \beta_{\mathbf{j}',j_s} a_{i+j_s} \right) Y_{j'_1} \cdots Y_{j'_k} \cdot w_{l-(j'_1+\cdots+j'_k)} \ . \end{aligned}$$

By the inductive hypothesis, the set $\{Y_{j'_1} \cdots Y_{j'_k} \,|\, 0 \le j'_1 < \cdots < j'_k \le N-1\}$ is linearly independent. Hence $\sum_{s\notin\{j'_1,\ldots,j'_k\}} \beta_{\mathbf{j}',j_s} (-1)^{s+1} a_{i+j_s} = 0$. If not all of the $\beta_{\mathbf{j}',j_s}$'s are zero, then we have a linear recurrence relation of degree $N-k$ on the a_i's. This implies that $\operatorname{rank} A \le N-k$, which is a contradiction. Hence $\beta_{\mathbf{j}',j_s} = 0$ for every $\mathbf{j}' \in J_k$. i.e, $\beta_{\mathbf{j}} = 0$ for every $\mathbf{j} \in J_{k+1}$.

2. For the second assertion, recall that $V(\mathbf{a})\langle j\rangle$ is the irreducible quotient of $U(\overline{\mathfrak{n}}^-) \otimes_{U(\overline{\mathfrak{b}})} L(\mathbf{a})\langle j\rangle$, where $L(\mathbf{a})\langle j\rangle = \operatorname{span}\{w_{rn+j} \,|\, n \in \mathbb{Z}\}$. Considering this and the definition of the $\mathbb{Z}\times\mathbb{Z}$ grading, (5.15), any element of $V(\mathbf{a})\langle j\rangle^{(-k,l)}$ is necessarily of the form $Y_{i_1}\cdots Y_{i_k} w_{rn+j}$ where $rn + j + i_1 + \cdots + i_k = l$.
By construction, $V(\mathbf{a})\langle j\rangle^{(-k,l)} \subset V(\mathbf{a})^{(-k,l)}$ and Proposition 5.2.7.1 tells us that $\mathcal{B}_k = \{Y_{i_1}\cdots Y_{i_k} \cdot w_{l-(i_1+\cdots+i_k)} \,|\, 0 \le i_1 < \cdots < i_k \le N-1\}$ is a basis of $V(\mathbf{a})^{(-k,l)}$. Hence to obtain a basis of $V(\mathbf{a})\langle j\rangle^{(-k,l)}$, we take precisely the elements $Y_{i_1}\cdots Y_{i_k} \cdot w_{l-(i_1+\cdots+i_k)} \in \mathcal{B}_k$ where $l - (i_1+\cdots+i_k) \equiv j \mod r$.

□

Lemma 5.2.8. *For $n \geq 0$ and for all $i_0, \ldots, i_n$; $j_0, \ldots, j_n \in \mathbb{Z}$*

$$X_{i_n} \cdots X_{i_0} Y_{j_0} \cdots Y_{j_n} w_l = \det \left[a_{i_s+j_t}\right]_{s,t=0,\ldots n} w_{l+i_0+\cdots+i_n+j_0+\cdots+j_n}.$$

Proof. We proceed by induction on n. When $n = 0$,

$$X_{i_0} Y_{j_0} \cdot w_m = Z_{i_0+j_0} w_l = a_{i_0+j_0} w_{l+i_0+j_0}.$$

Now suppose the result holds for some $n \geq 0$. Let $i_0, \ldots, i_{n+1}; j_0, \ldots, j_{n+1} \in \mathbb{Z}$. Then

$$\begin{aligned}
&X_{i_{n+1}} \cdots X_{i_0} Y_{j_0} \cdots Y_{j_{n+1}} w_l \\
&= (-1)^{n+1} X_{i_n} \cdots X_{i_0} X_{i_{n+1}} Y_{j_0} \cdots Y_{j_n} Y_{j_{n+1}} w_l \\
&= (-1)^{n+1} \sum_{k=0}^{n+1} (-1)^k X_{i_n} \cdots X_{i_0} Y_{j_0} \cdots [X_{i_{n+1}}, Y_{j_k}] \cdots Y_{j_{n+1}} w_l \\
&= (-1)^{n+1} \sum_{k=0}^{n+1} a_{i_{n+1}+k} (-1)^k X_{i_n} \cdots X_{i_0} Y_{j_0} \cdots \widehat{Y_{j_k}} \cdots y_{j_{n+1}} w_l \\
&= \sum_{k=0}^{n+1} (-1)^{n+2+k+1} a_{i_{n+1}+k} \det \left[a_{i_s+j_t}\right]_{s=0,\ldots,n,t=0,\ldots,\hat{k},\ldots,n+1} w_{l+i_0+\cdots+i_{n+1}+j_0+\cdots+j_{n+1}}.
\end{aligned}$$

Since $\det \left[a_{i_s+j_t}\right]_{s=0,\ldots,n,t=0,\ldots,\hat{k},\ldots,n+1}$ is the determinant of the submatrix obtained by removing the $(n+2)^{\text{th}}$ row and the $(k+1)^{\text{th}}$ column of $\det \left[a_{i_s+j_t}\right]_{s,t=0,\ldots,n+1}$, Laplace expansion gives us the result. $\square$

Chapter 6

The Lie superalgebra $\mathfrak{osp}(1|2) \otimes A$

We conclude by studying the highest weight modules of $\mathfrak{osp}(1|2) \otimes A$, where $A = \mathbb{C}[t]$, or $A = \mathbb{C}[t, t^{-1}]$. The simple roof of $\mathfrak{osp}(1|2)$ is δ, which an odd non-istropic root. Hence 2δ is an even root, so $\mathfrak{sl}_2 \subset \mathfrak{osp}(1|2)$ is a subalgebra. This allows us to reuse many results obtained for $\mathfrak{sl}_2 \otimes A$ from Chapter 4.

6.1 Highest Weight Modules

6.1.1 The current algebra $\mathfrak{osp}(1|2) \otimes A$

Let $\mathfrak{g} = \mathfrak{osp}(1|2)$ and recall that $g_n := g \otimes t^n \in \mathfrak{g} \otimes A$. The elements of $\mathfrak{g}$ are 3×3 complex matrices, and we may choose a basis of $\mathfrak{g}$ to be $\{f, h, e, x, y\}$ where

$$f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{g}_{\bar{0}} = \text{span}\{e, f, h\} \cong \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{g}_{\bar{1}} = \text{span}\{x, y\}$, and the other non-zero brackets are

$$[e, y] = -x,\ [x, y] = h,\ [h, y] = -y,\ [f, x] = -y, [h, x] = x,\ [x, x] = e, [y, y] = -f.$$

Recall the definition of the Verma module $M(\Lambda)$ from Definition 3.1.1.

Definition 6.1.1. Let $\mathbf{a} := (a_n)_{n\in\mathbb{Z}}$, where $a_n \in \mathbb{C}$. Let $\Lambda \in (\mathfrak{h} \otimes A)^*$ be such that $\Lambda(h_n) = a_n$ for all $h \in \mathfrak{h}$. Define $I(\Lambda, m) \subset M(\Lambda)$ to be the submodule generated by $F_0^{m+1} \cdot w_0$. Then

$$W(\Lambda, m) := M(\Lambda)/I(\Lambda, m)\,. \tag{6.1}$$

$W(\Lambda, m)$ is called a Weyl module of $\mathfrak{osp}(1|2) \otimes A$.

More informally, $W(\Lambda, m)$ is the highest weight $\mathfrak{osp}(1|2) \otimes A$-module generated by w_0 and defined by the relations

$$\begin{aligned} (\mathfrak{n}^+ \otimes A) \cdot w_0 &= 0, \\ H_n \cdot w_0 &= \Lambda(h_n) w_0 \quad \forall\, h_n \in \mathfrak{h} \otimes A, \\ F_0^{m+1} \cdot w_0 &= 0. \end{aligned} \tag{6.2}$$

Note that here we abuse notation by using w_0 as the generating vector of both $M(\Lambda)$ and $W(\Lambda, m)$.

Theorem 6.1.2. *Let $m \in \mathbb{Z}_{>0}$ and let $W(\Lambda, m)$ be as in Definition 6.1.1. Then*

1. *$W(\Lambda, m) \neq 0$ if and only if $a_n = \alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n$ for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$.*

2. *If $W(\Lambda, m) \neq 0$, then $\dim W(\Lambda, m) < \infty$.*

3. *If $W'(\Lambda, m)$ is a $\mathfrak{g} \otimes A$-module generated by a nonzero vector w_0' which satisfies the relations (6.2), then there exists a $\mathfrak{g} \otimes A$-submodule $V \subseteq W(\Lambda, m)$ such that $W'(\Lambda, m) \cong W(\Lambda, m)/V$.*

Proof of Theorem 6.1.2

We will prove this theorem in an analogous way as Theorems 4.1.3 and 5.1.2, by assuming a module which satisfies the conditions of 6.2 exists, and prove that properties 1, 2, and 3, of Theorem 6.1.2 hold. As in Chapter 3 and 4, we use the notation $E_I := E_{i_1} \cdots E_{i_n}$ and $F_I := F_{i_1} \cdots F_{i_n}$ where $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$. The following diagram outlines the sketch of the proof.

$$\begin{array}{ccccccc} \text{Conditions (6.2)} & \Rightarrow & \text{Prop. 6.1.3} & \Rightarrow & \text{Prop. 6.1.6} & \Leftrightarrow & \text{Prop. 6.1.7} \\ & & \Downarrow & & & & \Downarrow \\ & & \text{Prop. 6.1.8} & & & & \text{Thm. 6.1.2.1} \\ & & \Downarrow & & & & \\ & & \text{Thm. 6.1.2.2} & & & & \end{array}$$

Proposition 6.1.3. *Let $m \in \mathbb{Z}$ and suppose that $F_0^{m+1} \cdot w_0 = 0$. Then for all $I = (i_0, i_1 \ldots, i_m) \in \mathbb{Z}^{m+1}$, $j \in \mathbb{Z}$,*

1. *$F_I \cdot w_0 = 0$, and*

2. *$Y_j F_I \cdot w_0 = 0$.*

Proof. Since $\text{span}\{e, f, h\} \cong \mathfrak{sl}_2(\mathbb{C})$, Part 1. follows from Proposition 4.1.5.

For Part 2., let $j \in \mathbb{Z}$. Then by Lemma 6.1.4, setting $i_k = 0$ for all $i_k \in I$, $0 = X_j F_0^{m+1} \cdot w_0 = (m+1) Y_j F_0^m \cdot w_0$. Since $[f, y] = 0$, then $F_0 Y_j = Y_j F_0 \in U(\mathfrak{osp}(1|2))$. Hence $Y_j F_0^{m+1} \cdot w_0 = 0$ for all $j \in \mathbb{Z}$. Likewise, by Lemma 6.1.4, for all $i, j \in \mathbb{Z}$, $0 = X_j F_i F_0^m \cdot w_0 = Y_{j+i} F_0^m \cdot w_0 + m Y_j F_{i_1} F_0^{m-1} \cdot w_0$. Since we showed that $Y_{j+i} F_0^m \cdot w_0 = 0$, then $Y_j F_i F_0^{m-1} \cdot w_0 = 0$. Part 2. follows from Lemma 6.1.5. □

Lemma 6.1.4. *For all $I = \{i_1, \ldots, i_m\} \subset \mathbb{Z}$, $j \in \mathbb{Z}$,*

$$X_j F_I \cdot w_0 = \sum_{k=1}^{n} Y_{j+i_k} F_{I-\{i_k\}} \cdot w_0 \ .$$

Proof. We will induct on m. When $m = 1$, we simply have $X_j F_{i_1} \cdot w_0 = [X_j, F_{i_1}] \cdot w_0 + F_{i_1} X_j \cdot w_0 = Y_{j+i_1} \cdot w_0$. Now suppose the result holds for some $m \geq 1$. Then

$$\begin{aligned} X_j F_I \cdot w_0 &= X_j F_{I-\{i_{m+1}\}} F_{i_{m+1}} \cdot w_0 \\ &= [X_j, F_{i_{m+1}}] F_{I-\{i_{m+1}\}} \cdot w_0 + F_{i_{m+1}} X_j F_{I-\{i_{m+1}\}} \cdot w_0 \\ &= Y_{j+i_{m+1}} F_{I-\{i_{m+1}\}} \cdot w_0 + \sum_{k=1}^{m} F_{i_{m+1}} Y_{j+i_k} F_{I-\{i_l, i_{m+1}\}} \cdot w_0 \\ &= \sum_{k=1}^{m+1} Y_{j+i_k} F_{I-\{i_l\}} \cdot w_0 \ . \end{aligned}$$

□

Lemma 6.1.5. *Let $s \geq 1$, $j \in \mathbb{Z}$ and $I = \{i_1, \ldots, i_{s+1}\} \subset \mathbb{Z}$. Then $Y_j F_0^{m-s} F_{I-\{i_{s+1}\}} \cdot w_0 = 0$ if and only if $Y_j F_0^{m-s-1} F_I \cdot w_0 = 0$.*

Proof. The backwards direction is clear. For the forwards direction, using Lemma 6.1.4, we have

$$0 = X_j F_0^{m-s} F_I \cdot w_0 = (m-s) Y_j F_0^{m-s-1} F_I \cdot w_0 + \sum_{l=1}^{s+1} Y_{j+i_l} F_0^{m-s} F_{I-\{i_l\}} \cdot w_0 \ .$$

Since we assumed that $Y_{j+i_l} F_0^{m-s} F_{I-\{i_l\}} \cdot w_0 = 0$, it follows that $F_0^{m-s-1} F_I \cdot w_0 = 0$. □

Let M_n be the matrix

$$M_n := \begin{bmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & \cdot & & n-1 \\ p_n & p_{n-1} & \cdot & \cdots & p_1 \end{bmatrix}$$

and let M'_n be the matrix

$$M'_n := \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdot & & n-1 \\ a_n & a_{n-1} & \cdot & \cdots & a_1 \end{bmatrix}.$$

Proposition 6.1.6. *Let $n \geq m+1$ and I, J be subsets of $\mathbb{Z}$ of size n. Then $E_I F_J \cdot w_0 = 0$ if and only if $\det M'_n = 0$.*

Proof. Since $\operatorname{span}\{e, f, h\} \cong \mathfrak{sl}_2(\mathbb{C})$, then this follows from Proposition 4.1.8. □

Proposition 6.1.7. *Let $m \in \mathbb{Z}_{>0}$ and consider the sequence $\mathbf{a} = \{a_n \,|\, n \in \mathbb{Z}\}$. Then $\det M'_{m+1} = 0$ if and only if $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$.*

Proof. The direction $\det M'_{m+1} = 0$ implies $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}^*$ follows from Lemma 4.1.12 and Proposition 4.1.13.

Now suppose that $a_n = \alpha_1^n + \cdots + \alpha_m^n$ for some $\alpha_1, \ldots, \alpha_m \in \mathbb{C}^*$. Then

$$\det M'_{m+1} = \begin{vmatrix} \alpha_1 + \cdots + \alpha_m & 1 & 0 & \cdots & 0 \\ \alpha_1^2 + \cdots + \alpha_m^2 & \alpha_1 + \cdots + \alpha_m & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^m + \cdots + \alpha_m^m & \alpha_1^{m-1} + \cdots + \alpha_m^{m-1} & \cdots & \ddots & m \\ \alpha_1^{m+1} + \cdots + \alpha_m^{m+1} & \alpha_1^m + \cdots + \alpha_m^m & \cdots & \cdots & \alpha_1 + \cdots + \alpha_m \end{vmatrix}$$

Let $e_k(\alpha_1, \ldots, \alpha_m)$ be the k^{th} elementary symmetric function in m variables, $p_k(\alpha_1, \ldots, \alpha_m)$ be the k^{th} power-sum symmetric function in m variables, and recall the relation

$$ne_n(\alpha_1, \ldots, \alpha_m) = \sum_{r=1}^{n} (-1)^{r-1} e_{n-r}(\alpha_1, \ldots, \alpha_m) p_r(\alpha_1, \ldots, \alpha_m).$$

After applying the column operation $C(1) + \sum_{k=1}^{m} e_k(\alpha_1, \ldots, \alpha_m) C(k+1)$, the entry in row i, column 1 is

$$p_i(\alpha_1, \ldots, \alpha_m) + \sum_{k=1}^{i-1} (-1)^k e_k(\alpha_1, \ldots, \alpha_m) p_{i-k}(\alpha_1, \ldots, \alpha_m) + (-1)^{i+1} i e_i(\alpha_1, \ldots, \alpha_m).$$

Letting $r = i - k$, this becomes

$$\sum_{r=1}^{i}(-1)^{i-r}e_k(\alpha_1,\ldots,\alpha_m)p_{i-k}(\alpha_1,\ldots,\alpha_m) + (-1)^{i+1}e_i(\alpha_1,\ldots,\alpha_m)$$
$$= (-1)^i(-ie_i(\alpha_1,\ldots,\alpha_m) + ie_i(\alpha_1,\ldots,\alpha_m)).$$

Hence $\det M'_{m+1} = 0$. □

Proposition 6.1.8. *The module $W(\Lambda, m)$ is finite-dimensional.*

Proof. Consider W', the $U(\mathfrak{g}_{\bar{0}}\otimes A)$-submodule of $W(\Lambda, m)$ generated by w_0. Since $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}_2(\mathbb{C})$, then $\dim W' < \infty$ by Theorem 4.1.3, and so $\mathrm{Supp}W' < \infty$ and $\dim(W')^\mu < \infty$ for every $\mu \in \mathrm{Supp}W'$. Consider a non-zero odd weight space W^{2k+1} where $0 \neq u \in W^{2k+1}$. Since $2k+1$ is odd, then $X_i \cdot u \in W^{2k} \subset W'$ for every $i \in \mathbb{Z}$. Since W^{2k} is finite dimensional, we may fix a basis $\{v_1,\ldots,v_n\}$ of W^{2k}. Let $X_{i_1}, Y_{i_2} \in U(\mathfrak{osp}(1|2) \otimes A)$ such that $[X_{i_1}, Y_{i_2}] \cdot u \neq 0$. Then $0 \neq X_{i_1} \cdot u = \sum_{k=1}^{n} c_k v_k$, for some $c_k \in \mathbb{C}$. Since

$$Y_{i_2}X_{i_1} \cdot u = (H_{i_1+i_2} - X_{i_1}Y_{i_2}) \cdot u = \lambda u - X_{i_1}Y_{i_2} \cdot u,$$

for some $\lambda \in \mathbb{C}^*$, then

$$u = \frac{1}{\lambda}\left(\sum_{k=1}^{n} c_k Y_{i_2} v_k + X_{i_1}Y_{i_2} \cdot u\right) = \sum_{k=1}^{n+1} c'_k v'_k$$

where $c'_k v'_k = \frac{1}{\lambda} Y_{i_2} v_k$ for $k = 1,\ldots,n$ and $c'_{n+1}v'_{n+1} = \frac{1}{\lambda}X_{i_1}Y_{i_2}u$. Hence $\dim W^{2k+1} \leq \dim W^{2k} + 1 < \infty$ for all $k \geq 0$. Since $W(\Lambda, m)$ has finite support, it follows that $\dim W(\Lambda, m)$ is finite. □

Chapter 7

Future directions

In this chapter, we discuss possible directions for future research. The first direction in which one may go is to study Weyl modules of $\mathfrak{g} \otimes A$, where $A = \mathbb{C}[t]$ or $A = \mathbb{C}[t^{\pm 1}]$ and $\mathfrak{g}$ is a Lie superalgebra different from $\mathfrak{sl}(1\,|\,1)$, $\mathfrak{osp}(1\,|\,2)$, or $\mathfrak{sl}_2$ and find a character formula, irreducibility criterion, and classify all submodules of $W(m, \Lambda)$. A second direction is to study what happens when A is a more general associative algebra. As stated in the introduction, in [CLS19] Calixto, Lemay, and Savage proved finite dimensionality and universality of $W(m, \Lambda)$ when A is a finitely generated unital associative $\mathbb{C}$-algebra. One could go further and find a character formula, irreducibility criterion, and classify all submodules of $W(m, \Lambda)$.

7.1 The case when $\mathfrak{g} = \mathfrak{sl}(1\,|2)$

As a first example, we discuss the case of taking the rank 2 Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(1\,|2)$ and $A = \mathbb{C}[t]$ or $A = \mathbb{C}[t, t^{-1}]$.

The root system of $\mathfrak{g}$ is $\Delta = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \delta), \pm(\varepsilon_1 - \delta)\}$, and $\mathfrak{g}$ has two

simple roots. For any base Σ, let $\mathfrak{b}_\Sigma$ be the Borel subalgebra corresponding to Σ. The standard base of $\mathfrak{g}$ is $\Sigma = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta\}$, where $\varepsilon_1 - \varepsilon_2$ is even, and $\varepsilon_2 - \delta$ is odd. For $\alpha \in \Delta^+$, let $e_{\pm\alpha}$, $f_\alpha = e_{-\alpha}$, h_α, and $x_{\pm\beta}$, $y_\beta = x_{-\beta}$, z_β denote the Chevalley generators of $\mathfrak{g}$, where $\alpha, \beta \in \Delta^+$ are even and odd roots, respectively. Note that $\{e_\alpha, f_\alpha, h_\alpha\}$ is an $\mathfrak{sl}_2$-triple, while $\{x_\beta, y_\beta, z_\beta\}$ is an $\mathfrak{sl}(1|1)$-triple. We will use the notation $e_{\alpha,n} := e_\alpha \otimes t^n \in \mathfrak{g} \otimes A$. Each $\mathfrak{sl}_2$-triple and $\mathfrak{sl}(1|1)$-triple satisfies the commutation relations of $\mathfrak{sl}_2$ and $\mathfrak{sl}(1|1)$, respectively. The other commutation relations are

$$\begin{aligned}
&[x_{\varepsilon_2-\delta}, e_{\varepsilon_1-\varepsilon_2}] = -x_{\varepsilon_1-\delta}, \quad [y_{\varepsilon_2-\delta}, e_{\varepsilon_1-\varepsilon_2}] = 0, \quad [x_{\varepsilon_2-\delta}, f_{\varepsilon_1-\varepsilon_2}] = 0 \\
&[y_{\varepsilon_2-\delta}, f_{\varepsilon_1-\varepsilon_2}] = y_{\varepsilon_1-\delta}, \qquad [x_{\varepsilon_1-\delta}, e_{\varepsilon_1-\varepsilon_2}] = 0, \quad [y_{\varepsilon_1-\delta}, f_{\varepsilon_1-\varepsilon_2}] = y_{\varepsilon_2-\delta} \\
&[x_{\varepsilon_1-\delta}, f_{\varepsilon_1-\varepsilon_2}] = -x_{\varepsilon_2-\delta}, \quad [y_{\varepsilon_1-\delta}, f_{\varepsilon_1-\varepsilon_2}] = 0, \quad [x_{\varepsilon_1-\delta}, y_{\varepsilon_2-\delta}] = e.
\end{aligned}$$

Fix the standard base Σ and Borel subalgebra $\mathfrak{b}_\Sigma = \mathfrak{n}_\Sigma^+ \oplus \mathfrak{h}$. Let $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2_{>0}$. The Weyl module $W(\Lambda, \mathbf{m})$ of $\mathfrak{sl}(1\,|2)$ is defined as the highest weight $\mathfrak{g} \otimes A$-module generated by the highest weight vector w_0 such that

(i) $(\mathfrak{n}_\Sigma^+ \otimes A) \cdot w_0 = 0$;

(ii) $(h \otimes p(t)) \cdot w_0 = \Lambda(h \otimes p(t))w_0$ for every $h \otimes p(t) \in \mathfrak{h} \otimes A$;

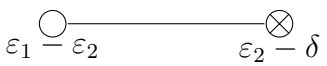
(iii) $F_{\alpha,0}^{m_1} \cdot w_0 = 0$, for every $\alpha \in \Sigma_{\bar{0}}$.

(iv) $Y_{\beta,0}Y_{\beta,1}\cdots Y_{\beta,m_2} \cdot w_0 = 0$, for every $\beta \in \Sigma_{\bar{1}}$.

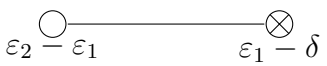
In this definition, w_0 is a highest weight vector with respect to $\mathfrak{b}_\Sigma \otimes A$. By changing the Borel subalgebra, we get a different highest weight vector. There are six different Borel subalgebras, hence there should be six different highest weights. The problem is showing that there are indeed six highest weights, proving that $W(\Lambda, m)$ has finite support.

This concept can be generalized to any algebra of the form $\mathfrak{sl}(m|n) \otimes A$.

The Dynkin diagram corresponding to Σ is



To change the Borel subalgebra, we must reflect Σ at a root α. If we choose to reflect at the even root $\varepsilon_2 - \varepsilon_1$, then $r_{\varepsilon_2-\varepsilon_1}(\Sigma) = \{\varepsilon_2 - \varepsilon_1, \varepsilon_1 - \delta\}$. The corresponding Dynkin diagram is



The algebra $\mathfrak{n}^+_{r_{\varepsilon_1-\varepsilon_2}(\Sigma)}$ is generated by $e_{-(\varepsilon_1-\varepsilon_2)} = f_{\varepsilon_1-\varepsilon_2}$ and $x_{\varepsilon_1-\delta}$. By the definition of $W(\Lambda, \mathbf{m})$ and by Proposition 4.1.5, $F_{\varepsilon_1-\varepsilon_2,n} \cdot \left(F^m_{\varepsilon_1-\varepsilon_2,0} \cdot w_0\right) = 0$. Furthermore, by the commutation relations of $\mathfrak{sl}(1\,|\,2)$, since $X_{\varepsilon_2-\delta} \in \mathfrak{n}^+_\Sigma$, and by the definition of $W(\Lambda, \mathbf{m})$, for every $n \in \mathbb{Z}$

$$\begin{aligned} X_{\varepsilon_1-\delta,n} \cdot \left(F^m_{\varepsilon_1-\varepsilon_2,0} \cdot w_0\right) &= F^m_{\varepsilon_1-\varepsilon_2,0} X_{\varepsilon_1-\delta,n} \cdot w_0 + \sum_{k=0}^{m} F^k_{\varepsilon_1-\varepsilon_2,0} [X_{\varepsilon_1-\delta,n}, F_{\varepsilon_1-\varepsilon_2,0}] F^{m-k}_{\varepsilon_1-\varepsilon_2,0} \cdot w_0 \\ &= 0 + (-1) \sum_{k=0}^{m} F^k_{\varepsilon_1-\varepsilon_2,0} X_{\epsilon_2-\delta,n} F^{m-k}_{\varepsilon_1-\varepsilon_2,0} \cdot w_0 \\ &= -\sum_{k=0}^{m} F^k_{\varepsilon_1-\varepsilon_2,0} F^{m-k}_{\varepsilon_1-\varepsilon_2,0} X_{\epsilon_2-\delta,n} \cdot w_0 = 0 \ . \end{aligned}$$

So $F^m_{\varepsilon_1-\varepsilon_2,0} \cdot w_0$ is a highest weight vector with respect to the base $r_{\varepsilon_1-\varepsilon_2}(\Sigma)$.

If we choose to reflect at the odd root $\delta - \varepsilon_1$, then $r_{\delta-\varepsilon_1}(\Sigma) = \{\epsilon_1 - \delta, \delta - \epsilon_2\}$. The Dynkin diagram corresponding to $r_{\delta-\varepsilon_1}(\Sigma)$ is

The algebra $\mathfrak{n}^+_{r_{\delta-\varepsilon_1}(\Sigma)}$ is generated by $x_{\varepsilon_1-\delta}$ and $x_{\delta-\varepsilon_2} = y_{\varepsilon_2-\delta}$. By the definition of $W(\Lambda, m)$ and by Proposition 5.1.4,

$$X_{\delta-\varepsilon_2,n} \cdot (Y_{\delta-\varepsilon_2,0}Y_{\delta-\varepsilon_2,1}\cdots Y_{\delta-\varepsilon_2,m_2} \cdot w_0) = Y_{\delta-\varepsilon_2,n}Y_{\delta-\varepsilon_2,0}Y_{\delta-\varepsilon_2,1}\cdots Y_{\delta-\varepsilon_2,m_2} \cdot w_0 = 0$$

for any $n \in \mathbb{Z}$. Furthermore, by the commutation relations of $\mathfrak{sl}(1\,|\,2)$, since $X_{\varepsilon_2-\delta} \in \mathfrak{n}^+_\Sigma$, and by the definition of $W(\Lambda, m)$, for every $n \in \mathbb{Z}$

$$\begin{aligned}
&X_{\varepsilon_1-\delta,n} \cdot (Y_{\delta-\varepsilon_2,0}Y_{\delta-\varepsilon_2,1}\cdots Y_{\delta-\varepsilon_2,m_2} \cdot w_0)\\
&= Y_{\delta-\varepsilon_2,0}Y_{\delta-\varepsilon_2,1}\cdots Y_{\delta-\varepsilon_2,m_2}X_{\varepsilon_1-\delta,n} \cdot w_0\\
&+ \sum_{k=0}^{m_2}(-1)^{k+1}Y_{\delta-\varepsilon_2,1}\cdots[X_{\varepsilon_1-\delta,n}, Y_{\delta-\varepsilon_2,k}]\cdots Y_{\delta-\varepsilon_2,m_2} \cdot w_0\\
&= 0 + \sum_{k=0}^{m_2}(-1)^{k+1}Y_{\delta-\varepsilon_2,1}\cdots E_{\varepsilon_1-\varepsilon_2,n+k}\cdots Y_{\delta-\varepsilon_2,m_2} \cdot w_0\\
&= \sum_{k=0}^{m_2}(-1)^{k+1}Y_{\delta-\varepsilon_2,1}\cdots \widehat{Y_{\delta-\varepsilon_2,k}}\cdots Y_{\delta-\varepsilon_2,m_2}E_{\varepsilon_1-\varepsilon_2,n+k} \cdot w_0 = 0\ .
\end{aligned}$$

So $Y_{\delta-\varepsilon_2,0}Y_{\delta-\varepsilon_2,1}\cdots Y_{\delta-\varepsilon_2,m_2} \cdot w_0$ is a highest weight vector with respect to the base $r_{\varepsilon_2-\delta}(\Sigma)$.

Let $Y_{\varepsilon_2-\delta,I} := Y_{\varepsilon_2-\delta,0}Y_{\varepsilon_2-\delta,1}\cdots Y_{\varepsilon_2-\delta,m_2-1}$. As a weight diagram, so far $W(\Lambda, \mathbf{m})$ looks as follows:

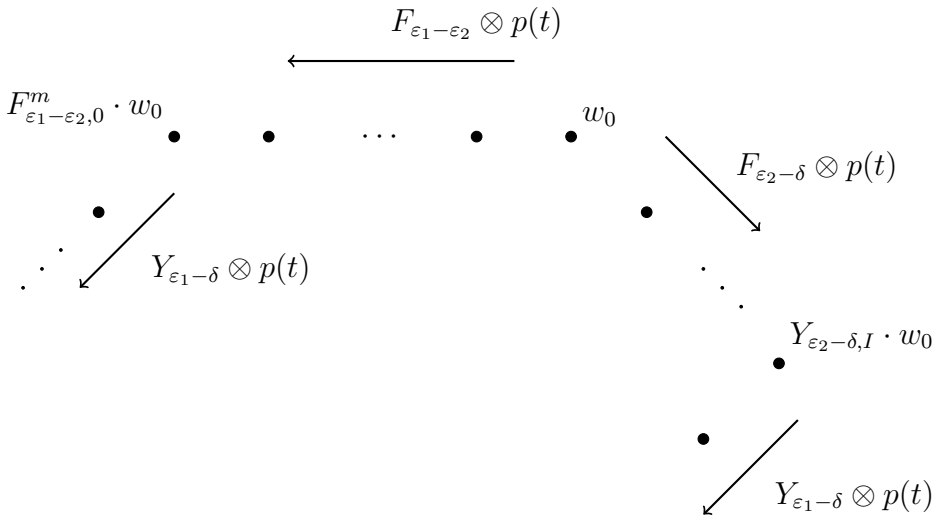


Figure 7.1: Incomplete weight diagram of $W(\Lambda, \mathbf{m})$

This tells us that the support of $W(\Lambda, \mathbf{m})$ is contained in the following shape:

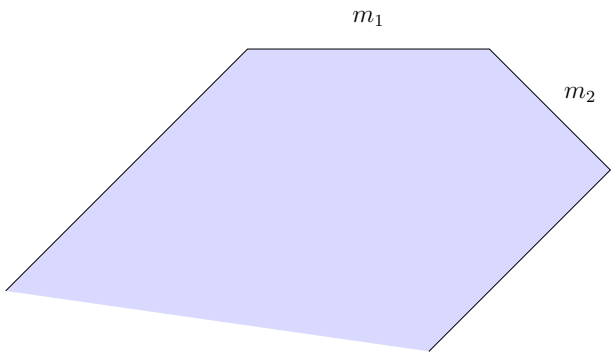


Figure 7.2: Support of $W(\Lambda, \mathbf{m})$

The challenge here is to show that $W(\Lambda, \mathbf{m})$ has finite support. That is, to show that for every root α, there exists $n \in \mathbb{Z}$ such that $F_{\alpha,I} \cdot w_0 = 0$ for every $I \subset \mathbb{Z}^n$. The algebra $\mathfrak{sl}(1\,|\,2)$ has six distinct Borel subalgebras, so we expect the weight diagram of $W(\Lambda, \mathbf{m})$ to be contained in a hexagon.

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