The Orthogonal Weingarten Calculus and Free Probability

by

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Abstract

We provide a self-contained derivation of the Weingarten calculus for compact matrix groups. We also provide a self-contained introduction to the combinatorics of the symmetric group needed for computations in free probability. We introduce partitioned pairings, an analogue of partitioned permutations. Using partitioned pairings, we will derive expressions for the leading and subleading terms of the Weingarten function for the orthogonal group. Using the combinatorial theory that we have developed, as well as what is called second-order real freeness, we will determine the first and second-order limiting distributions of the random matrix ensemble $X_m = O_1 + O_1^t + \cdots O_m + O_m^t$ where $\{O_i\}_{i=1}^m$ are independent Haar-distributed orthogonal matrices.

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Chapter 1

Introduction

1.1 Motivation and Summary of Results

Let $\{X_N\}_{N\in\mathbb{N}}$ be an ensemble of $N \times N$ random matrices with law dX_N. One of the central questions of random matrix theory is determining the ensemble's limiting eigenvalue distribution. That is to say; we wish to understand the measure $\mu_{X_n}(A) = \int_A \mathrm{tr}(X_N)dX_N$ where $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$ is the normalized trace as we let $N \to \infty$. This question was first looked at by Wigner, who observed that in the large N limit, the behaviour of eigenvalue distribution becomes greatly simplified. A method for approaching this problem is to determine the limiting moments of the distribution $\alpha_n := \lim_{N\to\infty} \mathbb{E}(\mathrm{tr}(X_N^n))$. For many ensembles, the limiting measure is compactly supported, thus knowing the moments is equivalent to knowing the measure.

One may also be interested in more finer questions about the eigen-

value distribution. A way to analyze problems in this direction is to look at the higher-order moments of the distribution. The r'th order moments $\{\alpha_{n_1,\ldots,n_r}\}_{n_i\in\mathbb{N}}$ of a random matrix ensemble (when they exist) are defined to be

$$\lim_{N\to\infty} N^{r-2} k_r(\mathrm{Tr}(A_N^{n_1}),\ldots,\mathrm{Tr}(A_N^{n_r}))$$

where k_r denotes the r'th classical cumulant. Note that the first cumulant is just the regular expectation, and the second cumulant is the covariance.

For many ensembles, one can write the moments as an expression in $\frac{1}{N}$ such that the leading order (order 1) terms can be written as a sum of non-crossing permutations. In contrast, with the r'th order moments, the leading order terms can be written as a sum over r-annular non-crossing permutations. This directly links random matrix theory and the combinatorial side of free probability. Free probability is a subject introduced by Voiculescu in the 1980s, who was studying the free group factor isomorphism problem in operator algebras [23]. Shortly after this, Voiculescu found links between free probability and random matrix theory by observing that random matrix models are asymptotically freely independent under certain assumptions [24]. Free independence gives a universal rule for limiting eigenvalue distributions of sums of freely independent ensembles. Due to Nica and Speicher, there is also a combinatorial interpretation of free independence using the lattice of non-crossing partitions and associated objects called free cumulants [22, 17]. In a series of papers ([14, 13, 3]), the ideas of free probability were extended

to include the higher-order moments of matrix ensembles. This led to the theory of second and higher-order freeness and extended free probability techniques to be able to answer finer questions about the eigenvalue distributions. In this higher-order theory, non-crossing partitioned and free cumulants are replaced by non-crossing annular objects and higher-order free cumulants.

The motivation for the theory of higher-order freeness comes from the study of complex matrix models. More specifically, matrix models which are invariant under conjugation by unitary random matrices. Such matrix models are very important in free probability as they serve as matrix models for free independence. In particular, if $\{A_N\}, \{B_n\}$ are two independent random matrix ensembles such that at least one is unitarily invariant, then they are asymptotically freely independent of all orders. Less well understood are the real matrix models. i.e. models which are invariant under conjugation by orthogonal matrices. The first-order theory in both cases is the same. Due to Redelmeier, there is a theory of second-order real freeness, but there is no theory for the higher-order cases [18]. The difference between these cases results from the difference between the unitary and orthogonal Weingarten functions. For a compact matrix group G, the associated Weingarten function tells one how to integrate polynomial functions with respect to the group's Haar measure. The Weingarten functions are rational functions in $\frac{1}{N}$, and the analysis of the leading order terms is key to many computations involving invariant ensembles.

In this thesis, we will study the ensemble $X_m = O_1 + O_1^t + \cdots + O_m + O_m^t$

where $\{O_i\}_{i=1}^m$ are independent Haar distributed orthogonal matrices. Using the combinatorial theory that we will develop and first and second-order real freeness, we will determine its first and second-order limiting distributions. The motivation for looking at this matrix ensemble is to find examples of second-order real cumulants, as there are few examples in the literature. However, the second-order real cumulants of our model end up being equal to zero.

We will also develop a combinatorial framework for analyzing the orthogonal Weingarten function called partitioned pairings. These can be considered as a real analogue of the partitioned permutations used in higher-order freeness. In this framework, we derive recursion formulas for the orthogonal Weingarten function's leading and subleading order terms. Although not done in this thesis, paired partitioned pairings should provide the correct framework to develop a theory of higher-order real freeness and potentially allow us to derive formulas for terms of all orders of the orthogonal Weingarten function.

The thesis will proceed as follows. In Chapter 2, we will construct the Haar measure for the orthogonal group and derive the orthogonal Weingarten calculus. A self-contained derivation of the Weingarten calculus requires certain nontrivial results from invariant theory. Proofs of these results will be relegated to the appendix. Chapter 3 provides a mostly self-contained account of the combinatorics of non-crossing and annular non-crossing diagrams understood as computations inside the symmetric group. We will then intro-

duce partitioned permutations, the combinatorial framework of higher-order freeness. Using this, we will define an analogous structure called partitioned pairings, which we will use in the next chapter. We end this chapter by introducing the concepts from free probability that will be needed in our computations of our random matrix model, namely, the needed results of first and second-order freeness. In Chapter 4, we derive the leading and sub-leading terms of the orthogonal Weingarten function using the framework of partitioned parings. Finally, in Chapter 5, we compute our matrix model's first and second-order limiting distributions.

Chapter 2

Haar Measure and the Weingarten Calculus

2.1 Introduction

In random matrix theory, the distributions of many real (resp. complex) ensembles of interest are invariant under conjugation by orthogonal (resp. unitary) matrices. Being invariant under conjugation by a compact matrix group implies that the matrix ensemble is invariant under integration with respect to the Haar measure of that group. Haar measure is the unique translation-invariant probability measure associated with that group.

$$X \overset{\text{dist}}{\sim} OXO^t \Rightarrow X \overset{\text{dist}}{\sim} \int_{O(N)} OXO^t d\mu(O)$$

This motivates one to understand how to integrate polynomial functions in

the matrix entries with respect to the Haar measure.

The first person to tackle this problem was physicist Don Weingarten in [25], who obtained results on integrating polynomial functions with respect to $U(N)$ in the large N limit. In [2], Collins found an explicit formula for the integrating polynomial functions with respect to Haar measure on $U(N)$ for all N, so long as the degree of the polynomial was no more than $2N$. In [4], Collins and Śniady generalized this result to all degrees. In addition, they the analogous formulas for the orthogonal and symplectic groups.

In this chapter, we will construct the Haar measure for the orthogonal group and derive an expression for integrating polynomial functions with respect to the orthogonal Haar measure. Our expression will be a rational expression in $\frac{1}{N}$. This expression will not be explicit. However, in a later chapter, we will determine the leading and subleading terms for the orthogonal Weingarten function. In random matrix computations, one is principally interested in the leading order term as the lower order terms will not contribute to the large N limit. For more refined questions about the eigenvalue distribution, one may also be interested in the subleading term as well.

2.2 Construction of Haar Measure for the Orthogonal Group

In this section, we will construct the Haar probability measure for the orthogonal group. The Haar probability measure is a translation-invariant

Radon probability measure. For a compact Hausdorff topological group, the existence and uniqueness of this measure are guaranteed. We will not investigate the general theory of this. Instead, we will focus on constructing the Haar measure for the orthogonal group. We will also comment on how to modify this construction for the unitary and symplectic groups, respectively. For alternative constructions of Haar measure on the orthogonal group and other compact matrix groups, see [9]. For a general account of the theory of Haar measure, one can consult a book such as [5].

Let $X = (x_{ij})$ be an $N \times N$ random matrix whose entries are i.i.d. standard Gaussian random variables. After identification of $\mathrm{Mat}_{N\times N}(\mathbb{R})$ with $\mathbb{R}^{N^2}$, X has the density

$$\frac{1}{(2\pi)^{\frac{N^2}{2}}} \exp\{-\frac{1}{2}\left(x_{11}^2 + x_{12}^2 + \cdots + x_{NN}^2\right)\}$$

Lemma 2.2.1. *Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonzero polynomial. Then, the set of all points $\vec{x} \in \mathbb{R}^n$ such that $p(\vec{x}) = 0$ has Lebesgue measure zero.*

Proof. Let λ denote the Lebesgue measure. When $n = 1$, the zero set of any nonzero polynomial is finite, so the assertion holds in this case. Now assume $n > 1$ and let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonzero polynomial.

Let $A = \{\vec{x} \in \mathbb{R}^n : p(\vec{x}) = 0\}$ denote the zero-set of p and let χ_A denote its indicator function. Then

$$\lambda(A) = \int_{\mathbb{R}^n} \chi_A(\vec{x}) d\vec{x} = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \chi(\vec{y}, x_n) d\vec{y} dx_n$$

where $\vec{y}$ denotes a generic vector in $\mathbb{R}^{n-1}$. By Fubini's theorem, we can switch the order of integration, yielding

$$\lambda(A) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_A(\vec{y}, x_n) dx_n d\vec{y} = \int_{\mathbb{R}^{n-1}} 0 d\vec{y} = 0$$

Where the final equality comes from the fact that for a fixed $\vec{y}$, $p(\vec{y}, x_n)$ is a univariate polynomial in the variable x_n. □

Corollary 2.2.2. *X is invertible, almost surely.*

Proof. Polynomial functions are nonzero almost everywhere. Since X is absolutely continuous with respect to Lebesgue measure, it follows that the probability that $\det(X) = 0$ is zero.

□

Lemma 2.2.3. *X is orthogonally invariant. i.e. for any $h \in O(N)$, hX and Xh have the same distribution as X.*

Proof. Given a measurable function f, let μ_f denote the pushforward of Lebesgue measure by f and let $\mathbb{E}_{\mu_f}$ denote the expectation taken with respect to μ_f.

Let $\varphi \in C_b(\mathbb{R}^{N^2})$ and let $h \in O(N)$. Write $X = (X_1, \ldots, X_N)$ where X_i is the i'th column of X. Then $hX = (hX_1, \ldots, hX_N)$. By noting that we can rewrite the density of X as $\frac{1}{(2\pi)^{\frac{N^2}{2}}} \exp\{-\frac{1}{2}\left(\|X_1\|^2 + \cdots + \|X_N\|^2\right)\}$ we obtain that $\mathbb{E}_{\mu_{hX}}(\varphi)$ equals

$$\int_{\mathbb{R}^{N^2}} \frac{1}{(2\pi)^{\frac{N^2}{2}}} \varphi(hX_1, \ldots, hX_N) \exp\{-\frac{1}{2}\left(\|hX_1\|^2 + \cdots + \|hX_N\|^2\right)\} dx_{11} \cdots dx_{NN}$$

Let $Y = hX$. Then as h is an isometry $\|Y_i\| = \|X_i\|$ for all i. Additionally, the absolute value of the Jacobian determinant of the change of variables is equal to 1. Thus, we obtain that $\mathbb{E}_{\mu_{hX}}(\varphi)$ equals

$$\int_{\mathbb{R}^{N^2}} \varphi(Y_1, \ldots, Y_N) \frac{1}{(2\pi)^{\frac{N^2}{2}}} \exp\{-\frac{1}{2}\left(\|Y_1\|^2 + \cdots + \|Y_N\|^2\right)\} dy_{11} \cdots dy_{NN}$$

Which is of the desired form. Similarly, by writing X row-wise, we see that X and Xh also have the same distribution. □

Define the mapping $T : GL(N) \to O(N)$ by $T(X_1, \ldots, X_N) = O$ where $O = (O_1, \ldots, O_N)$ is obtained from X by performing the Gram-Schmidt procedure on the columns of X. By our previous lemma, if $X = (x_{ij})$ is a random Gaussian matrix, $T(X)$ is defined almost surely. We will show that $T(X)$ is a Haar-distributed orthogonal matrix.

Lemma 2.2.4. *Let $X \in GL(N)$ be an arbitrary matrix. Then for any $h \in O(N)$, $T(hX) = hT(X)$.*

Proof. Let $T(X) = [O_1, \ldots, O_N]$ where O_i denotes the i'th column of $T(X)$. By performing the Gram-Schmidt procedure on X, one see that the i'th column of $hT(X)$ is given by

$$hO_i = \frac{hX_i - \sum_{k=1}^{i-1} \langle X_i, O_k \rangle hO_k}{\|X_i - \sum_{k=1}^{i-1} \langle X_i, O_k \rangle O_k\|}$$

As h is an isometry, this equals

$$\frac{hX_i - \sum_{k=1}^{i-1} \langle hX_i, hO_k \rangle hO_k}{\|hX_i - \sum_{k=1}^{i-1} \langle hX_i, hO_k \rangle hO_k\|}$$

By performing the Gram-Schmidt procedure on the column of hX, one sees that this is exactly the i'th column of $T(hX)$. Thus $T(hX) = hT(X)$. □

Corollary 2.2.5. *Let $X = (x_{ij})$ be a random matrix whose entries are i.i.d. standard Gaussian random variables, and let T be defined as above. Then $T(X)$ is defined almost surely and is a left-translation invariant orthogonal matrix.*

Proof. Let A be a Borel-measurable set, and let $h \in O(N)$. Then $\mathbb{P}(hT(X) \in A) = \mathbb{P}(T(hX) \in A) = \mathbb{P}(hX \in T^{-1}(A)) = \mathbb{P}(X \in T^{-1}(A)) = \mathbb{P}(T(X) \in A)$ where T^{-1} denotes the inverse-image of T. □

By performing Gram-Schmidt on the rows instead and by repeating the above proofs, we obtain a right translation-invariant orthogonal matrix. Now that we have shown that left and right translation-invariant probability measures exist, we will show that they are equal and unique.

Proposition 2.2.6. *Let G be a compact Hausdorff topological group, and let μ, ν be left (resp. right) translation-invariant probability measures. Then $\mu = \nu$*

Proof. Let $\varphi \in C(G)$. and let $h \in G$. Then

$$\mathbb{E}_\mu(\varphi) = \int_G \varphi(g)d\mu(g) = \int_G \varphi(hg)d\mu(g) = \int_G \int_G \varphi(hg)d\mu(g)d\nu(h)$$

As G is compact, we can apply Fubini's theorem and switch the order of integration, yielding

$$\int_G \int_G \varphi(hg)d\nu(g)d\mu(g) = \int_G \varphi(hg)d\nu(h) = \int_G \varphi(h)d\nu(h) = \mathbb{E}_\nu(\varphi)$$

This holds for any continuous function; thus, it follows that $\mu = \nu$. □

Since the orthogonal group is a compact Hausdorff topological group, we have just shown the existence and uniqueness of its Haar probability measure.

Remark 2.2.7. By replacing real standard Gaussian random variables with complex (resp. quaternionic) standard Gaussian random variables and repeating the above procedure, one can obtain the Haar measure of the unitary and symplectic groups. A complex standard Gaussian is of the form $\frac{1}{\sqrt{2}}(X + iY)$ where X and Y are independent standard Gaussians. Similarly, a quaternionic standard Gaussian is of the form $\frac{1}{2}(X + iY + jW + kZ)$ where X, Y, W and Z are independent standard Gaussians.

2.3 The Weingarten Calculus

2.3.1 Some Basic computations using Haar measure

The goal of developing the Weingarten calculus is to be able to integrate polynomial functions with respect to Haar measure. This section will be a warmup to this. We use elementary properties of Haar measure to integrate monomials of odd degree and of degree 2. Going past this requires more sophisticated tools, which will be the subject of the Weingarten calculus.

Proposition 2.3.1. *Let $O = (o_{ij})$ be a Haar distributed orthogonal matrix and let $u, u', v, v' \in \mathbb{R}^N$ be any four unit vectors. Then $\langle Ou, v\rangle \overset{\text{dist}}{\sim} \langle Ou', v'\rangle$. In particular $o_{ij} \overset{\text{dist}}{\sim} o_{kl}$ for all choices of indices i, j, k, l.*

Proof. Since $O(N)$ acts transitively on unit vectors, there exists $g, h \in O(N)$ such that $hu = u'$ and $gv = v'$. Then, by translation invariance of Haar measure, $\langle Ou, v\rangle \overset{\text{dist}}{\sim} \langle g^t Ohu, v\rangle \overset{\text{dist}}{\sim} \langle Ohu, gv\rangle \overset{\text{dist}}{\sim} \langle Ou', v'\rangle$ □

Proposition 2.3.2. *Let $O = (o_{ij})$ be an $N \times N$ Haar distributed Orthogonal matrix. Then*

1. *For all indices i, j, $\mathbb{E}(o_{ij}^2) = \frac{1}{N}$*
2. *For any odd k and indices $i_1, j_1, \ldots, i_k, j_k$, $\mathbb{E}(o_{i_1 j_1} \cdots o_{i_k j_k}) = 0$*

Proof.

1. Because O is orthogonal, $\sum_{i=1}^N o_{ij}^2 = 1$. Thus, by the previous proposition, $\mathbb{E}(\sum_{i=1}^N o_{ij}^2) = N\mathbb{E}(o_{ij}^2) = 1 \Rightarrow \mathbb{E}(o_{ij}^2) = \frac{1}{N}$

2. Since $-I \in O(N)$, $O \overset{\text{dist}}{\sim} -O$ thus $\mathbb{E}(o_{i_1 j_1} \cdots o_{i_k j_k}) = -\mathbb{E}(o_{i_1 j_1} \cdots o_{i_k j_k})$, which implies $\mathbb{E}(o_{i_1 j_1} \cdots o_{i_k j_k}) = 0$

□

2.3.2 Tensor products and Representation Theory

This section will recall some facts about tensor products and representation theory, which will be needed for the Weingarten calculus. In the general setup of the Weingarten calculus, we need to consider polynomial functions in both regular and conjugate variables; thus, in this section and in the next, we will do everything over a complex inner product space V. In the case of the orthogonal group and its subgroups, we can take V to be a real inner product space.

Definition 2.3.3.

1. Let V be a finite-dimensional complex inner product space. A **representation** of a group G is a group homomorphism $\varphi : G \to GL(V)$.

2. Given a representation $\varphi : G \to GL(V)$, we define the **dual representation** $\varphi^* : G \to GL(V^*)$ to be $\varphi^*(g) \cdot v^* = v^* \circ \varphi(g^{-1})$.

3. If $G \leq GL(V)$, then we call the inclusion map $i : G \to GL(V)$ the **defining representation**.

4. Given a representation $\varphi : G \to GL(V)$, we let V^G denote the subspace of all vectors $v \in V$ such that $\varphi(g)v = v$ for all $g \in G$.

When the mapping φ is clear, we shall denote $\varphi(g)$ simply by g, and the multiplication $\varphi(g)v$ by $g \cdot v$.

We are interested in studying the extension of the defining representation to tensor products. To this end, we will introduce some needed definitions and notation.

Let V be an N-dimensional inner product space. Recall that we have an anti-linear isomorphism between V and its dual space V^* via the mapping $v \to v^*$ where $v^*(w) := \langle w, v \rangle$ for all $w \in V$. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis of V. By defining the inner product on V^*, $\langle w^*, v^* \rangle := \langle v, w \rangle$, the dual basis $\{e_i^*\}_{i=1}^N$ becomes an orthonormal basis for V^*. Let $T^0(V) = \mathbb{C}$ and for $k \geq 1$, let $T^k(V) = V^{\otimes k}$ denote the k-fold tensor product and $T^{k,l}(V) = V^{\otimes k} \otimes V^{*\otimes l}$. Given multi-indices $I : [k] \to [N]$ and $J : [l] \to [N]$, we let $e_I \otimes e_J^* \in T^{k,l}(V)$ denote the tensor $e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e_{j_1}^* \otimes \cdots \otimes e_{j_l}^*$, where $i_t := I(t)$ and $j_s := J(s)$ for all $1 \leq t \leq k, 1 \leq s \leq l$.

Definition 2.3.4. We call $T(V) := \bigoplus_{i=0}^{\infty} T^k(V)$ the tensor algebra over V and $T(V \bigoplus V^*) \cong \bigoplus_{k,l=0}^{\infty} T^k(V) \otimes T^l(V^*)$ the mixed tensor algebra over V. Note that $T(V \bigoplus V^*)$ and $\bigoplus_{k,l=0}^{\infty} T^k(V) \otimes T^l(V^*)$ are naturally isomorphic due to the natural isomorphisms $(V \oplus W) \otimes U \cong V \otimes U \oplus W \otimes U$ and $V \otimes W \cong W \otimes V$. The multiplication rules are given by $e_I \cdot e_{I'} = e_I \otimes e_{I'}$ and $e_I \otimes e_J^* \cdot e_{I'} \otimes e_{J'}^* = (e_I \otimes e_{I'}) \otimes (e_J^* \otimes e_{J'}^*)$ respectively.

Given any representation φ and its dual representation φ^*, by functorality, we can extend these to a representation on the mixed tensor algebra

$T(V \bigoplus V^*)$ over V via the the rule $\widetilde{\varphi}(g) \cdot v_1 \otimes \cdots \otimes v_k \otimes w_1^* \otimes \cdots \otimes w_l^* := \varphi(g)v_1 \otimes \cdots \otimes \varphi(g)v_k \otimes \varphi^*(g)w_1^* \otimes \cdots \otimes \varphi^*(g)w_l^*$.

Lemma 2.3.5. *Let (g_{ij}) be the matrix representation of $\varphi(g)$ with respect to the orthonormal basis $e_1, \ldots, e_n$. Then $\varphi^*(g)$ has the matrix representation $((g_{ij})^{-1})^t$ with respect to the dual basis $e_1^*, \ldots, e_n^*$.*

Proof. Let g_{ij}^{-1} denote the entries of $(g_{ij})^{-1}$. i.e. $g^{-1} = \langle g^{-1}e_j, e_i\rangle$. Then $e_j^* \circ g^{-1}(e_i) = e_j^*(\sum_k g_{ki}^{-1}e_k) = \langle \sum_k g_{ki}^{-1}e_k, e_j\rangle = g_{ji}^{-1}$. Thus $e_j^* \circ g^{-1} = \sum_j g_{ji}^{-1}e_i^*$ which implies the result. □

Corollary 2.3.6. *If $\varphi(g) \in U(N)$ has matrix representation (g_{ij}) with respect to the orthonormal basis $\{e_i\}$ then $\varphi^*(g)$ has the matrix representation $(\overline{g_{ij}})$ with respect to the dual basis $\{e_i^*\}$.*

Corollary 2.3.7. *If $\varphi(G) \leq O(N)$, then $\varphi(g)$ and $\varphi^*(g)$ have the same matrix representations, thus (V, φ) and (V^*, φ^*) are equivalent as representations.*

This corollary tells us that when dealing with the orthogonal group and its subgroups, we only need to look at the regular tensor algebra over V when studying its invariants. This results in us not needing to consider polynomials in conjugate variables.

Corollary 2.3.8. *Let $i : G \to U(N)$ be the defining representation and $i^{k,l} : G \to GL(T^{k,l}(V))$ the extension of i to the k, l'th grading. Then $i^{k,l}(g)$ has the matrix representation $g^{\otimes k} \otimes \overline{g}^{\otimes l}$ with respect to the basis $\{e_I \otimes e_J^*\}$.*

2.3.3 The Weingarten Calculus

In this section, we will show how to integrate expressions of the form $\int_{O(N)} o_{i_1 j_1} \cdots o_{i_k j_k} d\mu$. We will initially develop the theory of the Weingarten calculus for a general compact matrix group $G \leq GL(V)$ where V is a finite-dimensional complex inner product space as the arguments follow the same line of reasoning. As stated before, the only difference is that in the complex case, we also need to understand monomials in the conjugate variables as well. i.e. we need to understand expressions of the form $\int_G g_{i_1 j_1} \cdots g_{i_k j_k} \overline{g_{i'_1 j'_1}} \cdots \overline{g_{i'_l j'_l}} d\mu(g)$. Note that without loss of generality, for any compact subgroup G, we may assume that $G \leq U(N)$ as given an inner product $\langle \cdot, \cdot \rangle$, we can define the group averaged inner product $\langle u, v \rangle_G := \int_G \langle gv, gu \rangle d\mu(g)$. One can check that $(V, \langle \cdot, \cdot \rangle_G)$ is still an inner product space, and that G is a unitary subgroup with respect to this inner product.

We will now rewrite our integral in terms of tensor notation and show how it connects to the representation theory of the underlying group.

Since $g_{i_1 j_1} \cdots g_{i_k j_k} \overline{g_{i'_1 j'_1}} \cdots \overline{g_{i'_l j'_l}} = \langle g^{\otimes k} \otimes \overline{g}^{\otimes l} e_{I'} \otimes e^*_{J'}, e_I \otimes e^*_J \rangle$, we can rewrite out integral as $\int_G g_{i_1 j_1} \cdots g_{i_k j_k} \overline{g_{i'_1 j'_1}} \cdots \overline{g_{i'_l j'_l}} d\mu(g) = \left(\mathbb{E} \left(g^{\otimes k} \otimes \overline{g}^{\otimes l} \right) \right)_{I,J}$.

Proposition 2.3.9. *Let $G \leq U(N)$ and let $Z_{k,l} = \mathbb{E} \left(g^{\otimes k} \otimes \overline{g}^{\otimes l} \right)$ where expectation is taken component-wise. Then $Z_{k,l}$ is the orthogonal projection operator from $T^{k,l}(V)$ onto $(T^{k,l}(V))^G$.*

Proof. First, we will check that $Z^2_{k,l} = Z_{k,l}$. Let $h \in G$. Then by the translation invariance of Haar measure, $Z_{k,l} = \mathbb{E} \left((hg)^{\otimes k} \otimes (\overline{hg})^{\otimes l} \right) = h^{\otimes k} \otimes$

$\overline{h}^{\otimes l} Z_{k,l}$. Integrating with respect to h, the result follows.

Next, we will check that $Z_{k,l}$ is self-adjoint.

$Z_{k,l}^* = (\mathbb{E}\left(g^{\otimes k} \otimes \overline{g}^{\otimes l}\right))^* = \mathbb{E}\left((g^{\otimes k} \otimes \overline{g}^{\otimes l})^*\right)$

$= \mathbb{E}\left((g^*)^{\otimes k} \otimes (\overline{g^*})^{\otimes l}\right) = \mathbb{E}\left((g^{-1})^{\otimes k} \otimes \overline{g^{-1}}^{\otimes l}\right) = \mathbb{E}\left(g^{\otimes k} \otimes \overline{g}^{\otimes l}\right)$.

The second equality comes from the fact that the conjugation and transpose operations both commute with $\mathbb{E}(\cdot)$ and the final equality comes from the fact that both g and $g^* = g^{-1}$ have the same distribution under Haar measure.

Finally, we must check that $\mathrm{Im}(Z_{k,l}) = (T^{k,l}(V))^G$. Let $v \in (T^{k,l}(V))^G$. Then $Z_{k,l}(v) = \mathbb{E}\left(g^{\otimes k} \otimes \overline{g}^{\otimes l}(v)\right) = \mathbb{E}(v) = v$. Thus $(T^{k,l}(V))^G \subseteq \mathrm{Im}(Z_{k,l})$. For the other inclusion, let $h \in G$. Then $h^{\otimes k} \otimes \overline{h}^{\otimes l} Z_{k,l}(v) = \mathbb{E}((hg)^{\otimes k} \otimes (\overline{hg})^{\otimes l}(v)) = \mathbb{E}(g^{\otimes k} \otimes \overline{g}^{\otimes l}(v)) = Z_{k,l}(v)$. □

The question of computing our matrix integral has now been reduced to understanding the projection operator onto the invariant subspaces $(T^{k,l}(V))^G$. Ideally, one would like to have orthonormal bases for our invariant subspaces. The natural spanning set that comes with the invariant subspace is not orthogonal and not always linearly independent. Thus, in order to be able to do computations, we need to understand how to compute the orthogonal projection when we are only given a spanning set. For this, we will use what is called the pseudo-inverse.

Proposition 2.3.10. *Let $T : V \to W$ be any linear transformation between finite-dimensional inner product spaces. Then there exists a unique linear*

operator $T^\dagger : W \to V$ *such that*

1. $TT^\dagger T = T$
2. $T^\dagger T T^\dagger = T^\dagger$
3. $TT^\dagger$ *is the orthogonal projection onto* $\mathrm{Im}(T)$
4. $T^\dagger T$ is the orthogonal projection onto $\mathrm{Ker}(T)^\perp$

$T^\dagger$ *is called the pseudo-inverse of* T

Proof. We will show how $T^\dagger$ is defined but omit the proof of the stated properties.

Write $V = \mathrm{Ker}(T) \bigoplus \mathrm{Ker}(T)^\perp$ and $W = \mathrm{Im}(T) \bigoplus \mathrm{Im}(T)^\perp$.

Then $T_{|\mathrm{Ker}(T)^\perp} : \mathrm{Ker}(T)^\perp \to \mathrm{Im}(T)$ is invertible. Let T' denote the inverse and define the pseudo-inverse $T^\dagger$ to be the extension of T' by having $T^\dagger(w) = 0$ for all $w \in \mathrm{Im}(T)^\perp$. Then, $T^\dagger$ is the unique operator that satisfies the stated properties. □

Let V be any finite-dimension inner product space and let W be a subspace of V spanned by the vectors $\{w_1, \ldots, w_n\}$. Let P be the orthogonal projection onto W. We are going to compute $P(v)$ for any arbitrary $v \in V$ using the spanning set $\{w_1, \ldots, w_n\}$. Let $G = (\langle w_i, w_j \rangle)^t$ denote the trans-

pose of the Gram matrix of $\{w_1, \ldots, w_n\}$. i.e.

$$G = \begin{bmatrix} \langle w_1, w_1 \rangle & \langle w_2, w_1 \rangle & \cdots & \langle w_n, w_1 \rangle \\ \langle w_1, w_2 \rangle & \langle w_2, w_2 \rangle & \cdots & \langle w_n, w_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle w_n, w_1 \rangle & \langle w_2, w_n \rangle & \cdots & \langle w_n, w_n \rangle \end{bmatrix}$$

Let Wg denote the pseudo-inverse of G. Denote $\alpha_j := \langle v, w_j \rangle$ and $\beta := \mathrm{Wg}\alpha$ were α is the column vector $(\alpha_i)_{i=1}^N$. We will show that $P(v) = \sum_i \beta_i w_i$.

Lemma 2.3.11. *Let G be defined as above and let $\gamma_i \in \mathbb{C}$ be a collection of scalars such that $P(v) = \sum_i \gamma_i w_i$. Then γ is unique up to elements of the kernel of G.*

Proof. Let $\gamma' \in \mathrm{Ker}(G)$. Then for all $1 \leq j \leq n$, $(G\gamma')_j = \sum_i \gamma_i' \langle w_i, w_j \rangle = \sum_i \langle \gamma_i' w_i, w_j \rangle = 0$. By linearity this implies $\langle \sum_i \gamma_i' w_i, \sum_i \gamma_i' w_i \rangle = 0$, which is equivalent to $\sum_i \gamma_i' w_i = 0$. Thus, $\sum_i (\gamma_i + \gamma_i') w_i = \sum_i \gamma_i w_i$

Conversely, assume that $\sum_i \gamma_i w_i = \sum_i \gamma_i' w_i$. Then $\sum_i (\gamma_i - \gamma_i') w_i = 0$, which implies that $\langle \sum_i (\gamma_i - \gamma_i') w_i, w_j \rangle = \sum_i (\gamma_i - \gamma_i') \langle w_i, w_j \rangle = (G(\gamma - \gamma'))_j = 0$ for all $1 \leq j \leq n$. Thus $\gamma - \gamma'$ is in the kernel of G, implying the result. □

Let $\gamma_i \in \mathbb{C}$ be any collection of scalars that satisfies $P(v) = \sum_i \gamma_i w_i$. Then $\langle v, w_j \rangle = \langle P(v), w_j \rangle = \sum_i \langle \gamma_i w_i, w_j \rangle = (G\gamma)_j$. Thus $G\gamma = \alpha$. But, as $G = G\mathrm{Wg}G$, we obtain $GWgG\gamma = G(\mathrm{Wg}\alpha)$. Since γ is unique up to elements in the kernel of G, we can take $\gamma = \mathrm{Wg}\alpha$.

Let $\{w_1, \ldots, w_n\}$ be a spanning set of $(T^{k,l}(V))^G$. Let G denote the transpose of the gram matrix and Wg its pseudo-inverse. Then, our matrix integral becomes

$$\int_G g_{i_1 j_1} \cdots g_{i_k j_k} \overline{g_{i'_1 j'_1}} \cdots \overline{g_{i'_l j'_l}} d\mu(g) = \langle Z_{k,l} e_J \otimes e^*_{J'}, e_I \otimes e^*_{I'} \rangle$$

$$= \langle \sum_{i=1}^{n} \sum_{j=1}^{n} \langle e_J \otimes e^*_{J'}, w_j \rangle \mathrm{Wg}(w_i, w_j) w_i, e_I \otimes e^*_{I'} \rangle$$

$$= \sum_{i,j=1}^{n} \langle e_J \otimes e^*_{J'}, w_j \rangle \langle e_I \otimes e^*_{I'}, w_i \rangle \mathrm{Wg}(w_i, w_j)$$

We will now specialize to the case of the orthogonal group. Note that due to the fact that $-I \in O(N)$, $(T^k(V))^{O(N)} = \{0\}$ for k odd.

Definition 2.3.12.

1. We call $p \in S_{2k}$ a pairing if p has no fixed points and $p^2 = e$. We denote the set of all pairings by $\mathcal{P}_2(2k)$

2. Given a pairing $p \in S_{2k}$, we define the pairing tensor

   $p(e) := \sum_{I \in [N]^{[2k]}} \left\langle \begin{smallmatrix} I \\ p \end{smallmatrix} \right\rangle e_I \in T^{2k}(V)$ where

$$\left\langle \begin{smallmatrix} I \\ p \end{smallmatrix} \right\rangle = \begin{cases} 1 & \text{if } I = I \circ p \\ 0 & \text{if } I \neq I \circ p \end{cases}$$

   and $\circ$ denotes function composition.

Remark 2.3.13. $\left\langle \begin{smallmatrix} I \\ p \end{smallmatrix} \right\rangle$ is a generalized Kronecker δ. If $I = I \circ p$, then $i_l = I(l) = I(p(l)) = i_{p(l)}$, thus $\left\langle \begin{smallmatrix} I \\ p \end{smallmatrix} \right\rangle = \prod_{(r,s)\in p} \delta_{i_r,i_s}$.

Proposition 2.3.14. *The pairing tensors are $O(N)$-invariants.*

Proof. Let $O = (o_{ij}) \in O(N)$. Then

$$O \cdot p(e) = \sum_{I \in [N]^{2k}} \left\langle \begin{smallmatrix} I \\ p \end{smallmatrix} \right\rangle (Oe_{i_1}) \otimes \cdots \otimes (Oe_{i_{2k}})$$

$$= \sum_I \sum_J \left\langle \begin{smallmatrix} I \\ p \end{smallmatrix} \right\rangle o_{j_1,i_1} \cdots o_{j_{2k}i_{2k}} e_{j_1} \otimes \cdots \otimes e_{j_{2k}}$$

$$= \sum_J e_J \prod_{(r,s)\in p} \left(\sum_{i=1}^{N} o_{j_r i_r} o_{j_s i_s} \right) = \sum_J e_J \prod_{(r,s)\in p} \delta_{j_r,j_s} = \sum_J \left\langle \begin{smallmatrix} J \\ p \end{smallmatrix} \right\rangle e_J$$

□

Theorem 2.3.15. *The pairing tensors form a spanning set for $(T^{2k}(V))^{O(N)}$. Further, if $N \geq k$, they are a basis.*

We will not prove this here. Instead, the proof will be relegated to the Appendix. Showing that the pairing tensors form a spanning set is nontrivial and requires us to develop more theory.

Given $p, q \in \mathcal{P}_2(2k)$, we compute the inner product of the corresponding pairing tensors. $\langle p(e), q(e) \rangle = \#\{I \in [N]^{[2k]} : p \vee q \leq \mathrm{Ker}(I)\} = N^{\#p\vee q}$ where $p \vee q$ denotes the smallest partition such that the cycles of both p and q are

contained inside of its blocks and $\mathrm{Ker}(I)$ denotes the partition $\pi \in \mathcal{P}(2k)$ defined by the rule $l \sim_\pi k \iff I(l) = I(k)$.

Thus, the matrix G of our paring tensors is given by $G = (N^{\#(p\vee q)})_{p,q\in\mathcal{P}_2(2k)}$ and the orthogonal Weingarten function, Wg, is the pseudo-inverse of this. Denote $\langle \mathrm{Wg}(p), q\rangle := \mathrm{Wg}(p,q)$. Rewriting the general expression in terms of the orthogonal Weingarten function, we obtain the formula

$$\int_{O(N)} o_{i_1 j_1}\cdots o_{i_{2k} j_{2k}} d\mu(O) = \sum_{p,q\in\mathcal{P}_2(2k)} \left\langle \begin{matrix} I \\ p \end{matrix} \right\rangle \left\langle \begin{matrix} J \\ q \end{matrix} \right\rangle \mathrm{Wg}(p,q)$$

One can see from the form of G, that $Wg(p,q)$ is a rational function in $\frac{1}{N}$. Computing $\mathrm{Wg}(p,q)$ completely is very challenging. We only need to know the leading order term for our computations involving the limiting eigenvalue distribution of random matrix models. Determining the leading order and the subleading order term will be the subject of chapter 4.

Chapter 3

Combinatorics and Free Probability

3.1 Introduction

When computing asymptotic eigenvalue quantities, one often gets a rational expression in one over the dimension. To compute asymptotic quantities, one only needs to determine the order 1 contributions as all lower order terms disappear in the large N limit. Under appropriate assumptions, one can use techniques of free probability to determine the joint eigenvalue distributions from the original ones using quantities called free cumulants. This section will provide a mostly self-contained development of the combinatorial tools needed for our matrix models. We will then introduce the needed concepts from free probability. For a more extensive introduction to the combinatorics

of free probability, one can consult [17]. For more on the relations between free probability and random matrix theory, one can consult [15].

3.2 Geodesics in S_n and non-crossing Diagrams

This section will develop the necessary theory related to the symmetric group and non-crossing diagrams. In the computation of the moments of orthogonally invariant random matrix models, one gets a finite expansion in $\frac{1}{N}$ where N is the dimension. It ends up being the case that the leading order term can be written as a sum over non-crossing permutations, while the sub-leading term can be written as a sum over non-crossing annular permutations that satisfy certain symmetry conditions. A permutation being non-crossing or non-crossing annular is equivalent to satisfying certain geodesic conditions with respect to a length function on S_n.

3.2.1 Cayley Graph of S_n and Length Functions on the Symmetric Group

Definition 3.2.1. Let G denote a group. We call $S \subseteq G$ a **symmetric subset** of G if

1. $S = S^{-1}$
2. $e \notin S$

Definition 3.2.2. Let $S \subseteq G$ be a symmetric subset. We define the **left Cayley graph** (V, E) of G with respect to S to be the graph such that $V = G$ and $(g, h) \in E \iff hg^{-1} \in S$. Equivalently, $h = sg$ for some $s \in S$. Similarly, we define the **right Cayley graph** (V, E) of G with respect to S to be the graph such that $V = G$ and $(g, h) \in E \iff g^{-1}h \in S$.

Note that if S is invariant under conjugation, then the left and right Cayley graphs are equal.

Proposition 3.2.3. *A Cayley graph of a group G with respect to S is connected iff S generates G.*

Proof. We prove this for a right Cayley graph. The proof for the left Cayley graph follows similarly. If S generates G, finding a path from g to h follows from the equation $gx = h$ being solvable for any group. Since S generates G, there exists $s_1, \ldots, s_n \in S$ such that $gs_1 \cdots s_n = h$. Thus, by definition of the edge set, $(g, gs_1), (gs_1, gs_1s_2), \ldots, (gs_1 \cdots s_{n-1}, gs_1 \cdots s_n)$ is a path in the Cayley graph. Conversely, if (V, E) is connected, Then there is a path from e to g for any $g \in G$, which, after unravelling the definition, implies that $g \in \langle S \rangle$. □

3.2.2 Length functions and metrics on G

Definition 3.2.4. Let G be a group. We call $|\cdot|$ a **length function** on G if

1. $|g| \geq 0$ for all $g \in G$.

2. $|g| = 0 \iff g = e$

3. $|g_1 g_2| \leq |g_1| + |g_2|$ for all $g_1, g_2 \in G$.

Definition 3.2.5. Let G be a group and (V, E) a connected Cayley graph induced by a symmetric subset S. We define the length function on G with respect to S, $|g|_S$ as the length of the smallest path from the identity element to g. Equivalently, we define $|g|_S$ to be the smallest integer k such that $g = s_1 s_2 \cdots s_k$ with $s_i \in S$.

Given a length function on G, we can define a corresponding metric.

Proposition 3.2.6. *Let $G, (V, E)$, and S all be as before. Then $d_s(g, h) := |g^{-1}h|_S$ is a metric on G.*

The only thing not immediate in the symmetric condition. However, S being symmetric implies that $|g|_S = |g^{-1}|_S$, which in turn implies that $d_S(g, h) = |g^{-1}h|_S = |h^{-1}g|_S = d_S(h, g)$.

3.2.3 Length function with respect to transpositions

In this section, take $G = S_n$ and S equal to the set of transpositions. From here on, we will only be interested in the case where S is the set of transpositions. Therefore, we will drop the subscript S. Note that because S is invariant under conjugation, we have $|gh| = |hg|$. We will now look at the relationship between our length function and the cycle decomposition of a permutation.

Given $\sigma \in S_n$, let $\#(\sigma)$ equal the number of cycles contained in its cycle decomposition.

Lemma 3.2.7. *Let $\sigma \in S_n$ and let (i, j) be a transposition. Then*

$$\#((i,j)\sigma) = \begin{cases} \#(\sigma) + 1 \text{ if } i, j \text{ occur in the same block of } \sigma \\ \#(\sigma) - 1 \text{ if } i, j \text{ occur in different blocks of } \sigma \end{cases}$$

This lemma follows from the observation that $(i,j)(i,a_1,\ldots,a_k,j,b_1,\ldots,b_s) = (i,a_1,\ldots,a_k)(j,b_1,\ldots b_s)$. Also note that $(i,a_1,\ldots,a_k,j,b_1,\ldots,b_s)(i,j) = (b_1,\ldots,b_s,i)(a_1,\ldots,a_k,j)$.

Corollary 3.2.8. *For any permutation $\sigma \in S_n$, $\#(\sigma) = n - |\sigma|$.*

Proof. Let $\sigma = \tau_1 \cdots \tau_k$ be a minimal decomposition of σ into transpositions. Then, by our above lemma and the fact the $\#(e) = n$, were e is the identity, we obtain $\#(\sigma) \geq n - k = n - |\sigma|$. For the other inequality, decompose $\sigma = C_1 \cdots C_s$ into disjoint cycles of sizes $n_1, \ldots, n_s$ respectively. Then by noting that a cycle of length k can be written as a product of $k-1$ transpositions, such as $(i_1,\ldots,i_k) = (i_1,i_2)(i_2,i_3)\cdots(i_{k-1},i_k)$ for example, and by noting that for disjoint cycles C, C' we have $|CC'| = |C| + |C'|$, we obtain $|\sigma| \leq \sum_{i=1}^{s}(n_i - 1) = n - s = n - \#(\sigma)$. combining these inequalities, we obtain the result. □

Corollary 3.2.9. *Let $\sigma \in S_n$ and write $\sigma = \tau_1 \cdots \tau_k$ as a minimal transposition decomposition. Then for each $1 \leq i \leq k$, τ_i always merges two different*

blocks of $\tau_{i-1} \cdots \tau_k$. *Moreover, if* $\tau_i = (r_i, s_i)$, *then* r_i *and* s_i *belong in the same cycle of* σ.

Proof. By the above corollary $\#(\tau_1 \cdots \tau_k) = n - k$. Additionally, by lemma 3.2.7, we have $\#(\tau_1 \cdots \tau_k) \geq \#(\tau_2 \cdots \tau_k) - 1 \geq \#(\tau_3 \cdots \tau_k) - 2 \geq \cdots \geq \#(e) - k = n - k$ with equality at each step iff τ_i merges two blocks of $\tau_{i-1} \cdots \tau_k$ for all i. Since $\tau_i = (r_i, s_i)$ merges two cycles of $\tau_{i-1} \cdots \tau_1$, r_i and s_i will be in the same block of $\tau_i \cdots \tau_1$ and therefore in the same block of $\sigma = \tau_1 \cdots \tau_k$. Thus, the results follow.

□

3.2.4 Geodesic paths in S_n

We are interested now in understanding the geodesics of S_n. i.e. we are interested in determining when the triangle inequality becomes equality.

Definition 3.2.10. We say that a permutation π lies on the geodesic from σ_1 to σ_2 if $d(\sigma_1, \sigma_2) = d(\sigma_1, \pi) + d(\pi, \sigma_2)$.

In terms of the length function, this condition becomes $|\sigma_1^{-1}\sigma_2| = |\sigma_1^{-1}\pi| + |\pi^{-1}\sigma_2|$.

It is easy to check that our metric is translation invariant. i.e. $d(\sigma_1, \sigma_2) = d(e, \sigma_1^{-1}\sigma_2)$. Thus, to understand the geodesics of S_n, it is enough to understand the geodesics between the identity and another permutation.

We now introduce the concept of a non-crossing partition. This is the combinatorial object needed to understand the geodesic structure.

Definition 3.2.11. Let $I \subset [n]$. Let S_I denote the set of all permutations of the set I and $\mathcal{P}(I)$ the set of all partitions of I.

1. We define the **first return map** $\varphi_I : S_n \ni \sigma \to \sigma_{|I} \in S_I$ by the rule $\sigma_{|I}(i) = \sigma^s(i)$ where s is the smallest positive integer such that $\sigma^s(i) \in I$

2. Similarly, we define the $f_I : \mathcal{P}(n) \ni \pi \to \pi_{|I} \in \mathcal{P}(I)$ by the rule $f_I(\pi) = f_I(\{B_1, \ldots, B_k\}) = \pi_{|I} = \{B_1 \cap I, \ldots, B_k \cap I\}$ where B_i denote the blocks of π.

As an example, if $I = \{2, 3, 5\} \subseteq [5]$ and $\sigma = (2, 4, 5)(1, 3)$, then $\sigma_{|I} = (2, 5)(3)$

Definition 3.2.12. Let $\gamma = (i_1, i_2, \ldots, i_n) \in S_n$ be an n-cycle and let $\mathcal{P}(n)$ denote the set of all partitions of $[n]$.

We say $\pi \in \mathcal{P}(n)$ is in **γ-cyclic order** if for all $\{j_1, j_2, j_3, j_4\}$ such that $\gamma_{|\{j_1,j_2,j_3,j_4\}} = (j_1, j_2, j_3, j_4)$, the conditions $\pi_{|\{j_1,j_3\}} = \{\{j_1, j_3\}\}$ and $\pi_{|\{j_2,j_4\}} = \{\{j_2, j_4\}\}$ implies that $\pi_{|\{j_1,j_2,j_3,j_4\}} = \{\{j_1, j_2, j_3, j_4\}\}$

Alternatively, the conditions $j_1 \sim_\pi j_3$ and $j_2 \sim_\pi j_4$, implies that j_1, j_2, j_3, j_4 all belong to the same block. The set of all such partitions is denoted by $NC(\gamma)$.

More generally, let $\sigma \in S_n$ be any permutation and write $\sigma = C_1 \cdots C_s$ as a product of disjoint cycles. Denote the i'th cycle as $(i_1, \ldots, i_k)$ We say that $\pi \in \mathcal{P}(n)$ is in **σ-cyclic order** if

1. For all i, $\pi_{|C_i} := \pi_{|\{i_1,\ldots,i_k\}} \in NC(C_i)$

2. $\pi = \bigcup_i \pi_{|C_i}$

For the case of $\gamma_n = (1, \ldots, n) \in S_n$, we will denote $NC(\gamma_n)$ by $NC(n)$.

Remark 3.2.13. The definition of cyclic ordering is motivated by the following reasons.

If we put the order $1 < 2 \cdots < n$ of $[n]$, One can alternatively define $NC(n)$ as the set of all permutations π such that for all $j_1 < j_2 < j_3 < j_4$, if $j_1 \sim_\pi j_3$ and $j_2 \sim_\pi j_4$, then j_1, j_2, j_3, j_4 all belong in the same block. In this definition, however, any cyclic permutation of the chosen order, such as $n < 1 < 2 < \cdots < n-1$, defines the same set of partitions.

Remark 3.2.14. By giving $NC(n)$ and $\mathcal{P}(n)$ and the partial ordering $\pi_1 \leq \pi_2$ iff the blocks of π_1 are contained inside of the blocks of π_2, $NC(n)$ and $\mathcal{P}(n)$ become a lattice. More than this, $NC(n)$, unlike $\mathcal{P}(n)$, is a self-dual lattice. For more on the structure of $NC(n)$, including proofs of the above-stated properties, one can consult [17] and [1]. [17] provides a comprehensive introduction to the combinatorics of non-crossing partitions, while [1] determines the skew-automorphisms of $NC(n)$.

Definition 3.2.15. Let $\sigma \in S_n$. We say that a permutation $\pi \in S_n$ is in **σ-cyclic order** if

1. The cycles of π are contained inside of the blocks of σ.

2. For all j_1, j_2, j_3, j_4 such that $\sigma_{|\{j_1,j_2,j_3,j_4\}} = (j_1, j_2, j_3, j_4)$, the conditions $\pi_{|\{j_1,j_3\}} = (j_1, j_3)$ and $\pi_{|\{j_2,j_4\}} = (j_2, j_4)$ implies $\pi_{|\{j_1,j_2,j_3,j_4\}} = (j_1, j_2, j_3, j_4)$.

Using this, we define the partial order on S_n by $\pi \leq \sigma$ if and only if π is in σ-cyclic order, and we denote the set of such permutations as $S_{NC}(\sigma)$ and we call the elements π of $S_{NC}(\sigma)$ **σ-noncrossing permutation**. It is immediate from the definitions that $S_{NC}(\sigma)$ and $NC(\sigma)$ are isomorphic as posets.

We will now show that membership in $S_{NC}(\sigma)$ is equivalent to lying on a geodesic from e to σ. For this, we prove a special case of σ being equal to the permutation $\gamma_n := (1, 2, \ldots, n)$, from which the general case will follow. Note that there is nothing special about γ_n versus any other n-cycle in the below proof. It is chosen just to make the proof below easier to follow.

Lemma 3.2.16. *Let $\gamma_n = (1, 2, \ldots, n)$ and let $C = (i_1, \ldots, i_k) \in S_n$ be a cycle. Then $C \leq \gamma_n$ iff $|C| + |C^{-1}\gamma_n| = |\gamma_n|$.*

Proof. Write $C = (i_1, i_2) \cdots (i_{k-1}, i_k)$ as a minimal transposition decomposition. Assume without loss of generality that i_1 is the smallest element that occurs inside of C with respect to the order induced by γ_n. Recall from lemma 3.2.7 that $(i, j)(i, a_1, \ldots, a_k, j, b_1, \ldots, b_s) = (i, a_1, \ldots, a_k)(j, b_1, \ldots, b_s)$. We see from this that multiplication by (i, j) "cuts" the cycle $(i, a_1, \ldots, a_k, j, b_1, \ldots, b_s)$ into two cycles $(i, a_1, \ldots, a_k)(j, b_1, \ldots, b_s)$ which are non-crossing relative to $(i, a_1, \ldots, a_k, j, b_1, \ldots, b_s)$. Specializing to the case of γ_n, we see that $(i, j)(i, i+1, \ldots j, j+1 \ldots, n, 1, \ldots, i-1) = (i, i+1, \ldots, j-1)(j, j+1 \ldots, n, 1, \ldots, i-1)$. Thus, if s is such that $i < s < j$, then s and j will be in different blocks of $(i, j)\gamma_n$.

By the triangle inequality, we have $|C| + |C^{-1}\gamma_n| \geq |\gamma_n|$. By corollary 3.2.9, equality to occurs if and only if (i_l, i_{l+1}) is contained inside of a block of $(i_{l-1}, i_l) \cdots (i_1, i_2)\gamma_n$ as it must be the case that for all i,
$|(i_{l-1}, i_l)(i_{l-1}, i_l) \cdots (i_1, i_2)\gamma_n| = |(i_{l-1}, i_l) \cdots (i_1, i_2)\gamma_n| - 1$

Trivially we see that $|(i_1, i_2)| + |(i_1, i_2)\gamma_n| = |\gamma_n|$. We now claim that $|(i_1, i_2)(i_2, i_3)| + |(i_2, i_3)(i_1, i_2)\gamma_n| = |\gamma_n| \iff i_1 < i_2 < i_3$. If $i_3 > i_2$, then by the above computation, i_2 and i_3 belong to the same cycle of $(i_1, i_2)\gamma_n$. Conversely, since (i_2, i_3) must split a cycle of $(i_1, i_2)\gamma_n$ and since in our above computation all s satisfying $i_1 < s < i_2$ are in a different cycle than i_2, the minimality of i_1 forces $i_3 > i_2$. Now assume that for all $1 \leq s \leq l-1$, we have $i_1 < i_2 < \cdots < i_{l-1}$. We shall show that $i_l > i_{l-1}$ is equivalent to $|(i_1, i_2) \cdots (i_{l-1}, i_l)| + |(i_l, i_{l-1}) \cdots (i_2, i_1)\gamma_n| = |\gamma_n|$.

By our above computation, the cycle containing i_{l-1} in $(i_{l-1}, i_{l-2}) \cdots (i_2, i_1)\gamma_n)$ is equal to $(i_{l-1}, i_{l-1} + 1, \ldots, n, 1, \ldots, i_1 - 1)$. If $i_l > i_{l-1}$, then (i_{l-1}, i_l) will split a cycles implying the equality.

Conversely, if the equality holds, (i_l, i_{l-1}) will split a cycle of $(i_{l-1}, i_{l-2}) \cdots (i_2, i_1)\gamma_n$. Thus i_l is contained inside of the cycle $(i_{l-1}, i_{l-1} + 1, \ldots, n, 1, \ldots, i_1 - 1)$. By assumption, as i_1 is the smallest element of C, it must be the case that $i_{l-1} + 1 \leq i_l \leq n$, thus the result follows.

□

Corollary 3.2.17. *Let $\pi, \sigma \in S_n$. Then $|\sigma| = |\pi| + |\pi^{-1}\sigma|$ iff $\pi \in S_{NC}(\sigma)$.*

Proof. Write $\sigma = C_1 \cdots C_s$ as a product of disjoint cycles. Both the conditions $\pi \in S_{NC}(\sigma)$ and $|\pi| + |\pi^{-1}\sigma| = |\sigma|$ require that the cycles of π be

contained in the cycles of σ, therefore, write $\pi = \pi_1 \cdots \pi_s$ as a product of permutations such that the π_i's have disjoint support and is contained inside of σ_i (note that π_i may equal the identity). Then $|\sigma| = |\pi| + |\pi^{-1}\sigma| \iff$ for all $1 \leq i \leq k$ $|\pi_i| + |\pi_i^{-1}C_i| = |C_i| \iff \forall 1 \leq i \leq k$ $\pi_i \in S_{NC}(C_i) \iff$ $\pi \in S_{NC}(\sigma)$

□

For any $\pi, \sigma \in S_n$ the triangle inequality tells us $|\pi| + |\pi^{-1}\sigma| \geq |\sigma|$. An important fact about the case of inequality is that its values can only change in steps of 2.

Corollary 3.2.18. *Let $\sigma, \pi \in S_n$. Then $d(e, \pi) + d(\pi, \sigma) = |\pi| + |\pi^{-1}\sigma| = d(e, \sigma) + 2m = |\sigma| + 2m$ where $m \in \{0, 1, 2, \ldots\}$.*

Proof. Write $\pi = \tau_1 \cdots \tau_k$ as a minimal transposition decomposition. If π is non-crossing relative to σ, then the equation is satisfied with $m = 0$. Therefore, assume that this is not the case. Let l be the smallest integer such that $\tau_1 \cdots \tau_{l-1}$ is non-crossing relative to σ, but $\tau_1 \cdots \tau_l$ is not.

By lemma 3.2.7 this means that τ_l must join a block in $\tau_{l-1} \cdots \tau_1 \sigma$. This gives us the equality $|\tau_1 \cdots \tau_l| + |\tau_l \cdots \tau_1 \sigma| = |\tau_1 \cdots \tau_{l-1}| + 1 + |\tau_{l-1} \cdots \tau_1 \sigma| + 1 = |\sigma| + 2$. If $l = k$, then we are done. If not define $\sigma' = \tau_l \cdots \tau_1 \sigma$ and $\pi' = \tau_{l+1} \cdots \tau_k$ and repeat the analysis with respect to π' and σ'. Continuing until our algorithm terminates, we obtain that $|\pi| + |\pi^{-1}\sigma| = |\sigma| + 2m$ where $m \in \mathbb{N}$

□

Remark 3.2.19. Expressing the above condition in terms of $\#(\cdot)$, we obtain $\#(\pi) + \#(\pi^{-1}\sigma) = n + \#(\sigma) - 2m$ with $m = 0$ iff π is non-crossing relative to σ.

We end this section with an equivalent condition for being a non-crossing permutation.

Proposition 3.2.20 (Local non-crossing conditions)**.** *Fix $\pi, \sigma \in S_n$. Then $\pi \in S_{NC}(\sigma)$ iff neither of the following conditions hold.*

1. *There exists i, j, k such that $\sigma_{|\{i,j,k\}} = (i, j, k)$ and $\pi_{|\{i,k,j\}} = (i, k, j)$*
2. *There exists i, j, k such that $\sigma_{|\{i,j,k\}} = (i, j)(k)$ and $\pi_{|\{i,j,k\}} = (i, j, k)$*

We omit the proof of this as it is straightforward. Condition (1) occurring meaning that π does not respect the order structure induced by σ while condition (2) occurring means that the blocks of π are not contained inside the blocks of σ.

Remark 3.2.21. A permutation σ being non-crossing relative to γ_n is equivalent to the following diagrammatic description. Draw a circle with points labelled 1 up to n. Then σ being non-crossing is equivalent to being able to draw the cycles of σ such that they respect to order induced by n and such that they do not cross each other.

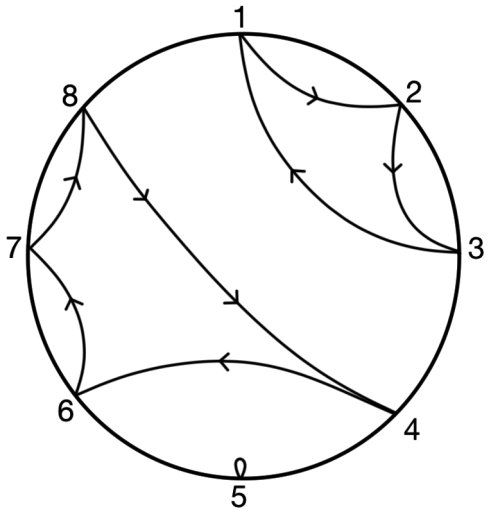

Figure 3.1: Example of a non-crossing permutation. $\sigma = (1, 2, 3)(4, 6, 7, 8)(5)$

3.2.5 Non-Crossing Annular Permutations

In this section, we shall look at an analogue of the non-crossing partitions, which we will call non-crossing annular diagrams. More specifically, we will be looking at the associated permutations, as while we had a bijection from $S_{NC}(\sigma)$ to $NC(\sigma)$, we only get a surjection in the case of the annular diagrams. This does not pose an issue for us, as the permutations, not the diagrams, will appear in our computations.

Definition 3.2.22. Let $n, m \in \mathbb{N}$ and let $\gamma_{n,m} = (1, \ldots, n)(n+1, \ldots, n+m) \in S_{n+m}$. We say that $\sigma \in S_{n+m}$ is a **non-crossing annular permutation** with respect to $\gamma_{n,m}$ if $\langle \sigma, \gamma_{n,m} \rangle$ acts transitively on $[n+m]$ and $\#(\sigma) + \#(\sigma^{-1}\gamma_{n,m}) = n + m$. The set of all such permutations is denoted as

$S_{NC}(n,m)$.

Note that in general, $\#(\sigma) + \#(\sigma^{-1}\gamma_{n,m}) \leq n+m+2$ with equality iff σ is non-crossing relative to $\gamma_{n,m}$.

We will now explain why this definition is the way that it is and then go on to prove the needed facts. We say that such permutations are non-crossing annular due to the following diagrammatic reasons. Draw a circle with points labelled $1, 2, \ldots, n$ going clockwise and in the exterior of that circle, draw a second one with labels $n+1, n+2, \ldots, n+m$ also going clockwise. Then, the non-crossing annular permutations are precisely the permutations whose cycles connect both circles and can be drawn in a non-crossing way.

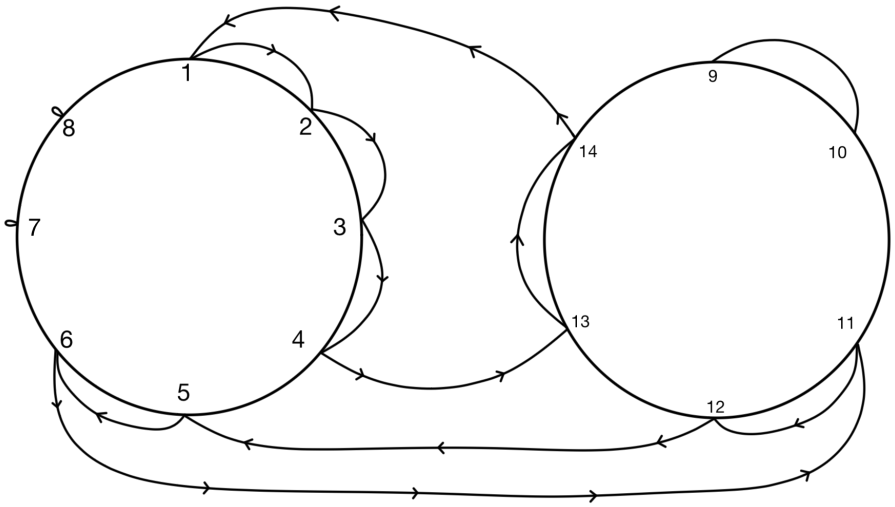

Figure 3.2: Example of a non-crossing Annular Permutation.
$\sigma = (1,2,3,4,13,14)(5,6,11,12)(9.10)$

Let us now analyze what the permutations in $\sigma \in S_{NC}(n,m)$ look like.

Proposition 3.2.23 ([17]). *Let $\gamma_{n,m}$ be as before and let $\sigma \in S_{n+m}$ be a permutation such that $\langle \sigma, \gamma_{n,m} \rangle$ acts transitively on $[n+m]$. Then σ is annular non-crossing with respect to $\gamma_{n,m}$ if and only if the following do not occur.*

1. $\gamma_{n,m|\{i,j,k\}} = (i,j,k)$ *and* $\sigma_{|\{i,j,k\}} = (i,k,j)$
2. $\gamma_{n,m|\{i,j,k,l\}} = (i,j)(k,l)$ *and* $\sigma_{|\{i,j,k,l\}} = (i,k,j,l)$

From this characterization, we immediately obtain the following corollary.

Corollary 3.2.24. *Let $\sigma \in S_{NC}(n,m)$. Then*

1. *If C is a cycle of σ that is not a through-cycle (i.e. the support of C lies in either $(1,\ldots,n)$ or $(n+1,\ldots,n+m)$), then C is non-crossing with respect to $\gamma_{n,m}$.*
2. *If C is a cycle of σ that is a through-cycle then $C = (i_1,\ldots,i_k,j_l,\ldots,j_l)$ where $(i_1,\ldots,i_k)$ is non-crossing with respect to $(1,\ldots,n)$ and $(j_1,\ldots,j_l)$ is non-crossing with respect to $(n+1,\ldots,n+m)$.*

3.2.6 Analogous Annular Constructions

In this section, we shall introduce a few analogous annular constructions. These don't differ in a fundamental way from $S_{NC}(n,m)$, but will needed in future sections as they show up in the computations of the fluctuation moments. Due to the fact that it will be notationally convenient in our computations, we will now work with the signed permutations $S_{\pm n}$.

Definition 3.2.25. Let $n \in \mathbb{N}$ and let

$\gamma_{n,-n} = (1, \ldots, n)(-n, -(n-1), \ldots, -1) \in S_{\pm n}$. We define $S_{NC}(n, -n)$ to be the set of all $\sigma \in S_{\pm n}$ such that $\langle \sigma, \gamma_{n,-n} \rangle$ acts transitively on $[\pm n]$ and $\#(\sigma) + \#(\sigma^{-1}\gamma_{n,-n}) = 2n$.

Note that the set $S_{NC}(n, -n)$ is exactly the same as $S_{NC}(n, n)$ except we the elements $n+1, \ldots, 2n$ get relabeled as $-n, \ldots, -1$.

Definition 3.2.26. Let $\gamma_{n,m} = (1, \ldots, n)(n+1, \ldots, n+m) \in S_{n+m}$. We define the set $S_{NC}(\gamma_{n,m}, -\gamma_{n,m})$ to be the set of all $\sigma \in S_{\pm(n+m)}$ such that $\sigma = \pi_1\pi_2$ where $\pi_1 \in S_{NC}(n, m)$ (or $\pi_1 \in S_{NC}(n, -m)$) and $\pi_2 \in S_{NC}(-n, -m)$ (or $\pi_2 \in S_{NC}(-n, m)$)

Graphically, the set $S_{NC}(\gamma_{n,m}, -\gamma_{n,m})$ corresponds to pairs of annular diagrams where either

1. $(1, \ldots, n)$ is connected to $(n+1, \ldots, n+m)$ and $(-n, \ldots, -1)$ to $(-(n+m), \ldots, -(n+1))$, or

2. $(1, \ldots, n)$ is connected to $(-(n+m), \ldots, -(n+1))$ and $(n+1, \ldots, n+m)$ to $(-n, \ldots, -1)$.

Due to the fact that our computations involve pairs of pairing and not pairs of general permutations, not all annular objects show up in our computations. The objects that do show up must satisfy extra symmetry conditions.

Definition 3.2.27. Let $\delta = (1, -1) \cdots (n, -n)$.

1. We denote $S^{\delta}_{NC}(n,-n)$ to be the set of all annular non-crossing permutations such $\sigma = \pi\delta\pi^{-1}\delta$ where $\pi \in S_{NC}(n,-n)$.

2. We denote $S^{\delta}_{NC}(\gamma_{n,m},\gamma_{n,m})$ to be the set of all $\sigma \in S_{NC}(\gamma_{n,m},\gamma_{n,m})$ such that $\sigma = \pi\delta\pi^{-1}\delta$ where $\pi \in S_{NC}(\gamma_{n,m},\gamma_{n,m})$.

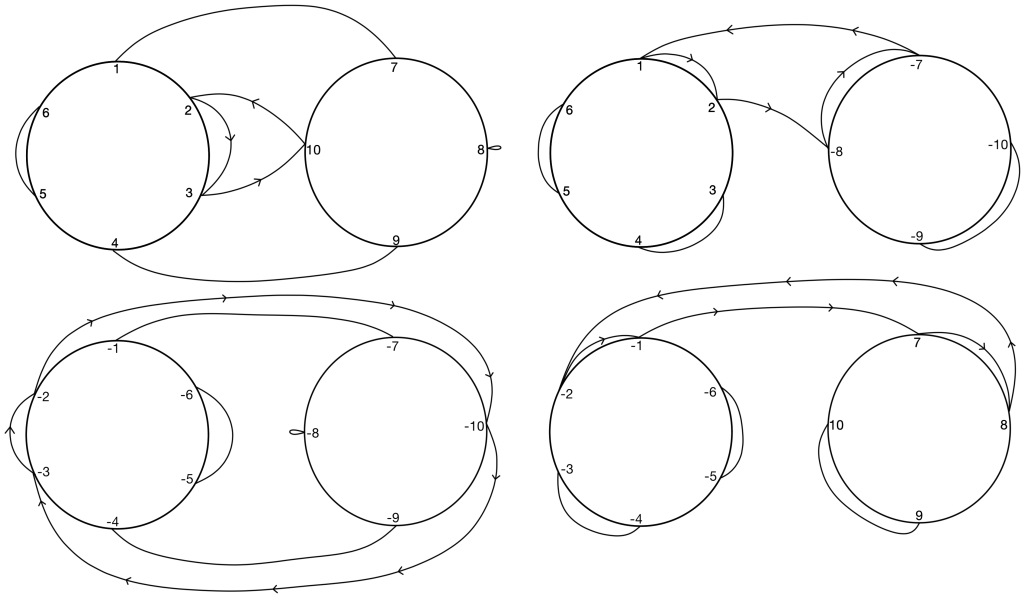

Figure 3.3: Example of two elements from $S^{\delta}_{NC}(\gamma_{n,m})$ for $n = 6, m = 4$

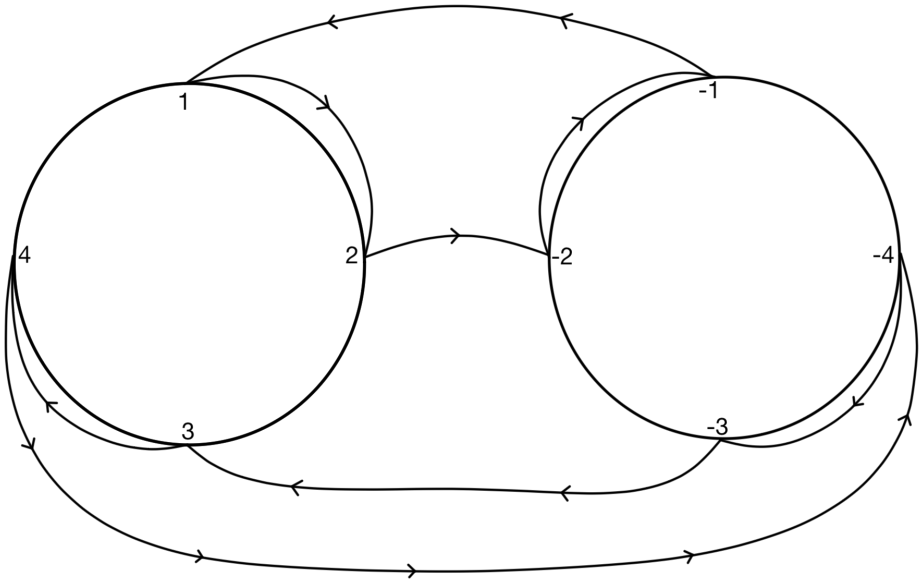

Figure 3.4: Examples of an elements of $S^{\delta}_{NC}(n,-n)$ for $n=4$

3.3 Computations on Pairings

In this section, we shall develop some needed computational results on pairs of pairings. The key idea in dealing with pairings in the context of real matrix models is to use the fact that we can convert computations involving pairs of pairings into computations inside the symmetric group.

Recall that a pairing is a permutation $p \in S_n$ such that $p^2 = e$ and p has no fixed points.

Proposition 3.3.1. *[Characterization of Pairs of Pairings] Let $p, q \in \mathcal{P}_2(n)$ and let pq denote there product. Then*

1. *pq has cycle decomposition* $C_1C_1' \cdots C_kC_k'$ *where* $C_l' = qC_l^{-1}q$ *(or equivalently* $pC_l^{-1}p$*). The pair of cycles* C_l, C_l' *are called* ***conjugate cycles****. More explicitly, if* $C = (i_1, \ldots, i_s)$ *is a cycle of* pq*, then so is* $(j_1, \cdots, j_s) = (q(i_s), q(i_{s-1}), \cdots, q(i_1))$.

2. *If* (i, j) *is a cycle of either* p *of* q*, then there exists an index* l *and a conjugate cycle pair* C_l, C_l' *such that* $i \in \mathrm{supp}(C_l)$ *and* $j \in \mathrm{supp}(C_l')$.

3. *Given any* $\pi \in S_n$ *such that it has an even number of cycles of each length, there exists* $p, q \in \mathcal{P}_2(n)$ *such that* $pq = \pi$.

4. *For each* l*, choose either* C_l *or* C_l' *and let* σ *denote the product of those cycles. Then* $\sigma p \sigma^{-1} = q$.

5. *Let* $C = (i_1, \ldots, i_s)$ *and* $C' = (j_1, \ldots, j_s)$ *be a conjugate cycle pair of* pq*. Then, viewed as partitions,* $p \vee q$ *has the block* $\{i_1, \ldots, i_k, j_1, \ldots, j_k\}$*. Consequently* $\#(p \vee q) = \frac{1}{2}\#(pq)$.

Proof. Let $(i_1, \ldots, i_s)$ be a cycle of pq. Then, by direct computation, one sees that q has the cycles $(i_1, q(i_1)), (i_2, q(i_2)), \ldots, (i_s, q(i_s))$ and p has cycles $(q(i_1), i_2), (q(i_2), i_3), \ldots, (q(i_{s-1}), i_s), (q(i_s), i_1)$. This again implies by direct computation that $(q(i_s), \ldots, q(i_1))$ is also a cycle of pq as $pq(q(i_s)) = p(i_s) = q(i_{s-1})$, $pq(q(i_{s-1}) = p(i_{s-1}) = q(i_{s-2})$ and so on. Also note that $i_l \neq q(i_{l'})$ for all indices l, l' and equality implies either that the cycle decompositions of p or q are not disjoint or that p or q have a fixed point, clear contradictions. Thus, we have proven (1) and (2). We also obtain (5) as an immediate corollary

as the cycle decompositions of p and q implies that $\{i_1, \ldots, i_k, j_1, \ldots, j_k\}$ are blocks of $p \vee q$.

For (3), let $\pi \in S_n$ be such that the cycles of each length occur an even number of times. Choose two cycles $C = (i_1, \ldots, i_s), C' = (j_1, \ldots, j_s)$ of the same length. Again, by direct computation, one sees that $((j_s, i_2)(j_{s-1}, i_3) \cdots (j_2, i_s) \cdot (j_1, i_1)) \cdot ((i_1, j_s)(i_2, j_{s-1}) \cdots (i_s, j_1)) = CC'$. Thus, we can always find $p, q \in \mathcal{P}_2(n)$ such that $pq = \pi$

For (4), let σ be defined as in the statement of the proposition. Then by (1) $pq = \sigma q \sigma^{-1} q$, thus by multiplying each side by q, we see that $p = \sigma q \sigma^{-1}$. □

Corollary 3.3.2. *Let $p, q \in \mathcal{P}_2(n)$ and let $\sigma \in S_n$ be such that $pq = \sigma p \sigma^{-1} p$. Then $pp' \leq pq$ iff there exists $\pi \in S_n$ such that $pp' = \pi p \pi^{-1} p$ and $\pi \leq \sigma$. Consequently, the set $\{p' \in \mathcal{P}_2(n) : pp' \leq pq\}$ under the partial order $p' \leq p'' \iff pp' \leq pp''$ is isomorphic to $NC(\sigma)$ as posets.*

Proof. If there exists $\pi \leq \sigma$ such that $pp' = \pi p \pi^{-1} p$, then the assertion is immediate. Conversely, assume $pp' \leq pq$. Write $pp' = C_1 C_1' \cdots C_s C_s'$. The condition $pp' \leq pq$ means that the cycles of pp' are contained inside the cycles of pq. Further, if C_l is contained inside of a cycle of σ, then $C_l' = p C_l^{-1} p$ must be contained inside of a cycle of $p\sigma^{-1}p$, thus without loss of generality we can assume that for all l, C_l is contained inside of a cycle of σ and C_l' inside of $p\sigma^{-1}p$. Let $\pi = C_1 \cdots C_s$. Then $pp' = \pi p \pi^{-1} p$. By the definition of our partial order, this gives us $\pi \leq \sigma$. □

3.4 Partitioned Permutations

In this section, we introduce the notion of partitioned permutations and some needed results. We state these results without proof. The proofs can be found in [3].

This framework is useful mainly in the context of complex matrix models. However, we will use it to define an analogous structure for analyzing the orthogonal Weingarten function.

Definition 3.4.1. A **partitioned permutation** on n elements is a pair $(\mathcal{V}, \pi)$ such that $\mathcal{V} \in \mathcal{P}(n), \pi \in S_n$ and $\pi \leq \mathcal{V}$. The set of all partitioned permutations on n elements is denoted by $\mathcal{PS}(n)$. Further, we denote $\mathcal{PS} := \bigcup_{n\in\mathbb{N}} \mathcal{PS}(n)$.

Notation 3.4.2. We denote $1_n \in \mathcal{P}(n)$ to be the partition with one block. Given $\pi \in S_n$, we define 0_π to be the partitioned obtained from π by forgetting the cycle structure and we denote $(0, \pi) := (0_\pi, \pi)$.

We have a length function and an associative multiplication rule on the set of partitioned permutations.

Definition 3.4.3. Let $\mathcal{P}(n)$, S_n, and $\mathcal{PS}(n)$ be as before. We define each set's **length function** $|\cdot|$ as follows.

1. $|\mathcal{V}| = n - \#(V)$
2. $|\pi| = n - \#(\pi)$

3. $|(\mathcal{V}, \pi)| = 2|\mathcal{V}| - |\pi|$

As the name implies, our length functions all satisfy a triangle inequality.

Proposition 3.4.4.

1. $|\mathcal{V} \vee \mathcal{U}| \leq |\mathcal{V}| + |\mathcal{U}|$ *for all* $\mathcal{V}$,$\mathcal{U} \in \mathcal{P}(n)$.

2. $|\sigma\pi| \leq |\sigma| + |\pi|$ *for all* $\sigma, \pi \in S_n$.

3. $|(\mathcal{V} \vee \mathcal{U}, \sigma\pi)| \leq |(\mathcal{V}, \sigma)| + |(\mathcal{U}, \pi)|$ *for all* $(\mathcal{V}, \sigma), (\mathcal{U}, \pi) \in \mathcal{PS}(n)$.

Using our length functions, we now define the multiplication of partitioned permutations.

$$(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) := \begin{cases} (\mathcal{V} \vee \mathcal{W}, \pi\sigma) & \text{if } |(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)| = |(\mathcal{V} \vee \mathcal{W}, \pi\sigma)| \\ 0 & \text{otherwise} \end{cases}$$

Multiplication inside of $\mathcal{PS}(n)$ is associative. Further, we have a simple geometric description for when a nontrivial multiplication occurs. The following definitions help to elucidate this.

Definition 3.4.5. Let $\gamma \in S_n$

1. $\pi \in S_n$ is called γ**-planar** if $|\pi| + |\pi^{-1}\gamma| + |\gamma| = 2|\pi \vee \gamma|$

2. $(\mathcal{V}, \pi) \in \mathcal{PS}(n)$ is called γ**-minimal** if $|\mathcal{V} \vee \gamma| - |\gamma \vee \pi| = |\mathcal{V}| - |\pi|$

The first condition means that we are able to draw the cycles of π so that they are noncrossing multi-annular with respect to the cycles of γ or vice versa. The second condition states that $\mathcal{V}$ can only connect blocks of π that γ does not.

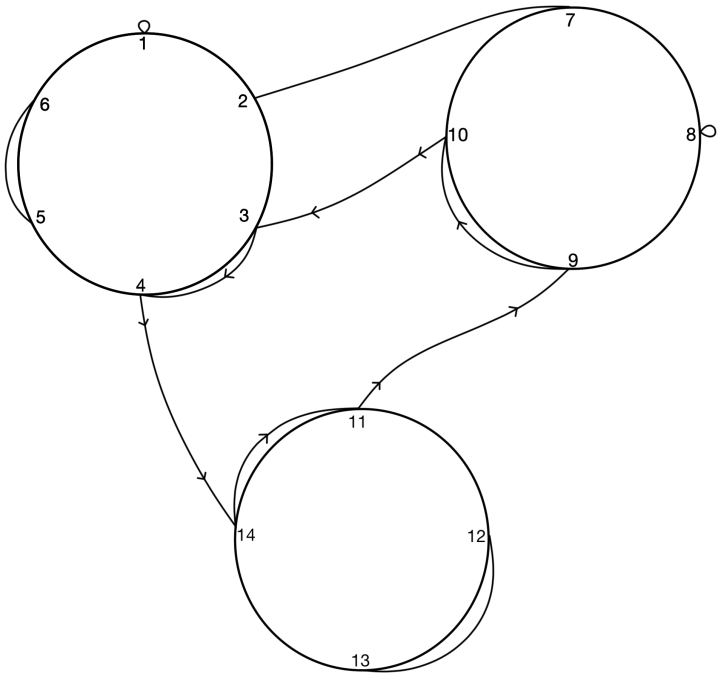

Figure 3.5: Example of a multi-annular permutation $\pi = ((2,7)(3,4,14,11,9,10)(12,13)$ with respect to the permutation $\gamma = (1,2,3,4,5,6)(7,8,9,10)(11,12,13,14)$

We obtain the following lemma via elementary manipulation using the triangle inequality of our length functions.

Lemma 3.4.6. *Let $\pi, \sigma \in S_n$ and let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2 \in \mathcal{P}(n)$ be such that*

$\mathcal{V}_2 \leq \mathcal{V}_1$ *and* $\mathcal{W}_2 \leq \mathcal{W}_1$*. Then, the following inequalities hold.*

1. $|\pi| + |\sigma| + |\pi\sigma| \geq 2|\pi \vee \sigma|$
2. $|\mathcal{W}_1| + |\mathcal{V}_1| + |\mathcal{V}_1 \vee \mathcal{W}_2| \geq |\mathcal{V}_1 \vee \mathcal{W}_1| + |\mathcal{W}_2| + |\mathcal{V}_2|$
3. $|\mathcal{V}_1 \vee \mathcal{W}_2| + |\mathcal{V}_2 \vee \mathcal{W}_1| \geq |\mathcal{V}_1 \vee \mathcal{W}_1| + |\mathcal{V}_2 \vee \mathcal{W}_2|$

Using this lemma, we obtain the following proposition (Theorem 5.6 of [3]), which lets us understand nontrivial factorizations in terms of our planar and minimality conditions.

Proposition 3.4.7. *For* $(\mathcal{V}, \pi), (\mathcal{W}, \sigma) \in \mathcal{PS}(n)$*,* $(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (\mathcal{V} \vee \mathcal{W}, \pi\sigma)$ *is equivalent to the following 4 conditions holding*

1. $|\pi| + |\sigma| + |\pi\sigma| = 2|\pi \vee \sigma|$
2. $|\mathcal{V}| + |\pi \vee \sigma| = |\pi| + |\mathcal{V} \vee \sigma|$
3. $|\mathcal{W}| + |\pi \vee \sigma| = |\sigma| + |\pi \vee \mathcal{W}|$
4. $|\mathcal{V} \vee \sigma| + |\pi \vee \mathcal{W}| = |\mathcal{V} \vee \mathcal{W}| + |\pi \vee \sigma|$

Corollary 3.4.8. *If* $(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (\mathcal{U}, \gamma)$*, then* π, σ *are* γ*-planar and* $(\mathcal{V}, \pi), (\mathcal{W}, \sigma)$ *are* γ*-minimal.*

We now focus on functions over $\mathcal{PS}$.

Definition 3.4.9. Given two functions $f, g : \mathcal{PS} \to \mathbb{C}$ and $(\mathcal{U}, \pi) \in \mathcal{PS}(n)$, we define their **convolution** $f * g : \mathcal{PS} \to \mathbb{C}$ to be

$$(f * g)(\mathcal{U}, \pi) := \sum_{\substack{(\mathcal{V}_1,\pi_1),(\mathcal{V}_2,\pi_2)\in\mathcal{PS}(n) \\ (\mathcal{V}_1,\pi_1)(\mathcal{V}_2,\pi_2)=(\mathcal{U},\pi)}} f(\mathcal{V}_1, \pi_1) g(\mathcal{V}_2, \pi_2)$$

Definition 3.4.10. A function $f : \mathcal{PS} \to \mathbb{C}$ is called **multiplicative** if $f(1_n, \pi)$ only depends on the conjugacy class of π and
$f(\mathcal{V}, \pi) = \prod_{V \in \mathcal{V}} f(1_V, \pi_{|V})$

A useful fact about multiplicative functions is that they commute with each other with respect to convolution. i.e. $f * g = g * f$. This follows from π and π^{-1} belonging to the same conjugacy class.

$\mathcal{PS}(n)$ has an identity element with respect to convolution defined by

$$\mathrm{Id}(1_n, \pi) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Key to our analysis will be understanding the nontrivial factorizations of special types of partitioned permutations.

We now introduce some more terminology. This terminology will not be motivated here, but one can see the appendix of [3] for the motivation.

Definition 3.4.11.

1. We call $(\mathcal{V}, \pi) \in \mathcal{PS}(n)$ a disk permutation if $\mathcal{V} = 0_\pi$.

2. We call $(\mathcal{V}, \pi) \in \mathcal{PS}(n)$ a tunnel permutation if $|\mathcal{V}| = |\pi| + 1$. i.e. $\mathcal{V}$ joins exactly one pair of cycles in π.

We will now study the factorizations of disk and tunnel permutations. Recall that $(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (\mathcal{V} \vee \mathcal{W}, \pi\sigma)$ if and only if

$$2|\mathcal{V}| - |\pi| + 2|\mathcal{W}| - |\sigma| = 2|\mathcal{V} \vee \mathcal{W}| - |\pi\sigma|$$

We may rewrite it as

$$(|\mathcal{V}| - |\pi|) + (|\mathcal{W}| - |\sigma|) + (|\mathcal{V}| + |\mathcal{W}| - |\mathcal{V} \vee \mathcal{W}|) = (|\mathcal{V} \vee \mathcal{W}| - |\pi\sigma|)$$

Note that everything bracketed is nonnegative.

Through a case-by-case analysis, one attains the following proposition.

Proposition 3.4.12.

1. *The solutions to the equation* $(\mathcal{V}, \pi) \cdot (\mathcal{W}\sigma) = (0, \gamma)$ *are all of the form* $(\mathcal{V}, \pi) = (0, \pi)$, $(\mathcal{W}, \sigma) = (0, \pi^{-1}\gamma)$ *where* π *is non-crossing relative to* γ.

2. *The solutions to the equation* $(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (1_{n+m}, \gamma_{n,m})$ *are of exactly 3 forms*

   (a) $(\mathcal{V}, \pi) = (0, \pi)$, $(0, \sigma) = (0, \pi^{-1}\gamma_{n,m})$ *where* $\pi \in S_{NC}(n, m)$

   (b) $(\mathcal{V}, \pi) = (0, \pi)$, $(0, \sigma) = (\mathcal{W}, \pi^{-1}\gamma_{n,m})$ *were* $\pi \in NC(n) \times NC(m)$ *and* $|\mathcal{W}| = |\pi^{-1}\gamma_{n,m}| + 1$

(c) $(\mathcal{V}, \pi) = (\mathcal{V}, \pi)$, $(0, \sigma) = (0, \pi^{-1}\gamma_{n,m})$ *where* $\pi \in NC(n) \times NC(m)$ *and* $|\mathcal{V}| = |\pi| + 1$

Remark 3.4.13. For multiplicative functions, we obtain the following formulas for convolutions.

$$f * g(1_n, \gamma_n) = \sum_{\pi \in NC(n)} f(0, \pi) g(0, \pi^{-1}\gamma_n)$$

$$f * g(1_{n+m}, \gamma_{n,m}) = \sum_{\pi \in S_{NC}(n,m)} f(0, \pi) g(0, \pi^{-1}\gamma_{n,m})$$

$$+ \sum_{\substack{\pi \in NC(n) \times NC(m) \\ |\mathcal{V}| = |\pi| + 1}} (f(0, \gamma_{n,m}\pi^{-1}) g(\mathcal{V}, \pi) + f(\mathcal{V}, \pi) g(0, \pi^{-1}\gamma_{n,m}))$$

Definition 3.4.14. Let $(\mathcal{U}, \gamma), (\mathcal{V}, \pi) \in \mathcal{PS}(n)$. We say that $(\mathcal{V}, \pi)$ is $(\mathcal{U}, \gamma)$ non-crossing if $(\mathcal{V}, \pi) \cdot (0, \pi^{-1}\gamma) = (\mathcal{U}, \gamma)$. We denote the set of non-crossing partitioned permutations by $\mathcal{PS}_{NC}(\mathcal{U}, \gamma)$.

Remark 3.4.15. By 3.4.12, we obtain the following equalities.

1. $\mathcal{PS}_{NC}(1_n, \gamma_n) = \{(0_\pi, \pi) : \pi \in NC(n)\}$
2. $\mathcal{PS}_{NC}(1_{n+m}, \gamma_{n,m}) = \{(0_\pi, \pi) : \pi \in S_{NC}(n, m)\} \cup \{(\mathcal{V}, \pi_1 \times \pi_2) : \pi_1 \in NC(n), \pi \in NC(m), |\mathcal{V}| = |\pi| + 1\}$

3.5 Partitioned Pairings

As stated before, in computations involving the orthogonal Weingarten function, the only permutations that appear are ones that can be written as a pair of pairings. Additionally, the conjugate cycle structure of these permutations plays a role. It is possible for $p_1p_2 = p_3p_4$ for four distinct pairings. This will result in the conjugate cycles differing. Take, for example, $p_1 = (1,3)(2,4)(5,7)(6,8), p_2 = (1,4)(2,3)(5,8)(6,7), p_3 = (1,5)(2,6)(3,7)(4,8)$ and $p_4 = (1,6)(2,5)(3,8)(4,7)$. Then $p_1p_2 = p_3p_4 = (1,2)(3,4)(5,6)(7,8)$, but the conjugate cycles are not the same.

Definition 3.5.1. We define the set of all **partitioned pairings** on $[n]$, $\mathcal{PS}_2(n)$ to be the triple $(\mathcal{U}, p, q)$ where $p, q \in \mathcal{P}_2(n)$ and $\mathcal{U} \in \mathcal{P}(n)$ is a permutation satisfying

1. $0_{pq} \leq \mathcal{U}$
2. If U is a cycles of $\mathcal{U}$, then so is $p(U)$ and $q(U)$. Note that the condition $0_{pq} \leq \mathcal{U}$ implies that $p(U) = q(U)$. We call U and $U' = p(U) = q(U)$ conjugate blocks of $\mathcal{U}$. Also, note that this definition allows for $U = U'$.

We call any partition satisfying both conditions a (p, q)**-invariant partition**. We denote $\mathcal{PS}_2 := \bigcup_{n\in\mathbb{N}} \mathcal{PS}_2(n)$. Note that because of the conjugate cycle structure, $|pq|$ is always even. Thus, we shall renormalize our metric on pairs of pairings to be $d(p, q) = \frac{n}{2} - \#(p \vee q) = \frac{1}{2}|pq|$.

We now wish to leverage the results of partitioned permutation to define

a length function and multiplication rule. For a multiplication rule to make sense, we need to ensure that joining partitions preserves pairing invariance.

Lemma 3.5.2. *Fix $p, p', q \in \mathcal{P}_2(n)$ and let $\mathcal{U}, \mathcal{W} \in \mathcal{P}(n)$ be (p, p') and (p', q)-invariant partitions respectively. Then $\mathcal{U} \vee \mathcal{W}$ is (p, q)-invariant.*

Proof. We will first show that $0_{pq} \leq 0_{pp'} \vee 0_{p'q}$. Since $0_{pp'} \vee 0_{p'q} \leq \mathcal{U} \vee \mathcal{W}$, this will imply the first condition.

Let $X \in 0_{pp'} \vee 0_{p'q}$ be a block, and let $x \in X$. Since $0_{p'q} \leq 0_{pp'} \vee 0_{p'q}$, $p'q(x) \in X$. Similarly, as $0_{pp'} \leq 0_{pp'} \vee 0_{p'q}$, $pp'(p'q(x)) = pq(x) \in X$, thus the blocks of 0_{pq} are contained inside of the blocks of $0_{pp'} \vee 0_{p'q}$, proving $0_{pq} \leq 0_{pp'} \vee 0_{p'q}$.

For the second condition, we will first show that $\mathcal{U} \vee \mathcal{W}$ is p'-invariant. Using this, we will show (p, q)-invariance. Recall that $U \in \mathcal{U}$ and $W \in \mathcal{W}$ being contained inside of a block X of $\mathcal{U} \vee \mathcal{W}$ is equivalent to the existence of $U_{i_1}, \ldots, U_{i_k} \in \mathcal{U}$ and $W_{i_1}, \ldots, W_{i_k} \in \mathcal{W}$ such that $U_{i_1} \cap W_{i_1} \neq \emptyset$, $U_{i_2} \cap W_{i_1} \neq \emptyset$, $U_{i_2} \cap W_{i_2} \neq \emptyset$, …, $U_{i_k} \cap W_{i_k} \neq \emptyset$ where $U_{i_1} = U$ and $W_{i_k} = W$.

Let $U \in \mathcal{U}$, $W \in \mathcal{W}$ be two blocks with nonempty intersection. Then $x \in U \cap W \iff p'(x) \in p'(U) \cap p'(W)$, thus as p' maps blocks of $\mathcal{U}$ to blocs of $\mathcal{U}$ and blocks of $\mathcal{W}$ to blocks of $\mathcal{W}$, we see that $U \in \mathcal{U}$ and $W \in \mathcal{W}$ are contained in the same block of $\mathcal{U} \vee \mathcal{W}$ if and only if $p'(U)$ and $p'(W)$ are contained in the same block of $\mathcal{U} \vee \mathcal{W}$, thus $X \in \mathcal{U} \vee \mathcal{W} \iff p'(X) \in \mathcal{U} \vee \mathcal{W}$, which shows that $\mathcal{U} \vee \mathcal{W}$ is p'-invariant.

We will now show that $\mathcal{U} \vee \mathcal{W}$ is p-invariant. Given two blocks $U, U' \in \mathcal{U}$, let $U \sim U'$ denote the condition that U, U' are contained in the same block

of $\mathcal{U} \vee \mathcal{W}$. Since $\mathcal{U} \vee \mathcal{W}$ is p'-invariant, $U \sim U' \iff p'(U) \sim p'(U') \iff p(U) \sim p(U')$ as $p(U) = p'(U)$ for all blocks of $\mathcal{U}$. Thus, as all blocks of $\mathcal{U} \vee \mathcal{W}$ are unions of blocks of $\mathcal{U}$, $\mathcal{U} \vee \mathcal{W}$ is p-invariant. Via an identical argument on the blocks of $\mathcal{W}$, $\mathcal{U} \vee \mathcal{W}$ is q-invariant, implying the result.

□

We can now define our multiplication rule in $\mathcal{PS}_2(n)$.

Definition 3.5.3.

1. We define the length function $|\cdot|$ on $\mathcal{PS}_2(n)$ to be $|(\mathcal{U}, p, q)| := |\mathcal{U}| - \frac{1}{2}|pq| = \frac{1}{2}|(\mathcal{U}, pq)|$

2. We define multiplication in $\mathcal{PS}_2(n)$ by the rule $(\mathcal{U}, p, p') \cdot (\mathcal{W}, p'', q) = 0$ if $p' \neq p''$ and

$$(\mathcal{U}, p, p') \cdot (\mathcal{W}, p'', q) = (\mathcal{U} \vee \mathcal{W}, p, q)$$

if $(\mathcal{U}, pp') \cdot (\mathcal{W}, p'q) = (\mathcal{U} \vee \mathcal{W}, pq)$

Note that by defining multiplication this way, we get that $\mathcal{U} \vee \mathcal{W}$ respects the conjugate cycle structure of p and q and that it is associative.

Following the terminology for partitioned permutations, we introduce the following definition.

Definition 3.5.4.

1. We call $(\mathcal{U}, p, q) \in \mathcal{PS}_2(n)$ a **disk partitioned pairing** if $|\mathcal{U}| = |pq|$ (i.e. $\mathcal{U} = 0_{pq}$) and we call $(\mathcal{U}, p, q)$ a **tunnel partitioned pairing** if $|\mathcal{U}| = |pq| + 1$ (i.e. $\mathcal{U}$ connect a conjugate pair of cycles of pq and otherwise equals 0_{pq}).

2. We call p' a **disk pairing** with respect to p and q if $|pp'| + |p'q| = |pq|$ and we call p' a **tunnel pairing** with respect to p and q if $|pp'| + |p'q| = |pq| + 2$

3. The set of all (p, q)-disk pairings is denoted by $NC(p, q)$ and the set of all (p, q)-tunnel pairings is denoted by $S_{NC}(p, q)$.

Proposition 3.5.5. *Let $p' \in S_{NC}(p, q)$. Then there exists conjugate cycles C, C' of pq such that $pp'|_{C \cup C'}$ is non-crossing annular with respect to CC' and is otherwise non-crossing with respect to pq.*

Proof. We must analyze the condition $|pp'| + |p'q| = |pq| + 2$

Note that $|\cdot|$ is additive on permutations with disjoint support. We first will show that the cycles of pp' are not contained inside of the cycles of pq. Assume for contradiction that the cycles of pp' are contained inside of the cycles of pq. Then the cycles of $(pp')^{-1}pq = p'q$ are also contained inside of the cycles of pq. Since $|pp'|+|(pp')^{-1}pq| = |pp'|+|p'q| = |pq|+2$, pp' is crossing relative to pq. Let C a a cycle of pq such that $pp'_{|C}$ is crossing with respect to $pq_{|C}$. Then $pp'_{pC^{-1}p}$ is also crossing with respect to $pq_{|pC^{-1}p}$. Since each crossing contributes at least $+2$ to the inequality $(|pp'_{|C}| + |p'q_{|C}| \geq |pq_{|C}| + 2$ and $|pp'_{|pC^{-1}p}| + |p'q_{|pC^{-1}p}| \geq |pq_{|pC^{-1}p}| + 2)$, we have $|pp'| + |p'q| = |pp'_{|C}| +$

$|p'q_{|C}| + |pp'_{|pC^{-1}p}| + |p'q_{|pC^{-1}p}| + |pp'_{|(C\cup pC^{-1}p)^c}| + |p'q_{|(C\cup pC^{-1}p)^c}| \geq |pq| + 4$, a contradiction. Thus, the cycles of pp' are not contained inside of the cycles of pq.

Since the cycles of pp' are not contained inside of the cycles of pq, pp' must connect exactly two cycles of pq and be otherwise noncrossing. It cannot connect more than two cycles of pq as each connection contributes $+2$ to the inequality (one transposition is needed to undo each connection). To see this, note that the condition $|\pi| + |\pi^{-1}\sigma| = |\sigma| + 2$ means that there exists a transposition τ such that $\pi = \tau\pi'$ and π' is noncrossing relative to σ (i.e. π is distance 1 from the geodesics of σ). Since a transposition can only join two cycles, pp' can only join two cycles of pq. This means that pp' must connect a conjugate pair of cycles C, C' of pq as if pp' connects the cycles C and D of pq, then it must also connect there conjugate cycles C' and D'. Since each connection contributes $+2$ to the inequality, this forces $C' = D$. Thus $|pp'_{|C\cup C'}| + |p'q_{C\cup C'}| = |pq_{C\cup C'}| + 2$. However, being transitive and satifying this equality is equivalent to $pp'_{C\cup C'}$ being noncrossing annular with respect to $pq_{|C\cup C'} = CC'$. Since $|\cdot|$ is additive on disjoint cycles, and $|pp'_{|\{C\cup C'\}^c}| + |p'q_{|(C\cup C)^c}| = |pq_{|(C\cup C')^c}|$, thus $pp'_{|(C\cup C')^c}$ is noncrossing with respect to $pq_{|\{C\cup C'\}^c}$.

□

By defining multiplication in $\mathcal{PS}_2(n)$ in terms of $\mathcal{PS}(n)$, we obtain from 3.4.12 the following proposition.

Proposition 3.5.6.

1. *Inside of* $\mathcal{PS}_2(n)$*, the solutions to the equation* $(0,p,q) = (\mathcal{V},p,p') \cdot (\mathcal{W},p'',q)$ *are exactly of the form* $(0,p,q) = (0,p,p') \cdot (0,p',q)$ *where* p' *is a disk pairing with respect to* p *and* q*.*

2. *Let* p,q *be pairings such that* pq *only has two blocks and let* $\mathcal{U} = 1_n$*. Then the solutions to the equation* $(1_n,p,q) = (\mathcal{V},p,p') \cdot (\mathcal{W},p'',q)$ *are of exactly 3 forms.*

   (a) $(1_n,p,q) = (0,p,p') \cdot (0,p',q)$ *where* p' *is a tunnel pairing with respect to* p *and* q*.*

   (b) $(1_n,p,q) = (0,p,p') \cdot (\mathcal{W},p',q)$ *where* p' *is a disk pairing and* $|\mathcal{W}| = |p'q| + 1$ *and* $\mathcal{W}$ *must connect a pair of conjugate blocks of* $p'q$*.*

   (c) $(1_n,p,q) = (\mathcal{V},p,p')(0,p',q)$ *where* p' *is a disk pairing and* $|\mathcal{V}| = |pp'| + 1$ *and* $\mathcal{V}$ *must connect a pair of conjugate blocks of* pp'*.*

Proof. We prove this, by rewriting our multiplication in $\mathcal{PS}_2(n)$ in terms of the multiplication on $\mathcal{PS}(n)$.

1. $(\mathcal{V},p,p') \cdot (\mathcal{W},p'',q) = (0,p,q)$ if and only if $p' = p''$ and $(\mathcal{V},pp') \cdot (\mathcal{W},p'q) = (0,pq)$. By 3.4.12, this is equivalent to $\mathcal{V} = 0_{pp'}$, $\mathcal{W} = 0_{p'q}$ and $|pp' + |p'q| = |pq|$. i.e. p' is a disk pairing.

2. $(\mathcal{V},p,p') \cdot (\mathcal{W},p'',q) = (1_n,p,q)$ is equivalent to $p' = p''$ and $(\mathcal{V},pp') \cdot (\mathcal{W},p'q) = (1_n,pq)$ By 3.4.12 there are three ways that this can occur.

(a) $\mathcal{V} = 0_{pp'}$, $\mathcal{W} = 0_{p'q}$ and pp' is noncrossing annular with respect to pq. i.e. p' is a tunnel pairing.

(b) $\mathcal{V} = 0_{pp'}$, $|\mathcal{W}| = |p'q| + 1$ and pp' is noncrossing with respect to pq, i.e. pp' is noncrossing with respect to pq, i.e. p' is a disk pairing. Since $|\mathcal{W}| = |p'q| + 1$, and since $0_{p'q} \leq \mathcal{W}$, the blocks of $\mathcal{W}$ consists of the union of two blocks of $0_{p'q}$ and is otherwise equal to $0_{p'q}$. Since $\mathcal{W}$ is $(p'q)$-invariant by assumption, the blocks joined must be conjgate blocks.

(c) $\mathcal{W} = 0_{p'q}$, $|\mathcal{V}| = |pp'| + 1$ and pp' is noncrossing with respect to pq, i.e p' is a disk pairing. By the same argument as above $\mathcal{V}$ joins a a conjugate pair of blocks of $0_{pp'}$ and is otherwise equal to $0_{pp'}$.

□

Definition 3.5.7.

1. We define the convolution of functions $f, g : \mathcal{PS}_2 \to \mathbb{C}$ by the rule

$$f * g(\mathcal{U}, p, q) = \sum_{\substack{(\mathcal{U}_1, p, p'), (\mathcal{U}_2, p', q) \in \mathcal{PS}_2(n) \\ (\mathcal{U}_1, p, p') \cdot (\mathcal{U}_2, p', q) = (\mathcal{U}, p, q)}} f(\mathcal{U}_1, p, p') g(\mathcal{U}_2, p', q)$$

.

2. Given a (p, q)-invariant partition $\mathcal{U}$, Let $C(U, U') \in \mathcal{P}(U \cup U')$ denote that partition $\{U, U'\}$ and let $\mathcal{C}(\mathcal{U}) := \{C(U, U') : U \in \mathcal{U}\}$. Recall U and U' may or may not be equal. We say that a function f is

multiplicative over $\mathcal{PS}_2(n)$

$$f(\mathcal{U}, p, q) = \prod_{C(U,U') \in \mathcal{C}(\mathcal{U})} f(C(U, U'), p_{|U \cup U'}, q_{|U \cup U'})$$

where U, U' are (p, q)-conjugate blocks and $f(C(U, U'), p_{|U \cup U'}, q_{|U \cup U'}) = f(C(U, U'), q_{|U \cup U'}, p_{|U \cup U'})$.

3. We define the multiplicative function Id on $\mathcal{PS}(n)$ by the rule

$$\mathrm{Id}(\mathcal{U}, p, q) = \prod_{C(U,U') \in \mathcal{C}(\mathcal{U})} \delta_{|C(U,U')|,0}$$

i.e. $\mathrm{Id}(\mathcal{U}, p, q) = 1$ if $p = q$ and $\mathcal{U} = 0_{pq}$.

Note that the definition of multiplicative function is well defined as $C(U, U')$ always has an even number of elements and p and q both restrict to pairings inside of $\mathcal{P}_2(U \cup U')$ by the (p, q)-invariance of $\mathcal{U}$.

Lemma 3.5.8. *The convolution of multiplicative functions is multiplicative and multiplicative functions commute with each other.*

Proof. Convolution preserving multiplicativity is immediate.

Let f be a multiplicative function. Then

$f(\mathcal{U}, p, q) = \prod_{C(U,U') \in \mathcal{C}(\mathcal{U})} f(C(U, U'), p_{|U \cup U'}, q_{|U \cup U'}) =$

$\prod_{C(U,U') \in \mathcal{C}(\mathcal{U})} f(C(U, U'), q_{|U \cup U'}, p_{|U \cup U'}) = f(\mathcal{U}, q, p)$

Now, let f and g be any two multiplicative functions. Then

$$f * g(\mathcal{U}, p, q) = \sum_{\substack{(\mathcal{U}_1,p,p'),(\mathcal{U}_2,p',q)\in\mathcal{PS}_2(n)\\(\mathcal{U}_1,p,p')\cdot(\mathcal{U}_2,p',q)=(\mathcal{U},p,q)}} f(\mathcal{U}_1, p, p')g(\mathcal{U}_2, p', q)$$

$$= \sum_{\substack{(\mathcal{U}_1,p,p'),(\mathcal{U}_2,p',q)\in\mathcal{PS}_2(n)\\(\mathcal{U}_1,p,p')\cdot(\mathcal{U}_2,p',q)=(\mathcal{U},p,q)}} g(\mathcal{U}_2, q, p')f(\mathcal{U}_1, p', p) = g * f(\mathcal{U}, q, p) = g * f(\mathcal{U}, p, q)$$

□

Lemma 3.5.9. *Id is the identity function with respect to convolution.*

Proof.

$$f * \mathrm{Id}(\mathcal{U}, p, q) = \sum_{\substack{(\mathcal{U}_1,p,p'),(\mathcal{U}_2,p',q)\in\mathcal{PS}_2(n)\\(\mathcal{V},p,p')\cdot(\mathcal{W},p',q)=(\mathcal{U},p,q)}} f(\mathcal{U}_1, p, p')\mathrm{Id}(\mathcal{U}_2, p', q)$$

Id will only be nonzero when $p' = q$ and $\mathcal{U}_2$ is the trivial partition. Since $\mathcal{U}_1 \vee \mathcal{U}_2 = \mathcal{U}$, this implies that $\mathcal{U}_1 = \mathcal{U}$, thus $f * \mathrm{Id}(\mathcal{U}, p, q) = f(\mathcal{U}, p, q)$

□

We now define the zeta and Möbius functions. Their definitions are inspired by the definitions of zeta and Möbius functions in the context of poset theory. These functions will be needed in our analysis of the orthogonal Weingarten function.

Definition 3.5.10. We define the multiplicative function ζ on $\mathcal{PS}_2$ by the

rule

$$\zeta(\mathcal{U}, p, q) = \begin{cases} 1 & \text{if } \mathcal{U} = 0_{pq} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3.5.11. *The ζ function is invertible.*

Proof. We must show that there exists a function μ such that $\zeta * \mu = \mathrm{Id}$. Recall that $\mathrm{Id}(\mathcal{U}, p, q) = 1$ if $p = q$ and $\mathcal{U} = 0_{pp}$ and zero otherwise.

We shall solve for the values of μ recursively by inducting on $d(p, q) = k$.

First assume that $d(p, q) = 0$, i.e. $p = q$. Then

$$\zeta * \mu(\mathcal{U}, p, p) = \sum_{(\mathcal{U}_1, p, p') \cdot (\mathcal{U}_2, p', p) = (\mathcal{U}, p, p)} \zeta(\mathcal{U}_1, p, p')\mu(\mathcal{U}_2, p', p)$$

The condition $(\mathcal{U}_1, p, p') \cdot (\mathcal{U}_2, p', p) = (\mathcal{U}, p, p)$ forces $p' = p$. Since $\zeta(\mathcal{U}_1, p, p) = 1$ if $\mathcal{U}_1 = 0_{pp}$ and zero otherwise, we can rewrite our sum as

$$\sum_{(0, p, p) \cdot (\mathcal{U}_2, p, p) = (\mathcal{U}, p, p)} \mu(\mathcal{U}_2, p, p)$$

Since $0_{pp} \leq \mathcal{U}_2$, $(0, p, p) \cdot (\mathcal{U}_2, p, p) = (\mathcal{U}, p, p)$ forces $\mathcal{U}_2 = \mathcal{U}$.

Thus $\zeta * \mu(\mathcal{U}, p, p) = \mu(\mathcal{U}, p, p)$

Therefore, we set $\mu(\mathcal{U}, p, p) = 1$ if $\mathcal{U} = 0_{pp}$ and $\mu(\mathcal{U}, p, p) = 0$ otherwise.

Now assume that we have determined $\mu(\mathcal{U}, p, q)$ for all p, q such that $d(p, q) < k$. We will determine $\mu(\mathcal{U}, p, q)$ for $d(p, q) = k$.

$$\zeta * \mu(\mathcal{U}, p, q) = \sum_{(\mathcal{U}_1, p, p') \cdot (\mathcal{U}_2, p', q) = (\mathcal{U}, p, q)} \zeta(\mathcal{U}_1, p, p') \mu(\mathcal{U}_2, p', q)$$

Like before, $\zeta(\mathcal{U}_1, p, p') = 1$ if $\mathcal{U}_1 = 0_{pp'}$ and zero otherwise. Conditioning the sum on $d(p', q)$, we obtain

$$\zeta * \mu(\mathcal{U}, p, q) = \sum_{l=1}^{k} \sum_{\substack{(0_{pp'}, p, p') \cdot (\mathcal{U}_2, p', q) = (\mathcal{U}, p, q) \\ d(p', q) = l}} \mu(\mathcal{U}_2, p', q)$$

The condition $(0_{pp'}, p, p') \cdot (\mathcal{U}_2, p', q) = (\mathcal{U}, p, q)$ implies that $d(p, p') + d(p', q) = d(p, q)$. Thus, if $d(p', q) = d(p, q) = k$, then $p' = p$. Since $0_{pp'} \leq \mathcal{U}_2$, this forces $\mathcal{U}_2 = \mathcal{U}$. Thus

$$\zeta * \mu(\mathcal{U}, p, q) = \mu(\mathcal{U}, p, q) + \sum_{l=1}^{k-1} \sum_{\substack{(0_{pp'}, p, p') \cdot (\mathcal{U}_2, p', q) = (\mathcal{U}, p, q) \\ d(p', q) = l}} \mu(\mathcal{U}_2, p', q)$$

By our inductive hypothesis, we know $\mu(\mathcal{U}_2, p', q)$ for all p' such that $d(p', q) \leq k - 1$, thus by setting

$$\mu(\mathcal{U}, p, q) = -\left(\sum_{l=1}^{k-1} \sum_{\substack{(0_{pp'}, p, p') \cdot (\mathcal{U}_2, p', q) = (\mathcal{U}, p, q) \\ d(p', q) = l}} \mu(\mathcal{U}_2, p', q) \right)$$

for all p, q such that $d(p, q) > 0$, we obtain the inverse μ of ζ. □

We end this section by noting the following formula for μ, which will be

needed later.

$$\mu(\mathcal{U},p,q) = (\zeta)^{-1}(\mathcal{U},p,q) = (\mathrm{I} + (\zeta - \mathrm{Id}))^{-1}(\mathcal{U},p,q)$$

$$= \sum_{k=0}^{\infty}(-1)^k(\zeta - \mathrm{Id})^{*k}(\mathcal{U},p,q) = \mathrm{Id}(\mathcal{U},p,q) + \sum_{k=1}^{\infty} \sum_{\substack{(\mathcal{U},p,q)=(0,p,p_1)\cdots(0,p_k,q) \\ p_i p_{i+1} \neq e}} (-1)^k$$

3.6 Free Probability and Random Matrix Theory

In this section, we will briefly introduce free probability and explain its connections to random matrix theory. The focus of this section will be the combinatorial side of the theory. Free probability and free independence can be thought of as noncommutative analogs of regular probability and classical independence. The usefulness of free probability in random matrix theory comes from the observation that the eigenvalue distributions of many random matrix ensembles are asymptotically freely independent. In particular, if two independent random matrix ensembles are such that at least one of them is either unitarily or orthogonally invariant, then they will be asymptotically freely independent.

We will also look at an extension of free probability called second-order freeness. Second-order freeness extends the ideas of free probability to what are called the fluctuation moments of a random matrix ensemble. While in the first-order theory, we do not have to make a distinction between real and

complex matrix models, we will for the second-order case.

3.6.1 First Order Freeness

Definition 3.6.1. A **noncommutative probability space** is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital $\mathbb{C}$-algebra and φ is a linear functional such that $\varphi(1) = 1$. We call elements $a \in \mathcal{A}$ (noncommutative) random variables.

Definition 3.6.2. Let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of unital subalgebras of $\mathcal{A}$.

1. We say that $a \in \mathcal{A}$ is **centered** if $\varphi(a) = 0$.

2. We say that $a_1, \ldots, a_k \in \mathcal{A}$ is **alternating** if $a_l \in \mathcal{A}_{j(l)}$ and $j(l) \neq j(l+1)$ for all $1 \leq l \leq k-1$ where $j : [k] \to I$ is a function from $[k]$ to our index set I.

3. We say that $a_1, \ldots, a_k \in \mathcal{A}$ is **cyclically alternating** if they are alternating and $j(1) \neq j(k)$.

4. We say that a collection of unital subalgebras $\{\mathcal{A}_i\}_{i \in I}$ of $\mathcal{A}$ are **free** if for all centered, alternating elements $a_1, a_2, \ldots, a_k$ we have $\varphi(a_1 \cdots a_k) = 0$.

5. We that $a_1, a_2, \ldots, a_k \in \mathcal{A}$ are **freely independent** if the the unital algebras they generate $\{\mathcal{A}_i := \mathrm{alg}(a_i, 1)\}_{i=1}^k$ are free.

Definition 3.6.3. Let $\{A_N\}_{N \in \mathbb{N}}$ be an ensemble of $N \times N$ random matrices. We say that $\{A_N\}$ has a **first order limiting moments** $\alpha_n \in \mathbb{C}$ if $\lim_{N \to \infty} \mathbb{E}\left(\mathrm{tr}(A_N^n)\right) = \alpha_n$ exists for all $n \in \mathbb{N}$.

Definition 3.6.4. Let $\{A_N\}$, $\{B_N\}$ be a collection of $N \times N$ random matrix ensembles with first order limiting moments $\{\alpha_n\}$ and $\{\beta_n\}$ respectively. We say that they are **asymptotically freely independent** if for all $n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N}$ we have $\lim_{N\to\infty} \mathbb{E}\left(\operatorname{tr}\left(\mathring{A}_N^{n_1}\mathring{B}_N^{m_1}\cdots\mathring{A}_N^{n_k}\mathring{B}_N^{m_k}\right)\right) = 0$

where $\mathring{A} = A - \mathbb{E}\left(\operatorname{tr}(A)\right) I$. Note that $\mathbb{E}\left(\operatorname{tr}(A)\right) = \alpha_1$.

Remark 3.6.5. Equivalent to this definition is that there exists a noncommutative probability space $(\mathcal{A}, \varphi)$ and random variables $a, b \in \mathcal{A}$ freely independent such that for all polynomials $p \in \mathbb{C}\langle x_1, x_2\rangle$ in the noncommuting variables x_1, x_2, we have $\lim_{N\to\infty} \mathbb{E}\left(\operatorname{tr}(p(A_n, B_n)) = \varphi(p(a, b))\right.$.

Free independence gives a rule for recursively computing the mixed moments of freely independent random variables. For example, let a and b be two freely independent random variables in a noncommutative probability space $(\mathcal{A}, \varphi)$. Then $0 = \varphi(\mathring{a}\mathring{b}) = \varphi((a - \varphi(a))(b - \varphi(b)) = \varphi(ab) - \varphi(a)\varphi(b) \Rightarrow \varphi(ab) = \varphi(a)\varphi(b)$. Continuing, one can derive the formulas $\varphi(aba) = \varphi(a^2)\varphi(b)$ and $\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(b^2)\varphi(a)^2 - \varphi(a)^2\varphi(b)^2$.

This computation quickly becomes cumbersome, and the general structure of the free independence rule is not obvious from these computations. The key idea discovered by Speicher in [21] is that given a moment sequence $\{\alpha_n\}$ of a random variable a, we can introduce a second sequence $\{\kappa_n\}$ called the free cumulants of a which linearize under free independence. i.e. given freely independent random variables a and b, $\kappa_n^{a+b} = \kappa_n^a + \kappa_n^b$. The free cumulants determine the moments and vice versa.

Definition 3.6.6. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $\varphi_n : \mathcal{A}^n \to \mathbb{C}$ be the n-linear functional defined by the rule $\varphi_n(a_1, \ldots, a_n) := \varphi(a_1 \cdots a_n)$.

1. Denote $NC := \cup_{n\in\mathbb{N}} NC(n)$. We define the multiplicative extension of $\{\varphi_n\}$ to NC, $\{\varphi_\pi\}_{\pi\in NC}$, by the rule

   $\varphi_\pi(a_1, \ldots, a_n) := \prod_{V\in\pi} \varphi_{|V|}(a_1, \ldots, a_n|V)$. Here $|V|$ denotes the number of elements in the block V and $\varphi_{|V|}(a_1, \ldots, a_n|V)$ means that we evaluate $\varphi_{|V|}(\cdot)$ on the a_i's whose indices are in the block V in the same order as they appeared in the original product.

2. We define the **free cumulants** $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ recursively by the rule $\varphi_n := \sum_{\sigma\in NC(n)} \kappa_\sigma$ and we define their multiplicative extension to NC, $\{\kappa_\pi\}_{\pi\in NC}$, analogously.

Note that in the multiplicative extension $\varphi_{1_n} = \varphi_n$. The fact that the moments and cumulant sequences determine each other follows from the fact that $\varphi_n := \sum_{\sigma\in NC(n)} \kappa_\sigma$ defines a triangular system of equations. When $n = 1$, $\kappa_1 = \varphi_1$. For $n = 2$, $\varphi_2 = \kappa_2 + \kappa_1^2$, thus, $\kappa_2 = \varphi_2 - \varphi_1^2$. More generally $\varphi_\sigma = \sum_{\pi\leq\sigma} \kappa_\pi$ and $\kappa_\sigma = \sum_{\pi\leq\sigma} \mu(\pi, \sigma)\varphi_\pi$. where μ denotes the Möbius function of $NC(n)$.

Theorem 3.6.7. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $\{\mathcal{A}_i\}_{i\in I}$ be a collection of unital subalgebras. Then $\{\mathcal{A}_i\}_{i\in I}$ are freely independent if and only if their mixed free cumulants vanish. i.e. if $a_1, \ldots, a_n$*

are such that $a_l \in \mathcal{A}_{j(l)}$ *and there exists indices* l, l' *such that* $j(l) \neq j(l')$, *then* $\kappa_n(a_1, \ldots, a_n) = 0$.

Remark 3.6.8. In free probability, free cumulants play the same role relative to free independence as the classical cumulants do relative to classical independence in probability theory. If, instead of non-crossing partitions, we defined a multiplicative extension to all partitions, we would end up with the classical cumulants. They have the property of linearizing under classical independence.

3.6.2 Second Order Freeness

Second-order freeness is an extension of free probability to what is called the fluctuation moments. Given a random matrix ensemble $\{A_N\}_{N\in\mathbb{N}}$, it having a first order limiting distribution $\{\alpha_k\}$ means that as $N \to \infty$, the random variables $\mathrm{tr}(A_n^k)$ tend to $\mathbb{E}(\mathrm{tr}(A_N^k))$ almost surely. When we consider the non-normalized trace however, $\mathrm{Tr}(A_n^k) - \mathbb{E}(\mathrm{Tr}(A_N^k))$ does not tend deterministically to zero. Instead, for many random matrix ensembles, this quantity tends to be a centred Gaussian random variable. Unique to Gaussian random variables is that in order to understand a family of Gaussians, it is enough to understand their covariances (the second classical cumulant). This is a consequence of the fact that for a centred Gaussian, only its second classical cumulant is nonzero. It was noticed in [10] that non-crossing annular objects showed up in the computations of fluctuation moments. This

idea was systematically developed in a series of papers ([14, 13, 3]), leading to second-order freeness. This theory extends the techniques of free probability to the limiting covariances of the family of random variables $\{\mathrm{Tr}(A_N^k)\}_k$. These limiting covariances are called the fluctuation moments. If two ensembles $\{A_N\}, \{B_N\}$ are what is called second-order free, then we get a universal rule for computing the mixed fluctuation moments. We also get objects called second-order free cumulants that linearize under second-order freeness. In the first-order case, there is no difference between the theory for real and complex matrix models. In the second-order cases, however, we must make a distinction. The real analogue for second-order free cumulants was determined by Redelmeier in [19]. We will now set up the general theory of second-order freeness and explain why the distinction must be made.

Definition 3.6.9. We call $(\mathcal{A}, \varphi, \varphi_2)$ A **second-order noncommutative probability space** if $(\mathcal{A}, \varphi)$ is a noncommutative probability space and $\varphi_2 : \mathcal{A}^2 \to \mathbb{C}$ is a bi-tracial map satisfying $\varphi_2(a, 1) = \varphi_2(1, a) = 0$ for all $a \in \mathcal{A}$.

Definition 3.6.10. Let $(\mathcal{A}, \varphi, \varphi_2)$ be second order noncommutative probability space, $\{\mathcal{A}_i\}_{i\in I}$ a collection of unital subalgebra and let $a_1, \ldots, a_n, b_1, \ldots, b_m \in \bigcup_{i\in I} \mathcal{A}_i$. We say that they are **second-order free** if

1. They are first-order free.
2. $\varphi_2(a_1, b_1) = 0$ if a_1, b_1 come from different subalgebras.
3. $\varphi_2(a_1 \cdots a_n, b_1 \cdots b_m) = \delta_{n,m} \sum_{k=1}^{n} \prod_{i=1}^{n} \varphi(a_i b_{k-i})$ where the indices are read mod n.

We will now define the second-order free cumulants that linearize under second-order freeness.

Definition 3.6.11. Let $\mathcal{PS}'(n,m)$ denote the set of all $(\mathcal{V},\pi) \in \mathcal{PS}(n+m)$ such that $\pi = \pi_1 \times \pi_2 \in NC(n) \times NC(m)$ and the blocks of $\mathcal{V}$ are equal to the cycles of π except for one cycles, which is the union of exactly one cycle from π_1 and one cycles from π_2. Let W_1, W_2 denote the blocks of π_1 and π_2 respectively that are joined by $\mathcal{V}$. We call W_1, W_2 **marked blocks** of π_1 and π_2.

Definition 3.6.12. Let $(\mathcal{A}, \varphi, \varphi_2)$ be a second order noncommutative probability space, fix $n, m \in \mathbb{N}$ and let $a_1, \ldots, a_{n+m} \in \mathcal{A}$. We define the **second-order free cumulants** $\kappa_{n,m}$ recursively by the relation

$$\varphi_2(a_1 \cdots a_n, a_{n+1} \cdots a_{n+m}) = \sum_{\pi \in S_{NC}(n,m)} \kappa_\pi + \sum_{(\mathcal{V},\pi) \in \mathcal{PS}'_{NC}(n,m)} \kappa_{(\mathcal{V},\pi)}$$

Where κ_π denotes the multiplicative extension of the first order free cumulants and $\kappa_{(\mathcal{V},\pi)} = \kappa_{|W_1|,|W_2|} \prod_{W \neq W_1 \in \pi_1} \kappa_{|W|} \prod_{U \neq W_2} \kappa_{|U|}$

Let us do a few example computations. For $n = m = 1$, there is only one element of $S_{NC}(n,m)$ and $\mathcal{PS}'_{NC}(n,m)$, names $(1,2)$ an $(\{1,2\},(1)(2))$. This gives us $\varphi_2(a,b) = \kappa_2(a,b) + \kappa_{1,1}(a,b)$. Thus $\kappa_{1,1}(a,b) = \varphi_2(a,b)$-$\kappa_2(a,b)$.

For $n = 1, m = 2$, There are 3 elements of $S_{NC}(n,m)$, namely $(1,2)(3)$, $(1,3)(2)$ and $(1,2,3)$. For $\mathcal{PS}'_{NC}(n,m)$, the elements are $(\{\{1,2\}\{3\}\}$,

$(1)(2)(3)), (\{\{1,3\}\{2\}\}, (1)(2)(3))$ and $(\{\{1,2,3\}\}, (1)(23))$. This gives us $\varphi_2(a,bc) = \kappa_3(a,b,c) + \kappa_1(c)\kappa_2(a,b) + \kappa_1(c)\kappa_2(a,c) + \kappa_1(c)\kappa_{1,1}(a,b) + \kappa_1(b)\kappa_{1,1}(a,c) + \kappa_{1,2}(a,b,c)$.

Theorem 3.6.13. *Let $(\mathcal{A}, \varphi, \varphi_2)$ be a second-order noncommutative probability space. and let $\{\mathcal{A}_i\}_{i\in I}$ be a collection of unital subalgebra. Then they are second-order free if and only if their mixed second-order free cumulants vanish.*

A distinction between the formulas of the real and complex cases is the appearance of the transpose in the real case. A real second-order space must, therefore, take this into account.

Definition 3.6.14. We call $(\mathcal{A}, \varphi, \varphi_2, t)$ a **real second order noncommutative probability space** if $t : \mathcal{A} \to \mathcal{A}$ is an order 2 anti-automorphism and $(\mathcal{A}, \varphi, \varphi_2)$ is a noncommutative probability space satisfying

1. $\forall a \in \mathcal{A}$, $\varphi(a) = \varphi(a^t)$

2. $\forall a, b \in \mathcal{A}$, $\varphi_2(a,b) = \varphi_2(a^t, b) = \varphi_2(a, b^t)$

Definition 3.6.15. Let $(\mathcal{A}, \varphi, \varphi_2, t)$ be second order noncommutative probability space, let $\{\mathcal{A}_i\}_{i\in I}$ be a collection of unital subalgebra and let $a_1, \ldots, a_n, b_1, \ldots, b_m \in \bigcup_{i\in I} \mathcal{A}_i$. We say that they are **real second-order free** if

1. They are first-order free.

2. $\varphi_2(a_1, b_1) = 0$ if a_1, b_1 come from different subalgebras.

3. $\varphi_2(a_1 \cdots a_n, b_1 \cdots b_m) = \delta_{n,m} \sum_{k=1}^{n} \prod_{i=1}^{n} \varphi(a_i b_{k-i}) + \prod_{i=1}^{n} \varphi(a_i b_{i-k}^t)$

   where the indices are read mod n.

Definition 3.6.16 ([19])**.** Let $(\mathcal{A}, \varphi, \varphi_2, t)$ be a real second order noncommutative probability space, fix $n, m \in \mathbb{N}$ and let $a_1, \ldots, a_{n+m} \in \mathcal{A}$. We then define the real second-order free cumulants recursively by the relation

$$\varphi_2(a_1 \cdots a_n, a_{n+1} \cdots a_{n+m}) = \sum_{\pi \in S_{NC}(n,m)} \kappa_\pi(a_1, \ldots, a_{n+m})$$

$$+ \sum_{\pi \in S_{NC}(n,m)} \kappa_\pi(a_1, \ldots, a_n, a_{n+1}^t, \ldots, a_{n+m}^t) + \sum_{(\mathcal{V},\pi) \in \mathcal{PS}'_{NC}(n,m)} \kappa_{(\mathcal{V},\pi)}$$

Where κ_π and $\kappa_{(\mathcal{U},\pi)}$ are defined the same as before.

Theorem 3.6.17 ([19])**.** *Let $(\mathcal{A}, \varphi, \varphi_2, t)$ be a real second-order noncommutative probability space. and let $\{\mathcal{A}_i\}_{i \in I}$ be a collection of unital subalgebra. Then they are real second-order free if and only if their mixed real second-order free cumulants vanish.*

Definition 3.6.18 ([12])**.** Let $\{A_{N,1}, \ldots, A_{N,s}\}_{N \in \mathbb{N}}\}$ be a collection of random matrices. We say that $\{A_{N,1}, \ldots, A_{N,s}\}_{N \in \mathbb{N}}$ has **second-order real limiting distribution** if there exists a real second-order probability space $(\mathcal{A}, \varphi, \varphi_2, t)$ and elements $a_1, \ldots, a_s \in \mathcal{A}$ such that for all polynomials p, q in the noncommuting random variables $\{x_1, \ldots, x_s, x_1^t, \ldots, x_s^t\}$

1. $\lim_{N\to\infty} \mathbb{E}\left(\mathrm{tr}(p(A_{N,1},\ldots,A_{N,s},A_{N,1}^t,\ldots,A_{N,s}^t)\right)$

   $= \varphi(p(a_1,\ldots,a_s,a_1^t,\ldots,a_s^t))$

2. $\lim_{N\to\infty} k_2(\mathrm{Tr}(p(A_{N,1},\ldots,A_{N,s}^t)),\mathrm{Tr}(q(A_{N,1},\ldots,A_{N,s}^t)))$

   $= \varphi_2(p(a_1,\ldots,a_s^t),q(a_1,\ldots,a_s^t))$

3. For all $k \geq 3$ and all polynomials $p_1,\ldots,p_n$

   $\lim_{N\to\infty} k_n(\mathrm{Tr}(p_1(A_{N,1},\ldots,A_{N,s}^t)),\ldots,\mathrm{Tr}(p_n(A_{N,1},\ldots,A_{N,s}^t))) = 0$

   where k_n denotes the n'th classical cumulant.

Equivalently, for a single ensemble $\{A_N\}_{N\in\mathbb{N}}$, let $A_N^{(1)} = A_N$ and $A_N^{(-1)} = A_N^t$. We say that $\{A_N\}$ has second-order limiting distribution if

1. $\lim_{N\to\infty} \mathbb{E}\left(\mathrm{tr}(A_N^n)\right) = \lim_{N\to\infty} \mathbb{E}\left(\mathrm{tr}((A_N^t)^n)\right) = \alpha_n \in \mathbb{C}$

2. $\lim_{N\to\infty} k_2(\mathrm{Tr}(A_N^n),\mathrm{Tr}(A_N^m)) = \lim_{N\to\infty} k_2(\mathrm{Tr}((A_N^t)^n),\mathrm{Tr}(A_N^m)) =$

   $\lim_{N\to\infty} k_2(\mathrm{Tr}(A_N^n),\mathrm{Tr}((A_N^t)^m)) = \alpha_{n,m} \in \mathbb{C}$

3. For all $n \geq 3$ and for all $\{\epsilon_1,\ldots,\epsilon_n\} \in \{-1,1\}^n$, $m_1,\ldots,m_n \in \mathbb{N}$

   $\lim_{N\to\infty} k_n(\mathrm{Tr}((A_N^{(\epsilon_1)})^{m_1}),\ldots,\mathrm{Tr}((A_N^{(\epsilon_n)})^{m_n})) = 0$

The condition that higher-order classical cumulants vanish is required for the convergence of the mixed fluctuation moments.

Theorem 3.6.19 ([12]). *Let $\{A_N\}$, $\{B_N\}$ be two random matrix ensembles with limiting second-order real distributions. If at least one of them is orthogonally invariant, then they are asymptotically second-order real free.*

Corollary 3.6.20. *Let $\{O_i\}_{i=1}^m$ be a collection of $N \times N$ Haar distributed orthogonal matrices and let $X_i = O_i + O_i^t$. Then $\{X_i\}$ is asymptotically second order real free.*

3.6.3 Real versus Complex Freeness

We will briefly explain why the second-order, and not the first-order theory differs.

In computations involving complex matrix models, one ends up with a sum over all pairs of permutations, while in the real case, one only sums over pairs of pairings. This is due to the differences between the unitary and orthogonal Weingarten functions. For the first-order complex case, the leading order terms are given by non-crossing diagrams, while in the real case, one gets pairs of non-crossing diagrams. Due to the conjugate cycle symmetry, however, knowing one diagram means that you automatically know the other. This causes the real case to reduce to the complex one. In the second-order case, we no longer get the same reduction.

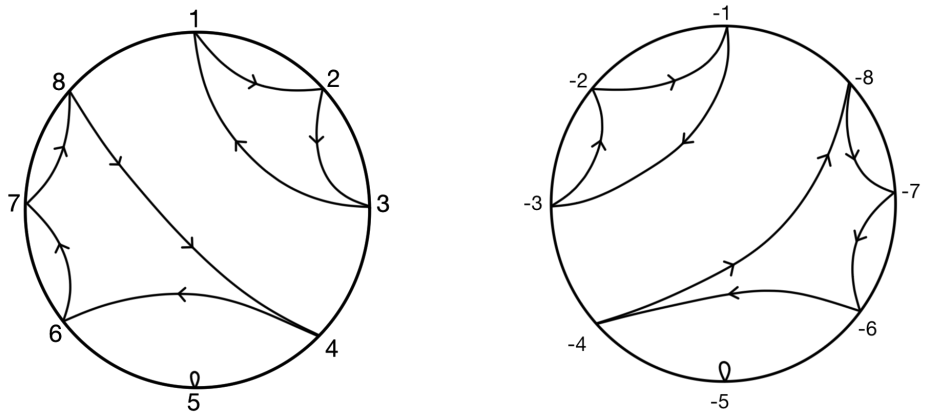

Figure 3.6: Example of a Leading Order Diagram in the Real Case.

Chapter 4

Orthogonal Weingarten Functions

4.1 Introduction

In this chapter, we will derive formulas for the leading and subleading terms of the orthogonal Weingarten functions. For computations in random matrix theory, it is typically enough to know just the leading order term; however, if one is interested in more refined questions about the eigenvalue distribution, such as its infinitesimal law, the subleading term may be of interest. See for example [11] or [16]. An expression for the subleading term was found by Féray in [6], who obtained an interpretation in terms of the area under Dyck paths. Using our partitioned pairings, we will find an alternative proof for the subleading term and an interpretation in terms of non-crossing

annular diagrams. The latter interpretation should be more amenable to computations in random matrix theory.

4.2 k-Chain Expansion of the Orthogonal Weingarten Function

In this section, we will get an expression for the orthogonal Weingarten function in terms of what we will call k-chains of pairing partitions. This will give us an expression for the leading and subleading terms of the orthogonal Weingarten function, which is amenable to analysis in the framework of the previous chapter.

Recall that the orthogonal Weingarten function is the inverse of the function $G : \mathbb{C}[\mathcal{P}_2(n)] \to \mathbb{C}[\mathcal{P}_2(n)]$ defined by $G(p) = \sum_{q \in \mathcal{P}_2(n)} N^{\#(p \vee q)} q$ where we take $N \in \mathbb{N}$. Note that while the Weingarten function is only invertible if $N \geq n$, because we are interested in asymptotics, we can always assume $N \geq n$.

Now define $\Psi : \mathbb{C}[\mathcal{P}_2(n)] \to \mathbb{C}[\mathcal{P}_2(n)]$ to be $N^{-\frac{n}{2}} G$. i.e. $\Psi(p) = \sum_{q \in \mathcal{P}_2(n)} N^{-d(p,q)} q$. Then $N^{\frac{n}{2}} Wg = \Psi^{-1} = (I + (\Psi - I))^{-1} = \sum_{k \geq 0} (-1)^k (\Psi - I)^k$.

Let us now compute $(\Psi - I)^k$. $(\Psi - I)^0$ is the identity by definition. A direct computation shows that $(\Psi - I)(p) = \sum_{p \neq p_1} N^{-d(p,p_1)} p_1$, $(\Psi - I)^2(p) =$

$\sum_{p\neq p_1\neq p_2} N^{-d(p,p_1)-d(p_1,p_2)}p_2$ Thus via a simple induction, we see that

$$\langle(\Psi - I)^k(p), q\rangle = \sum_{p\neq p_1\neq\cdots\neq p_{k-1}\neq q} N^{-(d(p,p_1)+d(p_1,p_2)+\cdots+d(p_{k-1},q))}$$

.

Note that the condition $p \neq p_1 \neq p_2 \neq \cdots \neq q$ makes Wg a rational function in $\frac{1}{N}$ which is what we expect.

Thus,

$$\langle N^{\frac{n}{2}}\mathrm{Wg}(p), q\rangle = \delta_{p,q} + \sum_{k\geq 1}(-1)^k \sum_{p\neq p_1\neq\cdots\neq q} N^{-(d(p,p_1)+\cdots+d(p_{k-1},q))}$$

This motivates the following definition.

Definition 4.2.1. Let $p_0, p_1, \ldots, p_{k-1}, p_k \in \mathcal{P}_2(n)$ be distinct pairings of $\mathcal{P}_2(n)$, We call the $k+1$ tuple $(p_0, p_1, \ldots, p_{k-1}, p_k)$ a **k-chain of excess** r starting at p_0 and ending at p_k if $|p_0p_1| + \cdots + |p_{k-1}p_k| = |p_0p_k| + 2r$.

Recall from the previous chapter, that we renormalized the metric d on pairings to be $d(p,q) = \frac{1}{2}|pq|$, thus written in terms of d, being a k-chain of excess r is equivalent to $d(p,p_1) + \cdots + d(p_{k-1},q) = d(p,q) + r$

It is clear that the coefficient of the leading order term is given by an alternating sum of excess zero k-chains, while the subleading term is given by an alternating sum of excess one k-chains.

4.3 Analysis of the Leading and Subleading Term

In this section, we put our excess zero and excess one k-chains in the context of our partitioned pairings and get explicit formulas.

4.3.1 Excess Zero case

Lemma 4.3.1. *If* $|p_0p_1| + |p_1p_2| + \cdots + |p_{k-1}p_k| = |p_0p_k|$ *then for all* i, $|p_0p_i| + |p_ip_k| = |p_0p_k|$ *and* $|p_0p_i| + |p_ip_{i+1}| = |p_0p_{i+1}|$

Proof. Assume that $|p_0p_1| + |p_1p_2| + \cdots + |p_{k-1}p_k| = |p_0p_k|$. Then $|p_0p_k| \leq |p_0p_i| + |p_ip_k| \leq |p_0p_1| + |p_1p_2| + \cdots + |p_{k-1}p_k| = |p_0p_k|$. Thus, $|p_0p_i| + |p_ip_k| = |p_0p_k|$. Similarly, for all i, $|p_0p_k| \leq |p_0p_{i+1}| + |p_{i+1}p_k| \leq |p_0p_i| + |p_ip_{i+1}| + |p_{i+1}p_k| \leq |p_0p_1| + |p_1p_2| + \cdots + |p_{k-1}p_k| = |p_0p_k|$ Thus, for all i, $|p_0p_i| + |p_ip_{i+1}| = |p_0p_{i+1}|$

□

Assume that $|p_0p_1| + |p_1p_2| + \cdots + |p_{k-1}p_k| = |p_0p_k|$, then $p_0p_1 \leq p_1p_2 \leq \cdots \leq p_{k-1}p_k$. By recalling that $(0, p, p') \cdot (0, p', q) = (0, p, q)$ if and only if $|pp'| + |p'q| = |pq|$, we see from our above lemma that we have a bijection between excess zero k-chains in $\mathcal{P}_2(n)$ and elements $(0, p_0, p_1), \ldots, (0, p_{k-1}, p_k) \in \mathcal{PS}_2(n)$ such that $(0, p_0, p_1) \cdot (0, p_1, p_2) \cdots (0, p_{k-1}, p_k) = (0, p_0, p_k)$. Thus the leading-order term for the $\mathrm{Wg}(p, q)$ is $\mu(0, p, q)$.

4.3.2 Excess One Case

In this section, we denote the set of excess 1 k-chains starting at p_0 and ending at p_k by $Z'(p_0, p_k)$.

Proposition 4.3.2.

Let $S_k(p_0, p_k) := \{((0, p_0, p_1), \ldots, (0, p_{k-1}, p_k) : (0, p_0, p_1) \cdots (0, p_{k-1}, p_k) = (\mathcal{V}, p_0, p_k)$ *where* $\#(\mathcal{V}) = \#(p_0 p_k) - 1\}$ *Then the map* $((0, p_0, p_1), \ldots, (0, p_{k-1}, p_k)) \to (p_0, p_1, \ldots, p_k)$ *defines a bijection between* S_k *and* $Z'_k(p_0, p_k)$.

Proof. Let $(p_0, \ldots, p_k) \in Z'(p_0, p_k)$

Then $|p_0p_1| + |p_1p_2| + \cdots + |p_{k-1}p_k| = |p_0p_k| + 2$

Let $\epsilon_i = |p_0p_{i-1}| + |p_{i-1}p_i| - |p_0p_i|$ for $2 \leq i \leq k$. Since $\epsilon_i \in \{0, 2, 4, \ldots\}$ and $\sum_i \epsilon_i = |p_0p_1| + |p_1p_2| + \cdots + |p_{k-1}p_k| - |p_0p_k| = 2$, there is exactly one index j such that $\epsilon_j = 2$ with the rest being equal to zero.

Because the other ϵ_i's are all zero, we immediately deduce the following facts

Since for all $i \neq j$, $|p_0p_{i-1}| + |p_{i-1}p_i| = |p_0p_{i-1}| + |(p_0p_{i-1})^{-1}p_0p_i| = |p_0p_i|$, we see that p_0p_{i-1} is noncrossing with respect to p_0p_i for all $i \neq j$, thus $p_0p_1 \leq p_0p_2 \leq \cdots \leq p_0p_{j-1}$ and $p_0p_j \leq \cdots \leq p_0p_k$ both hold.

Further, $p_0p_1 \leq \cdots \leq p_0p_{j-2} \leq p_0p_k$, as for all $j' \leq j - 2$, $|p_0p_k| \leq |p_0p_{j'}| + |p_{j'}p_k| = |p_0p_1| + \cdots + |p_{j'-1}p_{j'}| + |p_{j'}p_k| < |p_0p_1| + \cdots + |p_{k-1}p_k| = |p_0p_k| + 2$. Since, the inequality can only change in steps of 2, it follows that $|p_0p_{j'}| + |p_{j'}p_k| = |p_0p_k|$. Thus $p_0p_{j'} \leq p_0p_k$.

Combining these chains of inequalities, we see that $|p_0p_1|+\cdots+|p_{k-1}p_k| = |p_0p_{j-1}|+|p_{j-1}p_k| = |p_0p_k|+2$, thus p_{j-1} is a tunnel permutation with respect to p_0 and p_k.

By 3.5.6, and our first chain of inequalities, we see that

$(0, p_0, p_1) \cdot (0, p_1, p_2) \cdots (0, p_{j-2}, p_{j-1}) = (0, p_0, p_{j-1})$

Again by 3.5.6 and our second chain of inequalities, we see that

$(0, p_{j-1}, p_j) \cdots (0, p_{k-1}, p_k) = (0, p_{j-1}, p_k)$.

Since, p_{j-1} is a tunnel pairing with respect to p_0 and p_k, $(0, p_0, p_{j-1}) \cdot (0, p_{j-1}, p_k) = (\mathcal{V}, p_0, p_k)$ where $\mathcal{V}$ connects one conjugate pair of blocks of p_0p_k and is equal to the blocks of $0_{p_0p_k}$ otherwise. Thus, we have shown one direction.

Conversely, given partitioned pairings $(0, p_0, p_1), \ldots, (0, p_{k-1}, p_k) \in \mathcal{PS}_2(n)$ such that $(0, p_0, p_1) \cdots (0, p_{k-1}, p_k) = (\mathcal{V}, p_0, p_k)$ where $\mathcal{V}$ connects exactly one conjugate pair of blocks of p_0p_k, by 3.5.6, there must exist a smallest index j such that p_{j-1} is a tunnel pairing with respect to p_{j-2} and p_j. i.e. $|p_{j-2}p_{j-1}| + |p_{j-1}p_j| - |p_{j-1}p_j| = 2$. Then, as for all $l < j$, p_l is a disk pairing with respect to p_{l-1} and p_{l+1}, thus we obtain that $|p_0p_1| + \cdots + |p_{j-2}p_{j-1}| = |p_0p_{j-1}|$. Similarly, for all $l > j$, p_l is a disk pairing with respect to p_{l-1} and p_{l+1}, yielding $|p_jp_{j+1}|+\cdots+|p_{k-1}p_k| = |p_jp_k|$. Therefore, $|p_0p_1| + \cdots + |p_0p_k| = |p_0p_k| + 2$

Thus, our partitioned pairings define a k-chain of excess one, yielding the bijection. □

Corollary 4.3.3. *The coefficient for the subleading term of the orthogonal*

Weingarten function is given by

$$\sum_{\substack{\mathcal{V} \geq 0_{pq} \\ |\mathcal{V}| = |0_{pq}|+1}} \mu(\mathcal{V}, p, q)$$

4.4 Recursion Formulas for μ

We will derive some recursion formulas for μ. These formulas will allow us to relate the coefficients to non-crossing and non-crossing annular objects in the leading and subleading order cases, respectively. If $p = q$, then there is nothing to prove, so we will assume that $p \neq q$, which means that $pq \neq e$. Without loss of generality, in the coming computations, we will assume that $pq(1) \neq 1$.

Definition 4.4.1. Let $p, q \in \mathcal{P}_2(n)$. We say that pq is a **double transposition** if $pq = (i_1, j_1)(i_2, j_2)$ for distinct $i_1, i_2, j_1, j_2 \in [n]$.

Lemma 4.4.2. *Let $p, q \in \mathcal{P}_2(n)$ and suppose that 1 is not a fixed point of pq. Let $p' = q(1, q(p(1)))(p(1), q(1))$. Then*

1. *p' is a pairing*

2. *If pq is a double transposition, $p' = p$. Otherwise, $p' \neq p$.*

3. *1 and $p(1)$, are fixed points of pp'.*

4. *$d(p, p') + d(p', q) = d(p, q)$*

Proof. We will show that p' is a pairing. i.e. $(p')^2 = e$ and p' has no fixed points. First, by direct computation,

$(p')^2 = q(1, q(p(1)))(p(1), q(1))q(1, q(p(1)))(p(1), q(1))$

$= (q(1), p(1))(q(p(1)), 1)(1, q(p(1)))(p(1), q(1)) = e.$

Now, assume for contradiction that p' has a fixed point. Since q has no fixed points, the fixed points of p' must be one or more of the points $1, q(p(1)), p(1)$, and $q(1)$. By direct computation, the cycles containing these points in p' are $(1, p(1))$ and $(p(q(1)), q(1))$. Thus, either $p(1) = 1$ or $p(q(1)) = q(1)$, both contradictions as p has no fixed points. Thus p' is a pairing.

For the second assertion, first assume that pq is a double transposition. Then as $pq(1) \neq 1$, pq will have the cycle decomposition
$(1, p(q(1)))(p(1), q(1))$. Since the nontrivial cycles are transpositions $pq = (pq)^{-1} = qp = qpqq$. Thus,
$pq = q(1, p(q(1)))(p(1), q(1))q = (q(1), q(p(q(1))))(q(p(1)), 1)$.
Thus, $p(q(1)) = q(p(1))$. Then $pp' = pq(1, q(p(1)))(p(1), q(1))$
$= (1, q(p(1)))(p(1), q(1))(1, q(p(1)))(p(1), q(1)) = e$, Showing that $p = p'$.

Now assume that pq is not a double transposition. Then
$pp' = pq(1, q(p(1)))(p(1), q(1))$ cannot equal the identity as the inverse of a double transposition is clearly also a double transposition. Thus we have proven the second assertion.

For the third assertion,
$pp'(1) = pq(1, q(p(1)))(p(1), q(1))(1) = pq(q(p(1)) = 1$

Similarly, $pp'(p(1)) = pq(q, q(p(1)))(p(1), q(1))(p(1)) = pq(q(1)) = p(1)$

For the final assertion, note that $d(p', q) = \frac{1}{2}|(1, q(p(1))(p(1), q(1))q \cdot q| = \frac{1}{2}|(1, q(p(1))(p(1), q(1))| = 1$. Since 1 and $p(1)$ are fixed points of pp', but not pq we see that both $(1, q(p(1))$ and $(p(1), q(1))$ cuts the cycles of pq. Thus $d(p, p') = d(p, q) - 1$, which implies, $d(p, p') + d(p', q) = d(p, q)$ □

pp' in the above proposition is the permutation obtained from pq by removing 1 and $p(1)$ from the cycles of pq and otherwise leaving it unchanged. For example, if $p = (1, 5)(2, 8)(3, 7)(4, 6)$ and $q = (1, 8)(2, 7)(3, 6)(4, 5)$, then $p' = (1, 5)(2, 7)(3, 6)(4, 8)$, $pp' = (1)(5)(2, 3, 4)(6, 7, 8)$ and
$pq = (1, 2, 3, 4)(5, 6, 7, 8)$.

Proposition 4.4.3. *Let $p, q \in \mathcal{P}_2(n)$ be pairings such that $pq(1) \neq 1$ and define $\widetilde{Z}'_k(p, q) = \{(p_0, p_1, \ldots, p_k) \in Z'_k(p, q) : p_0 p_1 = (1, l)(p_0(1), p_0(l))\}$*

Then $\sum_{k\geq 2}(-1)^k|Z'_k(p, q)| = \sum_{k\geq 2}(-1)^k|\widetilde{Z}'_k(p, q)|$

Proof. Note that the sum starts at $k = 2$ as the sets are empty for $k = 1$. The proof will proceed as follows. We will split up $Z'_k(p, q)$ into three disjoint sets $\widetilde{Z}'_k(p, q), \hat{Z}'_k(p, q)$, and $\overline{Z}'_k(p, q)$. We will then show that $|\hat{Z}'_k(p, q)| = |\overline{Z}'_{k-1}(p, q)|$. Since $|\hat{Z}'_k(p, q)|$ and $|\overline{Z}'_{k-1}(p, q)|$ have opposite signs in the sum, they will cancel out, leaving only the contributions from $|\widetilde{Z}'_k(p, q)|$.

Let $(p_0, p_1, \ldots, p_k) \in Z'_k(p, q)$ and let r be the smallest integer such that $p_{r-1}p_r(1) \neq 1$. Such an r must exist as $pq = p_0p_1 \cdot p_1p_2 \cdots p_{k-1}p_k$. Let $\widetilde{Z}'_k(p, q)$ denote the set of all k-chains with excess 1 such that $r = 1$ and $p_{r-1}p_r$ is a double transposition. Let $\hat{Z}'_k(p, q)$ denote the excess 1 k-chains such that

$r > 1$ and $p_{r-1}p_r$ is a double transposition. Finally, let $\overline{Z}'_k(p,q)$ denote the excess 1 k-chains such that $r \geq 1$ and $p_{r-1}p_r$ is not a double transposition. Then we get the disjoint union $Z'_k(p,q) = \widetilde{Z}'_k(p,q) \cup \hat{Z}'_k(p,q) \cup \overline{Z}'_k(p,q)$. We shall show that $|\hat{Z}'_k(p,q)| = |\overline{Z}'_{k-1}(p,q)|$ via an explicit bijection.

Let $(p_0, \ldots, p_k) \in \hat{Z}'_k(p,q)$ with $r > 1$.

Send $(p_0, \ldots, p_k) \to (p_0, \ldots, p_{r-2}, p_r, \ldots, p_k)$. Since $p_{r-1}p_r$ is a double transposition, $d(p_{r-1}, p_r) = 1$. Additionally, as $p_{r-2}p_{r-1}$ fixes 1, multiplying $p_{r-2}p_{r-1}$ on the right by $p_{r-1}p_r$ joins 1 to l and $p_{r-1}(1)$ to $p_{r-1}(l)$, reducing the number of cycles by 2. Thus $d(p_{r-2}, p_r) = d(p_{r-2}, p_{r-1}) + 1 = d(p_{r-2}, p_{r-1}) + d(p_{r-1}, p_r)$. Thus $(p_0, \ldots, p_{r-2}, p_r, \ldots, p_k) \in \overline{Z}'_{k-1}(p,q)$.

Conversely, starting with $(p_0, \ldots, p_k) \in \overline{Z}'_k(p,q)$, we can apply the above lemma to p_{r-1} and p_r and get a mapping
$(p_0, \ldots, p_k) \to (p_0, \ldots, p_{r-1}, p', p_r, \ldots, p_k)$ such that $p_{r-1}p'$ fixes 1 and $p'p_r$ is a double transposition that does not fix 1. Note that because $d(p_{r-1}, p') + d(p', p_r) = d(p_{r-1}, p_r)$, this guarantees that $(p_0, \ldots, p_{r-1}, p', p_r, \ldots, p_k) \in \hat{Z}'_k(p,q)$ Thus,

$$\sum_{k\geq 2}(-1)^k|Z'_k(p,q)| = \sum_{k\geq 2}(-1)^k\left(|\widetilde{Z}'_k(p,q)| + |\hat{Z}'_k(p,q)| + |\overline{Z}'_k(p,q)|\right)$$

$$= \sum_{k\geq 2}(-1)^k\widetilde{Z}'_k(p,q)| + \sum_{k\geq 2}(-1)^k|\hat{Z}'_k(p,q)| + \sum_{k\geq 2}(-1)^k|\overline{Z}'_{k-1}(p,q)|$$

$$= \sum_{k\geq 2}(-1)^k|\widetilde{Z}'_k(p,q)|$$

□

Our bijection from the last section yields the following corollary.

Corollary 4.4.4. *Let $p, q \in \mathcal{P}_2(n)$ such that $pq(1) \neq 1$ and let $(\mathcal{V}, p, q) \in \mathcal{PS}_2(n)$ such that $|\mathcal{V}| = |0_{pq}| + 1$. Denote, by p_l, the pairing such that $pp_l = (1, l)(p(1), p(l))$ Then*

$$\mu(\mathcal{V}, p, q) = \sum_{\substack{(0,p_0,p_l)\cdot(\mathcal{U},p_l,q)=(\mathcal{V},pq) \\ l\neq 1}} -\mu(\mathcal{U}, p_l, q)$$

Proof. By definition

$$\mu(\mathcal{V}, p, q) = \sum_{k=1}^{\infty} \sum_{(0,p,p_1)\cdots(0,p_{k-1},q)=(\mathcal{V},p,q)} (-1)^k = \sum_{k=1}^{\infty} \sum_{\substack{(0,p,p_l)\cdot(0,p_l,p_2)\cdots(0,p_{k-1},p_k) \\ =(\mathcal{V},p,q)}} (-1)^k$$

$$= \sum_{k=2}^{\infty} \sum_{\substack{(0,p,p_l),(\mathcal{U},p_l,q) \\ (0,p,p_l)\cdot(0,p_l,q)=(\mathcal{V},p,q)}} \sum_{(0,p_l,p_2)\cdots(0,p_{k-1},p_k)=(\mathcal{U},p_l,q)} (-1)^p$$

$$= \sum_{\substack{(0,p,p_l),(\mathcal{U},p_l,q) \\ (0,p,p_l)\cdot(\mathcal{U},p_l,q)=(\mathcal{V},p,q)}} -\mu(\mathcal{U}, p_l, q)$$

□

μ, being the inverse of a multiplicative function, is also multiplicative. Thus, as V joins exactly, one pair of conjugate blocks, we need only determine

$\mu(1_n, pq)$ where pq has exactly two cycles. Then multiplicativity will allow us to extend to the general case.

Notation 4.4.5. Let $p, q \in \mathcal{P}_2(n)$ be such that pq only has two blocks. Let $m = \frac{n}{2}$, and let $\mu_m := \mu(0, p, q)$ and $\mu'_m = (1_n, p, q)$.

Now take a general $p, q \in \mathcal{P}_2(n)$. Write $pq = C_1 C'_1 \cdots C_k C'_k$, where C_l is a cycle of length i_l. Let $\mathcal{V}$ be any partition that joins the cycle pair C_l, C'_l of pq and is equal to 0_{pq} otherwise. Then by multiplicativity of μ, we obtain the expressions $\mu(0, p, q) = \prod_{s=1}^{k} \mu_{i_s}$ and $\mu(\mathcal{V}, p, q) = \mu'_{i_l} \prod_{s \neq l} \mu_{i_s}$

Proposition 4.4.6. *Let $p, q \in \mathcal{P}_2(n)$ be pairings such that pq has exactly two cycles. Then*

1. $-\mu_m = -\mu(0, p, q) = \sum_{k=1}^{m-1} \mu_k \mu_{m-k}$
2. $-\mu(1_n, p, q) = -\mu'_m = (m-1)\mu_m + \sum_{k=1}^{m-1} \mu'_k \mu_{m-k} + \mu_k \mu'_{m-k}$

Proof. 1. By our corollary

$$-\mu(0, p, q) = \sum_{\substack{(0,p,p_l)\cdot(\mathcal{U},p_l,q)=(0,p,q) \\ l \neq 1}} \mu(\mathcal{U}, p_l, q)$$

p_1 must be a disk permutation, with respect to p, q, thus $\mathcal{U} = 0_{p_l q}$. Let C denote the cycle of pq such that 1 and l are contained inside of it and let C' denote the cycles such that $p(1), p(1)$ are contained inside of it. Then $p_l q = (1, l)(p(1), p(l))pq$ must have two conjugate cycle pairs C_1, C'_1, C_2, C'_2 and $(1, l)$ cuts C and $(p(1), p(l))$ cuts C'. If C_1 has length

k, then C_2 must have length $m-k$, thus $\mu(0, p_l, q) = \mu_k \mu_{m-k}$. Since l cannot equal $p(1)$, there are $m-1$ choices for l, and each value for l corresponds to a different value of k for $1 \leq k \leq m-1$. Thus, we obtain the recursion $-\mu(0, p, q) = \sum_{l=1}^{m-1} \mu_k \mu_{m-k}$ giving us the result.

2. By our corollary,

$$-\mu(1_n, p, q) = \sum_{\substack{(0,p_0,p_l)\cdot(\mathcal{U},p_l,q)=(1_n,p,q) \\ l\neq 1}} \mu(\mathcal{U}, p_l, q)$$

Where p_l is that pairing such that $pp_l = (1,l)(p(1), p(l))$. Write $pq = CC'$. We shall do a case-by-case analysis.

Case 1: p_l is a tunnel pairing relative to p, q. Then, both $(1,l)$ and $(p(1), p(l))$ connect the blocks of pq. Since $p_l q = (p_l p)(pq)$, we deduce that $p_l q$ also only has two cycles. This follows from the fact that multiplication by $(1,l)$ will join the two cycles of pq into one cycle and further multiplication by $(p(1), p(l))$ will cut this newly joined cycle into two. By 3.5.6, we know that $\mathcal{U}$ must equal $0_{p_l q}$. There are $m-1$ possible choices for l, yielding $(m-1)\mu_m$.

Case 2: p_l is a disk pairing relative to p, q. Then we may assume without loss of generality that $1, l$ is contained inside of C and $p(1), p(l)$ is contained inside of C'. Thus, $p_l q = (1,l)(p(1), p(l))pq$ must have two conjugate cycle pairs C_1, C_1', C_2, C_2' as $(1,l)$ cuts C and $(p(1), p(l))$ cuts C'. Then by 3.5.6, $|U| = |0_{p_l q}| + 1$. i.e. it connects one pair of conjugate

cycles of $p_l q$. For each choice of l, there are exactly two choices for $\mathcal{U}$. Let $\mathcal{U}_1$ be the partition that joins C_1, C_1' and let $\mathcal{U}_2$ be the partition that joins C_2, C_2'. If C_1 has length k, then C_2, must have length $m-k$, thus $\mu(\mathcal{U}_1, p_l, q) = \mu_k' \mu_{m-k}$ and $\mu(\mathcal{U}_2, p_l, q) = \mu_k \mu_{m-k}'$.There are $m-1$ possible choices for l and each value for l corresponds to a different value of k for $1 \leq k \leq m-1$.

Thus, for the disk pairings, we get the term $\sum_{k=1}^{m-1} \mu_k' \mu_{m-k} + \mu_k \mu_{m-k}'$, implying the result.

□

Remark 4.4.7.

1. The first case of our above proposition tells us that $\mu(0, p, q)$ satisfies the same recursion as the signed and shifted Catalan numbers. By multiplicatively of μ, if $pq = C_1 C_1' \cdots C_s C_s'$ and C_l contains i_l elements, then $\mu(0, p, q)$ satisfies the same recursion and thus is equal to the product of signed and shifted Catalan numbers $\prod_{l=1}^{s} (-1)^{i_l - 1} C_{i_l - 1}$

2. μ_m' has the closed form $4^m - \binom{2m+1}{m}$. This was found by Féray in [7], who found an expression for the subleading term in terms of the area under Dyck paths.

3. The recursion for $|NC^{\delta}(n, -n)|$ was shown in [16] to be given by $m_n' = (n-1)m_{n-1} + \sum_{k=1}^{n-1} m_{k-1}' m_{n-k} + m_{k-1} m_{n-k}'$, where $m_n' = |NC^{\delta}(n, -n)|$ and $m_n = |NC(n)| = C_n$ thus the recursion for $\mu(1_n, p, q)$ is equal to the signed recursion of $|NC^{\delta}(n, -n)|$.

Chapter 5

The Asymptotic Distribution of X_m

5.1 Introduction

Let $\{O_i\}_{i=1}^m$ be a collection of independent $N \times N$ Haar distributed orthogonal matrices and let $X_m = O_1 + O_1^t + \cdots + O_m + O_m^t$. Using the machinery that we have built up, we will now determine its first and second-order limiting distributions. The structure of this chapter is as follows. First, we will prove some computational lemmas on products of traces. We will then determine the first order limiting distribution of X_m directly. After this, we will determine the second-order limiting distribution of $O + O^t$ and use real asymptotic second-order freeness to get the general case.

5.2 Pairings and Products of Traces

Let $A_k = (a_{i,j}^{(k)})$ be $N \times N$ matrix for $1 \leq k \leq n$. For any matrix A, denote $A^{(1)} = A$ and $A^{(-1)} = A^T$. In this section, we will be interested in expressions of the form $\mathrm{Tr}_\sigma(A_1^{(\epsilon_1)} \cdots A_n^{(\epsilon_n)}) := \prod_{(i_1,\ldots,i_n)\in\sigma} \mathrm{Tr}(A_{i_1}^{(\epsilon_{i_1})} \cdots A_{i_k}^{(\epsilon_{i_k})})$ where $\sigma \in S_n$ and $\epsilon = (\epsilon_1, \cdots, \epsilon_k) \in \mathbb{Z}_2^n$. In this section and in future sections, we will view $\mathbb{Z}_2^n$ as a subgroup of $S_{\pm n}$ where elements $\epsilon \in \mathbb{Z}_2^n$ permute elements of $[\pm n]$ by the rule $\epsilon(\pm k) = \pm\epsilon_k k$.

Recall, that given a multi-index, $I : [n] \to [N]$, we denote by $\mathrm{Ker}(I)$, the partition $\pi \in \mathcal{P}(n)$ defined by the rule $i_{l_1} \sim_\pi i_{l_2} \iff I(l_1) = I(l_2)$. We note that given a permutation $\sigma \in S_n$ and given a multi-index $I : [n] \to [N]$, $I = I \circ \sigma$ is equivalent to the condition that $0_\sigma \leq \mathrm{Ker}(I)$ i.e. I is constant on the cycles of σ.

Lemma 5.2.1. *Let $A_k = (a_{i,j}^{(k)})_{i,j=1}^N$ be an $N \times N$ matrix for $1 \leq k \leq n$, let $\delta = (1,-1)\cdots(n,-n) \in S_{\pm n}$ and let $(\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}_2^n$. Then*

$$\mathrm{Tr}(A_1^{(\epsilon_1)} \cdots A_n^{(\epsilon_n)}) = \sum_{\substack{I \\ \epsilon\gamma\delta\gamma^{-1}\epsilon \leq Ker(I)}} a_{i_1,i_{-1}}^{(1)} a_{i_2,i_{-2}}^{(2)} \cdots a_{i_n,i_{-n}}^{(n)}$$

Proof. Let $a_{i,j}^{(\epsilon_k)} = \begin{cases} a_{i,j}^{(k)} & \text{if } \epsilon_k = 1 \\ a_{j,i}^{(k)} & \text{if } \epsilon_k = -1 \end{cases}$

Then under this notation,our trace formula becomes,

$$\mathrm{Tr}(A_1^{(\epsilon_1)} \cdots A_n^{(\epsilon_n)}) = \sum_{j_1,\ldots,j_n=1}^{N} a_{j_1,j_2}^{(\epsilon_1)} a_{j_2,j_3}^{(\epsilon_2)} \cdots a_{j_n,j_1}^{(\epsilon_n)}$$

We can define a bijection between multi-indices $J : [n] \to [N]$ and multi-indices $L : [\pm n] \to [N]$ satisfying $\gamma\delta\gamma^{-1} \leq \mathrm{Ker}(I)$ via the mapping $(j_1, \ldots, j_n) \to (l_1, l_{-1}, \ldots, l_n, l_{-n})$ where $j_s = l_s$ for all $s \geq 1$ and $l_{-1} = l_2, l_{-2} = l_3$ and so on. Thus, we can rewrite our sum as

$$\sum_{\substack{L \\ \gamma\delta\gamma^{-1} \leq \mathrm{Ker}(L)}} a_{l_1,l_{-1}}^{(\epsilon_1)} a_{l_2,l_{-2}}^{(\epsilon_2)} \cdots a_{l_n,l_{-n}}^{(\epsilon_n)}$$

We now wish to undo the epsilons. Let $I = L \circ \epsilon$. Then as $\gamma\delta\gamma^{-1} \leq \mathrm{Ker}(L) = \mathrm{Ker}(I \circ \epsilon)$, $I \circ \epsilon = I \circ \epsilon \circ \gamma\delta\gamma^{-1} \Rightarrow I = I \circ \epsilon\gamma\delta\gamma^{-1}\epsilon$, thus $\epsilon\gamma\delta\gamma^{-1}\epsilon \leq \mathrm{Ker}(I)$. Under this re-indexing, we see that $a_{l_k,l_{-k}}^{(\epsilon_k)} = a_{i_k,i_{-k}}^{(k)}$, thus, we obtain

$$\mathrm{Tr}(A_1^{(\epsilon_1)} \cdots A_n^{(\epsilon_n)}) = \sum_{\substack{I \\ \epsilon\gamma\delta\gamma^{-1}\epsilon \leq \mathrm{Ker}(I)}} a_{i_1,i_{-1}}^{(1)} \cdots a_{i_n,i_{-n}}^{(n)}$$

□

We will now deal with products of traces. Let $\sigma = C_1 \cdots C_S \in S_n$. If $(l_1, l_2, \ldots, l_k)$ is a cycle of σ, then the condition $\sigma\delta\sigma^{-1} \leq \mathrm{Ker}(I)$ means that $I(-l_1) = I(l_2)$, $I(-l_2) = l_3$ and so on. Using a slight abuse of notation, we let $C \in \sigma$ denote the condition that C is a cycle of σ and let $\delta_{|C} :=$

$\delta_{|\{l_1,\ldots,l_k,-l_1,\ldots,-l_k\}} = (l_1,-l_1)\cdots(l_k,-l_k)$. Then under this notation,

$$\mathrm{Tr}_\sigma(A_1\cdots A_n) = \prod_{(l_1,\ldots,l_k)\in\sigma} \mathrm{Tr}(A_{l_1}\cdots A_{l_k})$$

$$= \prod_{(l_1,\ldots,l_k)\in\sigma} \left(\sum_{\substack{I:\{l_1,\ldots,l_k\}\to N \\ C\delta_{|C}C^{-1}\leq \mathrm{Ker}(I)}} a^{(l_1)}_{l_1,l_{-1}}\cdots a^{(l_k)}_{l_k,l_{-k}} \right) = \sum_{\sigma\delta\sigma^{-1}\leq \mathrm{Ker}(I)} a^{(1)}_{i_1,i_{-1}}\cdots a^{(n)}_{i_n,i_{-n}}$$

Allowing for transpositions, by doing the same reindexing, as in the above lemma, we see that this generalizes to

$$\mathrm{Tr}_\sigma(A_1^{(\epsilon_1)}\cdots A_n^{(\epsilon_n)}) = \sum_{\substack{I \\ \epsilon\sigma\delta\sigma^{-1}\epsilon\leq \mathrm{Ker}(I)}} a^{(1)}_{i_1,i_{-1}}\cdots a^{(n)}_{i_n,i_{-n}}$$

Note that $\epsilon\sigma\delta\sigma^{-1}\epsilon \in \mathcal{P}_2(\pm n)$.

Conversely, given an arbitrary pairing $p \in \mathcal{P}_2(\pm n)$, we can construct a corresponding $\sigma \in S_n$ and $\epsilon \in \mathbb{Z}_2^n$ so that the left-hand side above equation is satisfied. To this end, write $p\delta = C_1C_1'\cdots C_sC_s'$ where $C_l' = \delta C_l^{-1}\delta$. For each $1 \leq l \leq s$ denote $C_l = (i_1,\ldots,i_k)$ and define $\widetilde{C}_l = (r_1,\ldots,r_k)$ where $r_t = |i_t|$. Define $\epsilon_l \in \mathbb{Z}_2^n$ by $\epsilon_l(i_t) = \frac{i_t}{|i_t|}$ and $\epsilon_l(j) = j$ otherwise. Note that in this construction, it does not matter if we choose C_l or C_l' in the previous step. Set $\sigma = \widetilde{C_1}\cdots\widetilde{C}_l$ and ϵ as the product of the ϵ_l's corresponding to each cycle. Note that due to the cycles of δ, k being in the cycle C_l

implies that $-k$ is inside the cycle C'_l. i.e. $\sigma \in S_n$. By construction, we see that $\epsilon\sigma\epsilon = C_1 \cdots C_l$, thus $p\delta = (\epsilon\sigma\epsilon)(\delta\epsilon\sigma^{-1}\epsilon\delta)$. Multiplying each side by δ, $p = \epsilon\sigma\epsilon\delta\epsilon\sigma^{-1}\epsilon = \epsilon\sigma\delta\sigma^{-1}\epsilon$ as $\epsilon\delta\epsilon = \delta\epsilon^2\delta$. Thus, for our chosen p and constructed (σ, ϵ) pair, we obtain

$$\mathrm{Tr}_\sigma(A_1^{(\epsilon_1)} \cdots A_n^{(\epsilon_n)}) = \sum_{\substack{I \\ p \leq \mathrm{Ker}(I)}} a^{(1)}_{i_1,i_{-1}} \cdots a^{(n)}_{i_n,i_{-n}}$$

5.3 The First Order Distribution of X_m

We will now determine the asymptotic distribution of X_m. This will be done by determining the limiting moments. We will write out an expansion for the moments and show that the leading order term corresponds to the moments in the Kesten-McKay law with parameter m. Recall that the Kesten-Mckay Law with parameter m is a compactly supported $\mathbb{R}$-valued measure with density

$$\frac{1+m}{2\pi} \frac{\sqrt{4m - t^2}}{(1+m)^2 - t^2}$$

for $|t| \leq 2\sqrt{m}$ and zero otherwise. For $m = 1$, this becomes the arcsine law. The moments of the Kesten-Mckay law. written in terms of its free cumulants are given by $\sum_{\sigma \in NC(n)} m^{\#(\sigma)} \kappa_\sigma$ where $\kappa_{2k} = (-1)^{n-1} 2C_{n-1}$, $\kappa_{2k+1} = 0$ and κ_σ is the multiplicative extension. Note that for the case $m = 1$, we get the arcsine law. Because the Kesten-Mckay law is compactly supported, knowing the moments is equivalent to knowing the measure. The Riesz representation theorem tells us that knowing how to integrate continuous functions is

equivalent to knowing the measure. Since our measure has compact support, by the Stone-Weierstrass theorem, we can uniformly approximate continuous functions by polynomial functions. Since the moments are just monomials, one can show from this that the moments of a compactly supported measure determine the measure.

We now recall some basic computational lemmas.

Lemma 5.3.1.

1. *Let O be a Haar-distributed orthogonal matrix. Then*

$$\mathbb{E}[o_{i_1,i_{-i}} \cdots o_{i_n,i_{-n}}] = \sum_{p,q \in \mathcal{P}_2(n)} \left\langle \begin{matrix} I \\ p\delta q\delta \end{matrix} \right\rangle Wg(p,q)$$

2. *Let $\{O_1, \ldots, O_m\}$ be $N \times N$ independent Haar distributed orthogonal matrices and let $l : [n] \to [m]$. Then*

$$\mathbb{E}[o^{(l_1)}_{i_1,i_{-1}} \cdots o^{(l_n)}_{i_n,i_{-n}}] = \prod_{\substack{U \in Ker(l) \\ U=(m_1,\ldots,m_r)}} \mathbb{E}[o_{i_{m_1},i_{-m_1}} \cdots o_{i_{m_r},i_{-m_r}}]$$

.

Both claims follow immediately. The first claim is simply a rewriting of our original integration formula using only one multi-index I with domain $[\pm n]$ instead of two multi-indices I and J, both with domains $[n]$. The second claim follows from independence.

We now state and prove one final lemma before we begin the proof of the main result.

Lemma 5.3.2. *Let $p, q \in \mathcal{P}_2(n)$ and let $\delta = (1,-1)\cdots(n,-n)$. Then $\#(pq) = \#(p\delta q)$. Moreover, if $(i_1, \ldots, i_k)$ is a cycle of pq, then $(i_1, -q(i_1), i_2, -q(i_2), \ldots, i_k, -q(i_k))$ is a cycle of $p\delta q$.*

Proof. Let $p, q \in \mathcal{P}_2(n)$ and $\delta = (1,-1)(2,-2)\cdots(n,-n)$ and let $C = (i_1, \ldots, i_k)$ be a cycle of pq. We will show that $(i_1, -q(i_1), i_2, -q(i_2), \ldots, i_k, -q(i_k))$ is a cycle of $p\delta q$. Recall form 3.3.1 that $(i_1, \ldots, i_k)$ being a cycle of pq implies that $(q(i_1), i_2), (q(i_2), i_3), \ldots, (q(i_k), i_1)$ are blocks of p and $(i_1, q(i_1)), \ldots, (i_k, q(i_k))$ are blocks of q. Then, by direct computation, $p\delta q(i_l) = p(-q(i_l)) = -q(i_l)$, $p\delta q(-q(i_l)) = p(q(i_l)) = i_{l+1}$ where $l+1$ is understood mod k. Thus, $(i_1, -q(i_1), i_2, -q(i_2), \ldots, i_k, -q(i_k))$ and $(q(i_k), -i_k, \ldots, q(i_1), -i_1)$ are cycles of $p\delta q$. Since $p\delta q$ has the same number of cycles as pq, $\#(pq) = \#(p\delta q)$.

□

Proposition 5.3.3. $\mathbb{E}(\mathrm{tr}(X_m^n)) = \sum_{\sigma \in NC(n)} \kappa_\sigma + O(\frac{1}{N})$

Proof. Given numbers $x, y \in \mathbb{N}$, let $(x)_y := x(x-1)\cdots(x-(y-1))$. Expanding out the definitions and using 5.3.1, we obtain

$$\mathbb{E}(\mathrm{tr}(X_m^n)) = \frac{1}{N} \sum_{\epsilon \in \mathbb{Z}_2^n} \sum_{l:[n]\to[m]} \sum_{\substack{I \\ \epsilon\gamma\delta\gamma^{-1} \epsilon \leq \mathrm{Ker}(I)}} \mathbb{E}[o_{i_1,i_{-1}}^{(l_1)} \cdots o_{i_n,i_{-n}}^{(l_n)}]$$

$$= \frac{1}{N} \sum_{\epsilon \in \mathbb{Z}_2^n} \sum_{l:[n]\to[m]} \sum_{\substack{I \\ \epsilon\gamma\delta\gamma^{-1} \epsilon \leq \mathrm{Ker}(i)}} \sum_{\substack{p,q \in \mathcal{P}_2(n) \\ p \vee q \leq \mathrm{Ker}(l)}} \left\langle \begin{matrix} I \\ p\delta q\delta \end{matrix} \right\rangle \mathrm{Wg}(p,q)$$

$$= \frac{1}{N} \sum_{p,q \in \mathcal{P}_2(n)} \sum_{\substack{l:[n]\to[m] \\ \mathrm{Ker}(l) \geq p \vee q}} \mathrm{Wg}(p,q) \sum_{\epsilon \in \mathbb{Z}_2^n} \sum_{\substack{I \\ \epsilon\gamma\delta\gamma^{-1}\epsilon \leq \mathrm{Ker}(I)}} \left\langle \begin{matrix} i \\ p\delta q\delta \end{matrix} \right\rangle$$

$$= \sum_{p,q \in \mathcal{P}_2(n)} \sum_{\substack{\mathcal{U} \in \mathcal{P}(n) \\ p \vee q \leq U}} (m)_{\#(\mathcal{U})} \mathrm{Wg}(p,q) \sum_{\epsilon \in \mathbb{Z}_2^n} N^{\#(\epsilon\gamma\delta\gamma^{-1}\epsilon \vee p\delta q\delta)-1}$$

For p, q fixed, the leading order term for $\mathrm{Wg}(p,q)$ is given by $\#(p \vee q) - n$. Thus, to determine the asymptotic moments, we must determine when the quantity $\#(\epsilon\gamma\delta\gamma^{-1}\epsilon \vee p\delta q\delta) + \#(p \vee q) - (n+1)$ is maximized.

For this, we will rewrite our quantity in terms of permutations instead of partitions. $\#(\epsilon\gamma\delta\gamma^{-1}\epsilon \vee p\delta q\delta) + \#(p \vee q) - (n+1) = \frac{1}{2}(\#(\epsilon\gamma\delta\gamma^{-1}\epsilon p\delta q\delta) + \#(pq) - 2(n+1))$

Using the cyclic symmetry of $\#(\cdot)$ and the fact that ϵ and δ commute, we obtain $\#(\epsilon\gamma\delta\gamma^{-1}\epsilon p\delta q\delta) = \#((\epsilon p\delta q\epsilon)^{-1} \cdot \delta\gamma^{-1}\delta\gamma)$

By our above lemma, $\#(pq) = \#(p\delta q)$ and thus $\#(pq) = \#(\epsilon p\delta q\epsilon)$. Recall that $\epsilon p\delta q\epsilon = \epsilon p\delta q\delta\epsilon \cdot \delta$. i.e. it is a paired pairing.

Recall from our discussion on geodesics in S_n (remark 3.2.19, now replacing $[n]$ with $[\pm n]$) that $\#(\epsilon p\delta q\epsilon) + \#((\epsilon p\delta q\epsilon)^{-1} \cdot \delta\gamma^{-1}\delta\gamma) \leq 2n+2$ with equality iff $\epsilon p\delta q\epsilon$ is non-crossing relative to $\delta\gamma^{-1}\delta\gamma$. To be non-crossing relative to $\delta\gamma^{-1}\delta\gamma$, $\epsilon p\delta q\epsilon$ must decompose into $\sigma\sigma'$ where $\sigma \in S_n$ $\sigma' \in S_{-n}$ and $\sigma \leq \gamma, \sigma' \leq \delta\gamma^{-1}\delta$. Due to the conjugate cycle structure of paired pairings, $\epsilon p\delta q\epsilon$ breaking up in this way implies that $\sigma' = \delta\sigma^{-1}\delta$, thus if we can determine the necessary conditions on (p, q, ϵ) for this to happen, we can determine the lead-

ing order term. Recall from 3.3.1 that if $i, j \in [\pm n]$ occur in the same cycle of either $\epsilon p\delta q\delta\epsilon$ or δ, then they cannot appear in the same cycle as their product. Let $C = (i_1, \ldots, i_k)$ be a cycle of $\epsilon p\delta q\epsilon$. Then by 3.3.1, $\epsilon p\delta q\delta\epsilon$ has the cycles $(\delta(i_1), i_2), (\delta(i_2), i_3), \ldots, (\delta(i_k), i_1) = (-i_1, i_2), (-i_2, i_3), \ldots, (-i_k, i_1)$. For C to be a cycle in either S_n or S_{-n}, it immediately follows that all of the cycles of $\epsilon p\delta q\delta\epsilon$ must be composed of one element from $[n]$ and one from $[-n]$. Conversely, if all of the cycles of $\epsilon p\delta q\delta\epsilon$ are of this form, then $\epsilon p\delta q\delta\epsilon \cdot \delta$ decomposes into $\sigma\delta\sigma^{-1}\delta$ with $\sigma \in S_n$. To see this, let $i \in [n]$ and let $(-i, j)$ be a cycle of $\epsilon p\delta q\delta\epsilon$. Then $\epsilon p\delta q\delta\epsilon\delta(i) = \epsilon p\delta q\delta\epsilon(-i) = j$. Thus $\epsilon p\delta q\epsilon[n] = [n]$. Since for all $p, q \in \mathcal{P}_2(n)$, the blocks of $p\delta q\delta$ are either strictly positive or strictly negative, we obtain the following result.

Corollary 5.3.4. *Let $p, q \in \mathcal{P}_2(n)$ and let $\epsilon \in \mathbb{Z}_2^n$. Then $\epsilon p\delta q\epsilon = \sigma\delta\sigma^{-1}\delta$ if and only if for all blocks (r, s) of both p and q, $\epsilon_r = -\epsilon_s$. i.e. ϵ alternates on the blocks of both p and q.*

Recall that the conjugate cycle pairs of $p\delta q$ are of the form $(i_1, -q(i_1), \ldots i_k, -q(i_k))$, $(q(i_k), -i_k, \ldots, q(i_1), i_1)$ where $(q(i_1), i_2), \ldots, (q(i_k), i_1)$ are cycles of p and $(i_1, q(i_1)), \ldots, (i_k, q(i_k))$ are cycles of q. Thus, restricted to the pair of conjugate cycles, there are two possible choices of ϵ. Either ϵ switches the sign of all the i_l's and leaves the sign of the $q(i_l)$'s unchanged, or vice-versa. Thus, given a fixed $p, q \in \mathcal{P}_2(n)$ there are $2^{\frac{\#(p\delta q)}{2}}$ possible choices of ϵ such that ϵ alternates on the blocks of both p and q. Thus, if $p\delta q = \sigma\delta\sigma^{-1}\delta$, then there are $2^{\#(\sigma)}$ possible choices of ϵ.

For the geodesic condition to be satisfied, σ must also be non-crossing relative to γ. 5.3.2 implies that the σ's we can obtain are the σ's whose cycles only have an even length due to every cycle in pq corresponding to a cycle in $p\delta q$ with twice as many elements.

Let $w_1(p,q)$ be the leading order term for $\mathrm{Wg}(p,q)$ and note that

$$\sum_{\substack{\mathcal{U}\in\mathcal{P}(n)\\ p\vee q\leq\mathcal{U}}} (m)_{\#(\mathcal{U})} = \sum_{\substack{\mathcal{U}\in\mathcal{P}(n)\\ p\vee q\leq\mathcal{U}}} |\{I:[n]\to[m] \mid \mathrm{Ker}(f)=\mathcal{U}\}|$$

$$= |\{I:[n]\to[m] \mid \mathrm{Ker}(f)\geq p\vee q\}| = m^{\#(p\vee q)}$$

Then, we can rewrite our expectation $\mathbb{E}[\mathrm{tr}(X_m^n)]$ as

$$\sum_{p,q\in\mathcal{P}_2(n)} m^{\#(p\vee q)} w_1(p,q) \sum_{\epsilon\in\mathbb{Z}_2^n} N^{\#(p\vee q)+\#(\epsilon\gamma\delta\gamma^{-1}\epsilon\vee p\delta q\delta)-(n+1)} + O(\frac{1}{N})$$

$$= \sum_{\substack{\sigma\in NC(n)\\ \sigma \text{ is even}}} m^{\#(\sigma)} \prod_{i=1}^{\#(\sigma)} \mu_{r_i} \sum_{\substack{\epsilon\in\mathbb{Z}_2^n\\ \epsilon \text{ alternates on cycles}}} 1 + O(\frac{1}{N})$$

$$= \sum_{\substack{\sigma\in NC(n)\\ \sigma \text{ even}}} (2m)^{\#(\sigma)} \prod_{i=1}^{\#(\sigma)} \mu_{r_i} + O(\frac{1}{N}) = \sum_{\sigma\in NC(n)} m^{\#(\sigma)}\kappa_\sigma + O(\frac{1}{N})$$

Where we drop the condition that σ only has even cycles in the last

equality by recalling that the odd free cumulants are zero. □

5.4 The Second Order Distribution of X_m

Let $X = O + O^t$ where O is an $N \times N$ Haar distributed orthogonal matrix. In this section, we shall compute the asymptotic covariance $\mathrm{Cov}(\mathrm{Tr}(X^n), \mathrm{Tr}(X^m)) = \mathbb{E}[(\mathrm{Tr}(X^n)(\mathrm{Tr}(X^m))] - \mathbb{E}[\mathrm{Tr}(X^n)]\mathbb{E}[\mathrm{Tr}(X^m)]$. Using second-order real freeness, we will then obtain the second order limiting distribution for X_m. We have two cases. The first case is when both n and m are even, and the second is when both n and m are odd. In the second case, $\mathbb{E}[\mathrm{Tr}(X^n)]\mathbb{E}[\mathrm{Tr}(X^m)]$ is zero.

Expanding out the first term, we obtain that $\mathbb{E}[(\mathrm{Tr}(X^n)(\mathrm{Tr}(X^m))]$ equals

$$\mathbb{E}\left[\sum_{\epsilon\in\mathbb{Z}_2^n}\sum_{\substack{i\\ \epsilon\gamma_n\delta\gamma_n^{-1}\epsilon\leq\mathrm{Ker}(I)}} o_{i_1,i_{-1}}\cdots o_{i_n,i_{-n}}\sum_{\epsilon\in\mathbb{Z}_2^m}\sum_{\substack{i\\ \epsilon\gamma_m\delta\gamma_m^{-1}\epsilon\leq\mathrm{Ker}(I)}} o_{i_1,i_{-1}}\cdots o_{i_m,i_{-m}}\right]$$

Denote $\gamma_{n,m} = (1,\ldots,n)(n+1,\ldots,n+m)$. Then after reindexing and regrouping, the above sum becomes

$$\mathbb{E}[(\mathrm{Tr}(X^n)(\mathrm{Tr}(X^m))] = \sum_{\epsilon\in\mathbb{Z}_2^{n+m}}\sum_{\substack{I\\ \epsilon\gamma_{n,m}\delta\gamma_{n,m}\epsilon\leq\mathrm{Ker}(I)}} \mathbb{E}[o_{i_1,i_{-1}}\cdots o_{i_{n+m},i_{-(n+m)}}]$$

$$= \sum_{p,q \in \mathcal{P}_2(n+m)} \mathrm{Wg}(p,q) \left(\sum_{\epsilon \in \mathbb{Z}_2^{n+m}} N^{\#(\gamma_{n,m}^{-1} \delta \gamma_{n,m} \delta \vee \epsilon p \delta q \epsilon)} \right)$$

Applying our formula from the previous section to $\mathbb{E}[\mathrm{Tr}(X^n)]\mathbb{E}[\mathrm{Tr}(X^m)]$, after reindexing so that we can bring everything under one sum, we see that $\mathbb{E}[\mathrm{Tr}(X^n)]\mathbb{E}[\mathrm{Tr}(X^m)]$ is equal to

$$\sum_{\substack{p,q \in \mathcal{P}_2(n+m) \\ p,q \text{ don't connect the blocks of } \gamma_{n,m}}} \mathrm{Wg}(p,q) \left(\sum_{\epsilon \in \mathbb{Z}_2^{n+m}} N^{\#(\gamma_{n,m}^{-1} \delta \gamma_{n,m} \delta \vee \epsilon p \delta q \epsilon)} \right)$$

In the second term, all $p, q \in \mathcal{P}_2(n+m)$ that don't connect the blocks of $\gamma_{n,m}$ appear in the first term with the same coefficient. Thus we only need to understand how p, q connecting the blocks of $\epsilon p \delta q \epsilon$ affects the cycles of $\epsilon p \delta q \epsilon$.

Lemma 5.4.1. *Let $p, q \in \mathcal{P}_2(n+m)$. If p or q connects the blocks of $\gamma_{n,m}$, then $p\delta q$ connects $(1, \ldots, n, -1, \ldots, -n)$ with $(n+1, \ldots, n+m, -(n+1), \ldots, -(n+m))$.*

Conversely, if p, q does not connect the blocks of $\gamma_{n,m}$, then $p\delta q$ does not connect $(1, \ldots, n, -1, \ldots, -n)$ with $(n+1, \ldots, n+m, -(n+1), \ldots, -(n+m))$.

Proof. Recall from proposition 3.1 that if pq has the block $(i_1, \ldots, i_k)$, then p must contain the cycles $(q(i_1), i_2), (q(i_2), i_3), \ldots, (q(i_k), i_1)$ and q must contain the cycles

$(i_1, q(i_1)), (i_2, q(i_2), \ldots, (i_k, q(i_k))$. Also recall that by 5.3.2 that $(i_1, \ldots, i_k)$ being a cycle of pq implies that $(i_1, -q(i_1), i_2, -q(i_2), \ldots, i_k, -q(i_k))$ is a cycle of $p\delta q$. It is then immediate from the cycle decomposition of $p\delta q$ in terms of the cycles of pq that $p\delta q$ connects $(1, \ldots, n, -1, \ldots, -n)$ with $(n+1, \ldots, n+m, -(n+1), \ldots, -(n+m))$ iff p or q connects the cycles of $\gamma_{n,m}$. □

Note that this lemma is also true for $\epsilon p\delta q\epsilon$ for any choice of ϵ as ϵ only changes the sign of entries in a block. We are now ready to perform our analysis on the surviving leading order terms.

Proposition 5.4.2. *Let $p, q \in \mathcal{P}_2(n+m)$ be such that p or q connects the the blocks $\gamma_{n,m}$ and let $\epsilon \in \mathbb{Z}_2^n$. Then $\#(\epsilon q\delta p\epsilon\gamma_{n,m}\delta\gamma_{n,m}^{-1}\delta) + \#(\epsilon p\delta q\epsilon) \leq 2(n+m)$*

Proof. By noting that $\gamma_{n,m}\delta\gamma_{n,m}^{-1}\delta$ has 4 blocks, remark 3.2.19 tells us that for any choice of p, q, ϵ we have $\#(\epsilon q\delta p\epsilon\gamma_{n,m}\delta\gamma_{n,m}^{-1}\delta) + \#(\epsilon p\delta q\epsilon) \leq 2(n+m) + 4$. Write $\epsilon p\delta q\epsilon = \sigma\delta\sigma^{-1}\delta$. By our above lemma, p, q connecting the blocks of $\gamma_{n,m}$ implies that $\epsilon p\delta q\epsilon$ contains a cycle which connects $(1, \ldots, n, -1, \ldots, -n)$ with $(n+1, \ldots, n+m, -(n+1), \ldots, -(n+m))$. Without loss of generality, assume that $(1, \ldots, n)$ gets connected to $(n+1, \ldots, n+m)$ and that this connecting cycle occurs inside of σ. Then by the conjugate cycle structure, $\delta\sigma^{-1}\delta$ connects $(-1, \ldots, -n)$ to $(-(n+1), \ldots, -(n+m))$. By remark 3.2.19, both of these connections contribute a -2, thus $\#(\epsilon q\delta p\epsilon\gamma_{n,m}\delta\gamma_{n,m}^{-1}\delta) + \#(\epsilon p\delta q\epsilon) \leq 2(n+m)$. □

Recall from the main proof of the previous section that for a fixed p, q, ϵ

the leading order of N is given by $\#(\gamma_{n,m}^{-1}\delta\gamma_{n,m}\delta \vee \epsilon p\delta q\epsilon) + \#(p \vee q) - (n+m) = \#(\epsilon p\delta q\epsilon \vee \gamma_{n,m}\delta\gamma_{n,m}^{-1}\delta) + \#(p \vee q) - (n+m) = \frac{1}{2}(\#(\epsilon q\delta p\epsilon\gamma_{n,m}\delta\gamma_{n,m}^{-1}\delta) + \#(\epsilon p\delta q\epsilon) - 2(n+m))$. Since our above corollary shows that $\#(\epsilon q\delta p\epsilon\gamma_{n,m}\delta\gamma_{n,m}^{-1}\delta) + \#(\epsilon p\delta q\epsilon) \leq 2(n+m)$, the covariance converges.

From our above discussion, we know that the only terms that contribute are those from $S_{NC}^{\delta}(\gamma_{n,m})$.

In the case of both n and m being odd, the second term is zero. In the first term, we can't have any order N^2 or N terms due to parity. Since $\epsilon p\delta q\epsilon$ only has blocks of even cycle length, it must connect $(1,\ldots,n,-n,\ldots,-1)$ with $(n+1,\ldots,n+m,-(n+m),\ldots,-(n+1))$. Since only connected things appear, The order 1 term also only has contributions from $S_{NC}^{\delta}(\gamma_{n,m})$

Thus, $\mathrm{Cov}(\mathrm{Tr}(X^n), \mathrm{Tr}(X^m))$ equals

$$\sum_{\substack{p,q\in\mathcal{P}_2(n+m)\\ \epsilon\in\mathbb{Z}_2^{n+m}\\ \epsilon p\delta q\epsilon\in S_{NC}^{\delta}(\gamma_{n,m})}} \mathrm{Wg}(p,q)\left(\sum_{\epsilon\in\mathbb{Z}_2^{n+m}} N^{\#(\gamma_{n,m}^{-1}\delta\gamma_{n,m}\delta\vee\epsilon p\delta q\epsilon)}\right) + O(\frac{1}{N})$$

$$= \sum_{\substack{\sigma\in S_{NC}^{\delta}(\gamma_{n,m})\\ \sigma \text{ is even}}} \prod_{i=1}^{\#(\sigma)} \mu_{r_i} \sum_{\substack{\epsilon\in\mathbb{Z}_2^{n+m}\\ \epsilon \text{ alternates on the}\\ \text{cycles of } \sigma}} 1 + O\left(\frac{1}{N}\right)$$

$$= \sum_{\sigma\in S_{NC}^{\delta}(\gamma_{n,m})} 2^{\#(\sigma)} \prod_{i=1}^{\#(\sigma)} \mu_{r_i} + O\left(\frac{1}{N}\right)$$

$$= \sum_{\sigma \in S_{NC}^{\delta}(\gamma_{n,m})} \kappa_\sigma + O\left(\frac{1}{N}\right)$$

Rewriting in the notation of Redelmeier in theorem 3.6.17, The elements $\sigma \in S_{NC}^{\delta}(\gamma_{n,m})$ which decomposes into $\pi\delta\pi^{-1}\delta$ such that $\pi \in S_{NC}(n,m)$ correspond to the summand $\sum_{\sigma \in S_{NC}(n,m)} \kappa_\sigma(a_1, \ldots, a_n, a_{n_1}, \ldots, a_{n+m})$ and the σ's such that $\pi \in S_{NC}(n,-m)$ correspond to the summand

$\sum_{\sigma \in S_{NC}(n,-m)} \kappa_\sigma(a_1, \ldots, a_n, a_{n+1}^t, \ldots, a_{n+m}^t)$. Since $X = O + O^t$ is self-adjoint, this is of the desired form. Note that second-order real cumulants are zero.

Corollary 5.4.3. *Let $\{O_i\}_{i=1}^m$ be a collection of independent $N \times N$ Haar distributed orthogonal matrices. $X_m = O_1 + O_1^t + \cdots + O_m + O_m^t$. Then X_m has second order limiting distribution $\alpha_{n,m} = \sum_{\sigma \in S_{NC}^{\delta}(\gamma_{n,m})} m^{\#(\sigma)} \kappa_\sigma$ where κ_n denotes the n'th free cumulant of the arcsine law.*

Chapter 6

Appendix

6.1 Introduction

In this appendix, we will characterize the invariant tensors of the orthogonal group. This is called the tensor first fundamental theorem (FFT) of invariant theory for the orthogonal group. We will prove something stronger. We will provide general criteria for determining the invariant tensors of a matrix subgroup and recover the FFT of the orthogonal group as a special case. The proof is based mainly on [20] and will proceed as follows. We will first prove the tensor FFT for the unitary group. Using this, we characterize invariant tensors as special subalgebras of the mixed tensor algebra $T(V \bigoplus V^*)$. In doing so, we will obtain a criterion for a subalgebra to correspond to the invariant tensor of a group. We will then apply it to the special case of the orthogonal group and obtain its tensor FFT. For more on

invariant theory, one can consult standard references such as [8] or [26].

6.2 Unitary FFT

In this section, we will prove the FFT for the unitary group. The proof will proceed by identifying $T^{k,k}(V)$ with $\mathrm{End}(T^k(V)) = \mathrm{End}(V^{\otimes k})$. Left multiplication of tensors by elements of $U(N)$ will shift to conjugation in the image. This will create a one-to-one correspondence between invariant tensors and elements of the commutator algebra. We then characterize the commutator algebra and obtain the unitary FFT. Note that for $k \neq l$, $T^{k,l}(V)^{U(N)} = \{0\}$. This follows by noting that for all $z \in S^1$, $\mathrm{diag}(z) \in U(N)$. Thus for all $v \in T^{k,l}(V)$, $\mathrm{diag}(z) \cdot v = z^{k-l}v \neq v$ unless $k = l$ or $v = 0$.

Definition 6.2.1. Let $\mathcal{A}$ be an algebra and $\mathcal{B} \subseteq \mathcal{A}$ a subset. We define define the **commutator** of $\mathcal{B}$, $\mathcal{B}'$ to be $\{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{B}\}$.

Definition 6.2.2. Let $\rho : G \to GL(V)$ be a representation. We call the linear span of $\rho(G)$ inside $\mathrm{End}(V)$ the **enveloping algebra** of $\rho(G)$ and denote it by $\rho[\mathbb{C}[G]]$.

Proposition 6.2.3. *Let $G \leq GL(V)$ and define the representation $\rho : G \to GL(End(V^{\otimes k}))$ by the rule $\rho(g) \cdot m := g^{\otimes k} \cdot m \cdot (g^{-1})^{\otimes k}$. Identify $T^{k,k}(V)$ with $End(V^{\otimes k})$ by the rule $e_I \otimes e_J^* \cdot (v) := \langle v, e_J^* \rangle e_I$. Let i denote the defining representation and $i^{k,k}$ the extension to $T^{k,k}(V)$. Then, under this mapping, multiplication by $i^{k,k}(g)$ in $T^{k,k}(V)$ shifts to conjugation by $g^{\otimes k}$ in the image. i.e. $i^{k,k}$ and ρ are equivalent as representations.*

Proof. Fix $g \in G$ and let $v \in V^{\otimes k}$. Then $i^{k,k}(g)e_I \otimes e_J^* \cdot v = ((g^{\otimes k}e_I) \otimes (e_J^* \circ (g^{-1})^{\otimes k})) \cdot v = \langle (g^{-1})^{\otimes k}v, e_J\rangle g^{\otimes k}e_I = g^{\otimes k} \cdot e_I \otimes e_J^* \cdot (g^{-1^{\otimes k}}v)$ □

From this, we immediately obtain the following corollary.

Corollary 6.2.4. *Let $G \leq GL(V)$ and let i denote its defining representation. Let i^k and $i^{k,k}$ denote the extensions to $T^k(V)$ and $T^{k,k}(V)$ respectively. Denote $\mathcal{A} = i^k[\mathbb{C}[G]]$. Then $J \in (T^{k,k})^G$ if and only if there exists $B = (b_{I,J}) \in \mathcal{A}'$ such that $J = \sum_{I,J} b_{I,J}e_I \otimes e_J^*$.*

This follows from the fact that $i^k(g) = g^{\otimes k}$.

Note that generic elements of $\mathrm{End}(V^{\otimes k})$ are of the form $\sum_S m_S$ where $m_{s_l} \in \mathrm{End}(V)$ and $(\sum_S m_S)e_{j_1} \otimes \cdots \otimes e_{j_k} = \sum_S (m_{s_1}e_{j_1}) \otimes \cdots \otimes (m_{s_k}e_{j_k})$. i.e. $\mathrm{End}(V^{\otimes k})$ is naturally isomorphic to $\mathrm{End}(V)^{\otimes k}$.

Now that we have a correspondence between invariant tensors $T^{k,k}(V)^{U(N)}$ and the commutator algebra of $i^k[\mathbb{C}[U(N)]]$, we will characterize both $i^k[\mathbb{C}[U(N)]]$ and its commutant. It will end up being the case that the commutator algebra is the enveloping algebra of the representation of S_k on $T^k(V)$ obtained by permuting the tensors arguments.

Definition 6.2.5.

1. Fix $k \in \mathbb{N}$. Given a multi-index $I : [k] \to [N]$, we define the action of S_k be the rule $\sigma \cdot I(l) = I(\sigma(l))$. Note that if $I = (i_1, \ldots, i_k)$ and $\sigma \cdot I = J = (j_1, \ldots, j_k)$, then $j_l = i_{\sigma^{-1}(l)}$.

2. With this action, we define the representation ρ of S_k on $T^k(V)$ via the rule $\sigma \cdot e_I = e_{\sigma \cdot I}$.

3. We call the elements $(T^k(V))^{S_k}$ the symmetric tensors of $T^k(V)$. Note that the orthogonal projection onto $(T^k(V))^{S_k}$ is given by $\mathrm{Sym}(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} v_{\sigma^{-1}(1)} \otimes \cdots v_{\sigma^{-1}(k)}$

Proposition 6.2.6. *Let ρ denote the above representation of S_k on $T^k(V)$. Under the identification of $T^{k,k}(V)$ with $End(V^{\otimes k})$, the preimage of $\rho(\sigma)$ is the tensor $\sum_I e_{\sigma \cdot I} \otimes e_I^*$.*

Proof. Fix $e_J \in T^k(V)$. Then $(\sum_I e_{\sigma\cdot I} \otimes e_I^*) \cdot e_J = \sum_I \langle e_J, e_I\rangle e_{\sigma\cdot I} = e_{\sigma\cdot J} = \rho(\sigma)\cdot e_J$ □

Lemma 6.2.7. *Let $m_S = m_1 \otimes \cdots \otimes m_k \in End(V^{\otimes k})$. If m_S has the matrix representation $\{m_{I,J}\}$ with respect to the basis $\{e_I\}$, Then $m_{\sigma\cdot S}$ has the matrix representation $\{m_{\sigma^{-1}\cdot I,\sigma^{-1}\cdot J}\}$*

Proof. Fix e_J. For each $1 \le s \le k$, let $\{m_{i,j}^{(s)}\}$ denote the matrix representation of m_s with respect to the basis $\{e_i\}$ Then $m_{\sigma\cdot S}e_J = m_{(\sigma^{-1}(1))} \otimes \cdots \otimes m_{(\sigma^{-1}(k))}e_J = \sum_I m_{i_1,j_1}^{(\sigma^{-1}(1))} \cdots m_{i_k,j_k}^{(\sigma^{-1}(k))} e_J$. $\sigma^{-1}(s) = 1 \iff \sigma(1) = s$, thus, after rearranging the m's, our sum becomes $\sum_I m^{(1)}_{i_{\sigma(1)},j_{\sigma(1)}} \cdots m^{(k)}_{i_{\sigma(k)},j_{\sigma(k)}} e_J = \sum_I m_{\sigma^{-1}\cdot I,\sigma^{-1}\cdot J}e_J$ □

Proposition 6.2.8. *Let ρ be as before. Then $\rho[\mathbb{C}[S_k]]' = \{\sum_S m_S : \forall \sigma \in S_K \ \sum_S m_S = \sum_S m_{\sigma\cdot S}\}$. i.e. $\rho[\mathbb{C}[S_k]]'$ is equal to the symmetric tensors inside of $End(V)^{\otimes k}$.*

Proof. We must check the condition that $\rho(\sigma)\sum_S m_S \rho(\sigma^{-1}) = \sum_S m_S$ for all $\sigma \in S_k$. Write $m_s = \{m_{I,J}^{(S)}\}$ as a matrix with respect to the basis $\{e_J\}$. Then $m_{\sigma\cdot S}$ will have the matrix representation $\{m^{(S)}_{\sigma^{-1}\cdot I,\sigma^{-1}\cdot J}\}$.

Then $\rho(\sigma)\sum_S m_S\rho(\sigma^{-1})e_J = \rho(\sigma)\sum_S m_S e_{\sigma^{-1}\cdot J} = \rho(\sigma)\sum_{I,S} m^s_{I,\sigma^{-1}\cdot J}e_I = \sum_{S,I} m^S_{I,\sigma^{-1}\cdot J}e_{\sigma\cdot I} = \sum_{I,S} m^S_{\sigma^{-1}\cdot I,\sigma^{-1}\cdot J}e_I = \sum_S m_{\sigma\cdot S}e_J$. Thus, as σ was arbitrary, the result follows.

□

We now wish to show that $i^k[\mathbb{C}[U(N)]]$ is equal to the symmetric tensors inside of $\mathrm{End}(V)^{\otimes k}$. For this, we will first show that the symmetric tensors can be written in a special form.

Lemma 6.2.9 (Polarization for symmetric tensors)**.** *Let V be any vector space and let $v_1, \ldots, v_k \in V$. Let $Sym(v_1 \otimes \cdots \otimes v_k) := \frac{1}{k!}\sum_{\sigma\in S_k} v_{\sigma^{-1}(1)} \otimes \cdot \otimes v_{\sigma^{-1}(k)}$ denote the orthogonal projection onto the subspace of symmetric tensors. Then $Sym(v_1\otimes\cdots\otimes v_k) = \frac{1}{k!2^k}\sum_{\epsilon_1,\ldots,\epsilon_k=\pm1}(\prod_{j=1}^k \epsilon_j)(\epsilon_1 v_1+\epsilon_2 v_2+\cdots+\epsilon_k v_k)^{\otimes k}$. Thus, the set all symmetric tensors in $V^{\otimes k}$ is equal to $span\{v^{\otimes k} : v \in V\}$*

Proof. Expanding out the sum on the right-hand side:

$$\sum_{\epsilon_1,\ldots,\epsilon_k=\pm1}\left[\prod_{j=1}^k \epsilon_j\right](\epsilon_1 v_1+\cdots+\epsilon_k v_k)^{\otimes k} = \sum_{\epsilon_1,\ldots,\epsilon_k=\pm1}\left[\prod_{j=1}^k \epsilon_j\right]\sum_{I:[k]\to[k]}\epsilon_I v_I =$$

$$\sum_{\epsilon_1,\ldots,\epsilon_k=\pm1}\left(\left[\prod_{j=1}^k \epsilon_j\right]\sum_{\substack{I:[k]\to[k]\\ I([k])=[k]}} v_I + \left[\prod_{j=1}^k \epsilon_j\right]\sum_{\substack{I:[k]\to[k]\\ I([k])\neq[k]}}\epsilon_I v_I\right)$$

$$= 2^k \sum_{s \in S_k} v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(k)} + \sum_{\substack{I:[k]\to[k] \\ I([k]) \neq [k]}} \left[\sum_{\epsilon_1,\ldots,\epsilon_k = \pm 1} \prod_{j=1}^{k} \epsilon_j \epsilon_I v_I \right]$$

Notice that in the case where I is not a bijection, there is at least one ϵ_i that does not appear in the product ϵ_I. Thus, we can rewrite our sum as

$$\sum_{\substack{I:[k]\to[k] \\ I([k]) \neq [k]}} \sum_{\substack{\epsilon_i = \pm 1 \\ i \notin I([k])}} \epsilon_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{k} \sum_{\epsilon_j \pm 1} \prod_{\substack{j=1 \\ j \neq i}} \epsilon_j \epsilon_I v_I \right)$$

Since the summand inside is independent of ϵ_i, it is equal to zero thus the result follows.

□

We will now show that $i^k[\mathbb{C}[(N)]] = \rho[\mathbb{C}[S_k]]'$ and $\rho[\mathbb{C}[S_k]] = i^k[\mathbb{C}[U(N)]]'$. For this, we recall the double-commutant theorem for finite-dimensional $*$-algebras.

Definition 6.2.10. A **$*$-algebra** is an algebra $\mathcal{A}$ over $\mathbb{C}$ with an map $* : \mathcal{A} \to \mathcal{A}$ satisfying for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$

1. $(a^*)^* = a$
2. $(ab)^* = b^* a^*$
3. $(a + b)^* = a^* + b^*$

4. $(\lambda a)^* = \overline{\lambda} a^*$

Theorem 6.2.11 ($*$-algebra double commutant theorem)**.** *Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $\mathcal{A} \subseteq End(V)$ be a unital $*$-algebra. Then $\mathcal{B} = \mathcal{A}'$ is also a $*$-algebra and $\mathcal{B}' = \mathcal{A}$.*

Since $(u^{\otimes k})^* = (u^{-1})^{\otimes k}$ and $\rho(\sigma)^* = \rho(\sigma^{-1})$ for all $u \in U(N)$ and all $\sigma \in S_k$, it follows that $i^k[\mathbb{C}[U(N)]]$ and $\rho[\mathbb{C}[S_k]]$ are finite-dimensional $*$-algebras.

Lemma 6.2.12. *Let $\mathcal{D} := span\{D^{\otimes k} : D \text{ is a diagonal matrix}\}$. Then $\mathcal{D} \subseteq$ $\imath^k[\mathbb{C}[U(N)]]$*

Proof. To prove this, we shall show that $i^k[\mathbb{C}[U(N)]]' \subset \mathcal{D}'$.

Let $D = (d_{I,J}) \in \mathcal{D}, B = (b_{I,J}) \in \mathcal{D}'$. Note that $d_{I,J} = 0$ unless $I = J$ and $d_{I,I} = d_{\sigma \cdot I, \sigma \cdot I}$ for all choices of σ. Fix a basis element e_J. Then $DBe_J = BDe_J \iff \sum_I d_{I,I} b_{I,J} e_I = \sum_I b_{I,J} d_{J,J} e_I \iff d_{I,I} b_{I,J} = d_{J,J} b_{I,J}$ for all choices I, J. Unless there exists σ such that $I = \sigma \cdot J$, we can always choose a diagonal matrix such that $d_{I,I} \neq d_{J,J}$, thus $(b_{I,J}) \in \mathcal{D}'$ iff $b_{I,J} \neq 0$ implies that there exists σ such that $I = \sigma \cdot J$ Now, let $D^{\otimes k} \in \mathcal{D} \cap i^k[U(N)]$. i.e. $D = \text{diag}(z_1, \ldots, z_n)$ such that $|z_i| = 1$ for all i. By the same argument as above, if there does not exist $\sigma \in S_K$ such that $I = \sigma \cdot J$ we can always find $D^{\otimes k} \in \mathcal{D} \cap i^k[U(N)]$ such that $d_{I,I} \neq d_{J,J}$. Thus $(\mathcal{D} \cap i^k[U(N)])' = \mathcal{D}'$. This gives us, $(i^k[\mathbb{C}[U(N)]])' \subseteq (\mathcal{D} \cap i^k[U(N)])' = \mathcal{D}'$, which by the double commutant theorem implies $\mathcal{D} \subseteq i^k[\mathbb{C}[U(N)]]$

□

Corollary 6.2.13. *span*$\{u^{\otimes k} : g \in U(N)\} = span\{m^{\otimes k} : m \in End(V)\}$. *Thus* $\mathcal{A} = \mathcal{B}'$ *and* $\mathcal{B} = \mathcal{A}'$

Proof. Let $m \in \mathrm{End}(V)$. Then, by singular value decomposition, we can write $m = U\Sigma V$ where U, V are unitary matrices and Σ is a diagonal matrix. Thus $m^{\otimes k} = U^{\otimes k}\Sigma^{\otimes k}V^{\otimes k}$, since $U^{\otimes k}, \Sigma^{\otimes k}, V^{\otimes k}$ are all contained $i^k[\mathbb{C}[U(N)]]$, so is $m^{\otimes k}$, thus the result follows. □

6.3 The Tensor FFT

Let V be an inner product space and let $T := \bigoplus_{k,l\geq p} T^{k,l}(V)$ denote the mixed tensor algebra over V. In this section, we will determine general criteria for a subalgebra of T to be equal to the algebra of invariant tensors for some subgroup G of $U(N)$. We will then use this to determine the invariant tensors of $O(N)$. This section will follow [20].

Recall that since V is an inner product space, we have a natural identification of V with V^* via the mapping $v \to v^*$ where v^* is the linear functional defined by the rule $v^*(w) := \langle w, v\rangle$ for all $w \in V$. We can define an inner product on V^* via $\langle v^*, w^*\rangle := \langle w, v\rangle$. Both the $*$-mapping and the inner product extent to T via the rules $(e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^*_{j_1} \otimes \cdots e^*_{j_l})^* = e_{j_l} \otimes \cdots \otimes e_{j_1} \otimes e^*_{i_k} \otimes \cdots \otimes e^*_{i_1}$ and

$\langle e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^*_{j_1} \otimes \cdots \otimes e_{j_l}, e_{i'_1} \otimes \cdots \otimes e_{i'_{k'}} \otimes e^*_{j'_1} \otimes \cdots \otimes e^*_{j'_{l'}}\rangle = \delta_{k,k'}\delta_{l,l'} \prod\langle e_{i_l}, e_{i'_l}\rangle \prod\langle e^*_{j_l}, e^*_{j'_l}\rangle$

We remark that after identification of $T^{1,1}(V)$ with $\mathrm{End}(V)$, $*$ becomes

the adjoint map and the inner product the Hilbert-Schmidt inner product $\langle A, B\rangle_{HS} = \mathrm{Tr}(AB^*)$.

Definition 6.3.1. Let $1 \leq i \leq k$ $1 \leq j \leq l$. We define the **contraction** $C^{k,l}_{s,t} : T^{k,l}(V) \to T^{k-1,l-1}(V)$ by the rule

$$C^{k,l}_{s,t}(e_I \otimes e^*_J) = e^*_{j_t}(e_{i_s})e_{i_1} \otimes \cdots \otimes \widehat{e}_{i_s} \otimes \cdots \otimes e_{i_k} \otimes e^*_{j_1} \otimes \cdots \otimes \widehat{e^*}_{j_t} \otimes \cdots \otimes e^*_{j_l}$$

Remark 6.3.2. The inner product of two elements $v, w \in T^{k,l}(V)$ can be realized as an appropriate series of contractions of $v \otimes w^*$. For example, $\langle v, w\rangle = C^{1,1}_{1,1} \circ C^{2,2}_{2,1} \circ \cdots \circ C^{k+l,k+l}_{k+1,1}(v \otimes w^*)$.

Definition 6.3.3. Let T be as before and let $\mathcal{A}$ be a subalgebra of T.

1. Let $\mathcal{A}^{k,l} = \mathcal{A} \cap T^{k,l}(V)$. We say that $\mathcal{A}$ is **graded** if $\mathcal{A} = \bigoplus_{k,l\geq 0} \mathcal{A}^{k,l}$

2. We say that $\mathcal{A}$ is **nondegenerate** is there exists no proper subspace W of V such that $\mathcal{A} \subseteq \bigoplus_{k,l\geq 0} T^{k,l}(W)$

3. Given $v \in T$, we say that $w \in T$ is a **mutation** of V, if w can be obtained from v by permuting its arguments. i.e if $v = \sum \alpha_{I,J} e_I \otimes e^*_J$, then there exists $(\sigma, \pi) \in S_k \times S_l$ such that $w = \sum_{I,J} \alpha_{I,J} e_{\sigma\cdot I} \otimes e^*_{\pi\cdot J}$. We say that $\mathcal{A}$ is closed under mutations if $v \in \mathcal{A}$ implies that all mutations of v are also contained inside of $\mathcal{A}$.

Proposition 6.3.4. *Let $\mathcal{A}$ be a nondegenerate, graded, contraction-closed $*$-subalgebra of T. Then*

1. $\sum_{t=1}^{N} e_t \otimes e_t^* \in \mathcal{A}$

2. *$\mathcal{A}$ is closed under mutations*

Proof.

1. Let $\mathcal{A}$ be as above. Then, after identification $\mathcal{A}^{1,1}$ as a subalgebra of $\mathrm{End}(V)$, it is a finite-dimensional $*$-algebra and thus contains a unit e. We shall show that $e = \sum_{t=1}^{N} e_t \otimes e_t^*$. Let $W = eV$. i.e. W is the image of V under e. Without loss of generality, we may assume that $W = \mathrm{span}\{e_1, \ldots, e_m\}$ for some $m \leq N$. Assume for contradiction that $m < N$. Since $\mathcal{A}$ is nondegenerate, there exists a tensor $v \in \mathcal{A}$ that contains e_l in a component for some $l > m$. After applying an appropriate series of contractions to $v \otimes v^*$, we obtain a tensor $v' \in \mathcal{A}^{1,1}$ that contains e_l in a component, a contradiction, thus $\sum_{t=1}^{N} e_t \otimes e_t^* \in \mathcal{A}^{1,1}$

2. To show that $\mathcal{A}$ is closed under taking mutations, it is enough to show that each grading is closed under mutations. Fix $k, l \in \mathbb{N}$. and let $v = \sum_{I,J} \alpha_{I,J} e_I \otimes e_J^* \in \mathcal{A}$. Then as $\sum_{t=1}^{N} e_t \otimes e_t^* \in \mathcal{A}$ and as $\mathcal{A}$ is closed under contractions,

$$C_{s,l+1}^{k+1,l+1}\left((\sum_{I,J} \alpha_{I,J} e_I \otimes e_J^*) \cdot (\sum_{t} e_t \otimes e_t^*)\right)$$

$$= C^{k+1,l+1}_{s,l+1} \left(\sum_t \sum_{I,J} \alpha_{I,J} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e_t \otimes e^*_{j_1} \otimes \cdots \otimes e^*_{j_l} \otimes e^*_t \right)$$

$$= \sum_t e^*_t(e_{i_s}) \sum_{I,J} \alpha_{I,J} e_{i_1} \otimes \cdots \otimes \widehat{e_{i_s}} \otimes \cdots \otimes e_{i_k} \otimes e_t \otimes e^*_{j_1} \otimes \cdots \otimes e^*_{j_l}$$

$$= \sum_{I,J} \alpha_{I,J} e_{i_1} \otimes \cdots \otimes \widehat{e_{i_s}} \otimes \cdots \otimes e_{i_k} \otimes e_{i_s} \otimes e^*_{j_1} \otimes \cdots \otimes e^*_{j_k}$$

Thus, the covariant arguments are closed under the permutations $(s, s+1, \ldots, k)$. These clearly generate S_k, however. A similar argument shows we can permute contravariant arguments, implying the result.

□

Notation 6.3.5. Given a subset S of T, we let G^S denote the largest subgroup of $U(N)$ that leaves S invariant. Similarly, given a subset G of $U(N)$, we let T^G denote the set of all tensors invariant under G

Recall from the previous section that the invariant tensors of $U(N)$ in $T^{k,k}(V)$ can be written as a linear combination of the tensors $\sum_I e_{\sigma \cdot I} \otimes e^*_I$ where $\sigma \in S_k$. All such tensors are mutations of $\left(\sum_{s=1}^N e_s \otimes e^*_s\right)^{\otimes k}$. These tensor are closed under the $*$-operation as $\left(\sum_I e_{\sigma \cdot I} \otimes e^*_I\right)^* = \sum_I e_{\sigma^{-1} \cdot I} \otimes e^*_I$. Further, if we contract an invariant tensor of $T^{k,k}(V)$, it's easy to see that

we obtain an invariant tensor in $T^{k-1,k-1}(V)$. Thus $T^{U(N)}$ is the smallest contraction closed, graded $*$-subalgebra of T containing $\sum_{s=1}^{N} e_s \otimes e_s^*$. More generally, for any subgroup G, T^G is a contraction-closed as $v^* \circ g^{-1}(g \cdot w) = v^*(w)$. It is graded as the action of G preserves gradings, and it is nondegenerate as for any subgroup G, $\sum_{s=1}^{N} e_S \otimes e_S^*$ is an invariant tensor as $T^{U(N)} \subseteq T^G$. It is also clear the multiplying, taking linear combinations and applying the $*$-operation all preserves invariant tensors, thus, T^G is a graded, nondegenerate, contraction closed $*$-subalgebra of T. We will now show that the other direction is also true.

Proposition 6.3.6. *Let $N \geq 1$ and let $\mathcal{A} \subset T$. Then there exists a subgroup $G \leq U(N)$ such that $\mathcal{A} = T^G$ if and only if $\mathcal{A}$ is a nondegenerate, graded contraction-closed subalgebra of T.*

Proof. As remarked above, if $\mathcal{A} = T^G$, then it satisfies the stated conditions.

Let $\mathcal{A}$ be a graded, nondegenerate, contraction closed $*$-subalgebra of T and let $G = U(N)^{\mathcal{A}}$. We shall show that $T^G = \mathcal{A}$.

Let $X = G/U(N)$ be the left coset space of G. Since $U(N)$ is compact, X will be compact in the quotient topology. Fix $v \in \mathcal{A}$, $w \in T$ and define $\varphi_{v,x} : X \to \mathbb{C}$ via the rule $\varphi_{v,w}(UG) := \langle U \cdot v, w \rangle$. Note that $\varphi_{v,w}$ is well defined as if $UG = U'G$, then $U^{-1}U' \in G$ which as $\mathcal{A} \subseteq T^G$ implies $\langle U' \cdot v, w \rangle = \langle U^{-1}U' \cdot v, U^{-1} \cdot w \rangle = \langle U \cdot v, w \rangle$

We will show that $\{\varphi_{v,w} : v \in \mathcal{A}, w \in T\}$ satisfies the conditions of the Stone-Weierstrass theorem. It is a $*$-subalgebra of $C(X)$ as $\varphi_{v,w} \cdot \varphi_{v',w'} = \varphi_{v \otimes v', w \otimes w'}$ and $\overline{\varphi_{v,w}} = \varphi_{v^*,w^*}$, thus we only need to check that it separates

points. This is immediate, however, as if $UG \neq U'G$, then there exists at least one $v \in \mathcal{A}$ such that $U \cdot v \neq U' \cdot v$. Thus, we can find $w \in T$ such that $\langle U \cdot v, w \rangle \neq \langle U'v, w \rangle$ and such that both quantities are nonzero. Then, by Stone-Weierstrass theorem, $\overline{\{\varphi_{v,w} : v \in A, w \in T\}} = C(X)$ with respect to the supremum norm.

We will now show that $\mathcal{A} = T^G$. Assume for contradiction that there exists $z \in T^G/\mathcal{A}$. Without loss of generality, we may assume that $z \in \mathcal{A}^\perp$. Then $UX \to \langle U \cdot z, z \rangle$ is a well-defined continuous function over X. Thus, as $\langle z, U \cdot z \rangle = \overline{\langle U \cdot z, z \rangle}$, $\int_{U(N)} \langle U \cdot z, z \rangle \langle z, U \cdot z \rangle d\mu(U) > 0$. Since we can uniformly approximate continuous functions over $C(X)$ be elements of $\{\varphi_{v,w}\}$, there exists $v \in A, w \in T$ such that $\int_{U(N)} \langle U \cdot v, w \rangle \langle z, U \cdot z \rangle > 0$

Since integrating over U^* instead of U does not change the value of the integral, we can shift U to the other side of the inner product, yielding.

$\int_{U(N)} \langle U \cdot v, w \rangle \langle z, U \cdot z \rangle d\mu(U) = \int_{U(N)} \langle v, U \cdot w \rangle \langle U \cdot z, z \rangle d\mu(U) = \langle \int_{U(N)} \langle v, U \cdot w \rangle U \cdot z, v \rangle > 0$

We claim that $\int_{U(N)} \langle v, U \cdot w \rangle U \cdot z d\mu(U) \in \mathcal{A}$. Recall that the inner product $\langle v, U \cdot w \rangle$ can be obtained by performing an appropriate series of contractions on $v \otimes (U \cdot w)^*$. Let C denote the series of contractions such that $C\left(v \otimes (U \cdot w)^* \otimes (U \cdot z)\right) = \langle v, U \cdot w \rangle U \cdot z$ and let $w = \int_{U(N)} (U \cdot w)^* \otimes (U \cdot z) d\mu(U)$. Then $w \in T^{U(N)}$ and $\int_{U(N)} \langle v, U \cdot w \rangle U \cdot z d\mu(U) = \int_{U(N)} C\left(v \otimes (U \cdot w)^* \otimes (U \cdot z)\right) = z \otimes w$. Since $z \in \mathcal{A}$ and since $T^{U(N)} \subseteq \mathcal{A}$, it follows that $z \otimes w \in \mathcal{A}$, a contraction. Thus $T^G = \mathcal{A}$

□

By analyzing the proof, we obtain the following corollaries.

Corollary 6.3.7. *Let $\mathcal{A}$ be any nondegenerate contraction-closed, graded $*$-subalgebra of $\mathcal{A}$ and let $G = U(N)^{\mathcal{A}}$. Then $T^G = \mathcal{A}$*

Corollary 6.3.8. *Let $S \subseteq T$ and let $G = U(N)^S$. Then T^G is the smallest nondegenerate, contraction closed, graded $*$-subalgebra of T containing S.*

We now specialize in the case of the orthogonal group. Let $f = \sum_{s=1}^{N} e_i \otimes e_i$. Recall from chapter two that f is invariant under the action of $O(N)$. Thus by our above corollary, $T^{O(N)}$ is the smallest contraction-closed graded $*$-subalgebra generated by $\{f\}$. Specializing to the invariant tensors of $\bigoplus_{k\geq 0} T^K(V)$, we obtain precisely the pairing tensors of chapter two as all pairing tensors can be realized as mutations of the tensors $\left(\sum_{s=1}^{N} e_s \otimes e_s\right)^{\otimes k}$. As an example, take the case $k = 2$. Then

$$\left(\sum_{s=1}^{N} e_s \otimes e_s\right)^{\otimes 2} = \sum_{s_1,s_2} e_{s_1} \otimes e_{s_1} \otimes e_{s_2} \otimes e_{s_2} = \sum_{I} \left\langle \begin{matrix} I \\ p \end{matrix} \right\rangle e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}$$

Where $p = (1,2)(3,4)$. If we permute the second and third indices (act on our tensor via the transposition $\tau = (2,3)$), we obtain the tensor

$$\sum_{s_1,s_2} e_{s_1} \otimes e_{s_2} \otimes e_{s_1} \otimes e_{s_2} = \sum_{I} \left\langle \begin{matrix} I \\ p' \end{matrix} \right\rangle e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}$$

Where $p' = (1,3)(2,4)$.

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