On the Dynamical and Algebraic Properties of Bratteli Diagrams

by

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Abstract

A variety of dynamical systems, in topological and measurable settings, can be realized as Bratteli-Vershik systems. This document introduces the basic notions of Bratteli diagrams and provides detailed descriptions of Bratteli-Vershik models in both abstract and concrete settings. We also introduce the algebraic properties of Bratteli diagrams that interact with the dynamical realizations. In particular, we explore the dynamical and algebraic features of Bratteli diagrams derived from the action of an irrational rotation on the circle with a distinguished parameter that further cuts the partitions generated by the rotation.

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Chapter 1

Introduction

1.1 Motivation

Bratteli diagrams are a class of infinite graphs that exhibit both dynamical and algebraic properties. In fact, these properties have been shown to be intertwined. In 1972 the Norwegian mathematician Ola Bratteli introduced the notion of a Bratteli diagram in [3] as a model for approximately finite dimensional C*-algebras (AF-algebras). Four years later it was presented in [11] by George Elliott that, given a Bratteli diagram, one could compute an algebraic invariant that classified the AF-algebra it modeled up to (C*-)isomorphism. This invariant is called a dimension group.

In the field of dynamics, Anatole Vershik published the papers [23] and [24] in 1981 and 1982 respectively. In these papers, Vershik developed what is called an adic transformation to model ergodic automorphisms acting on a standard measure space. The adic model is a dynamical system consisting of a transformation acting on the path space of a Bratteli diagram. The notion of an adic transformation has a topological analogue. This was developed by Richard Herman, Ian Putnam, and

Christian Skau (HPS) in [17], which was published in 1992. They showed that every minimal $\mathbb{Z}$-action on a Cantor space is topologically conjugate to a Bratteli-Vershik system. The techniques employed there have been extended to aperiodic homeomorphisms acting on a Cantor space by Konstantin Medynets in [18] and Borel spaces by Sergey Bezuglyi, Anthony Dooley, and J. Kwiatkowski in [1].

It is the work of HPS that was published in 1992 that began to bridge the dynamical and algebraic properties of Bratteli diagrams. Over the course of that decade, Thierry Giordano, Ian Putnam, and Christian Skau furthered that work in [12] and [13], which were published in 1995 and 1999 respectively. In the latter paper they showed that the dimension group classified a Bratteli-Vershik system up to strong orbit equivalence.

This is only a sliver of the development that has been made in the theory of dynamical systems using the Bratteli-Vershik model. For a broad tour of Bratteli diagrams, we recommend the surveys by Sergey Bezuglyi and Olena Karpel ([2]) and Fabien Durand ([6]).

This thesis introduces and expounds on the basic applications of Bratteli diagrams. It begins by presenting the Bratteli-Vershik system in a topological setting and covers the details of the proof of the conjugacy result in [17]. The dynamical tower model detailed there has found uses in other areas of dynamics. In 2017, Michael Bromberg and Corinna Ulcigrai proved a temporal central limit theorem for irrational rotations in [4] using techniques that are analogous to those found in [17]. After covering the basics of Bratteli diagrams, this thesis studies the constructions found in [4] under a topological, Bratteli diagram-centric lens.

1.2 Content of Thesis

Below is a break down of the mathematical content of this thesis by chapter.

- Chapter 3 introduces Bratteli diagrams, constructs a clopen topology on its path space, and ultimately presents the Bratteli-Vershik dynamical system.
- Chapter 4 details the conjugacy result between a Cantor minimal system and a Bratteli-Vershik system achieved in [17].
- Chapter 5 presents a Bratteli model for irrational rotations on the circle from the work of Michael Bromberg and Corinna Ulcigrai in [4].
- Chapter 6 introduces dimension groups and computes examples for instances of the Bratteli model derived in Chapter 5.

Chapter 2

Literature Review

Most of the relevant papers have been referenced in Section 1.1 of the introduction. We take the opportunity here to mention a handful of papers that are unrelated to dynamics, but nonetheless influenced chapter 6 of this thesis.

While dimension groups were pioneered by George Elliott in [11], their geometric and algebraic properties were uncovered largely by Edward Effros and Chao-Liang Shen between 1978 and 1981 ([9], [8], [16], [10]). Although, they were primarily interested in an object called a Riesz group. A Riesz group is an unperforated ordered group with something called the Riesz interpolation property (see any of the papers cited above for a proper definition). It is not difficult to show that all dimension groups are Riesz groups and for some time it was believed that the latter was a larger class of algebraic objects. We will see in chapter 6 that the definition of a dimension group (Definition 6.4) is less abstract than that of a Riesz group. The nontrivial converse was proved in collaboration with David Handelman in [16].

We make special mention of the following theorem in [9].

Theorem 2.1 (Effros and Shen 1980)**.**

Suppose γ is an irrational number and has continued fraction expansion $[c_0; c_1, c_2, \ldots]$. Let $C_n = \begin{pmatrix} c_{n-1} & 1 \\ 1 & 0 \end{pmatrix}$. Then, the inductive limit $\varinjlim(\mathbb{Z}^2, C_n)$ is order isomorphic to $\mathbb{Z}^2$ with positive cone

$P_\gamma = \{v \in \mathbb{Z}^2 \mid v \cdot (\gamma, 1) > 0\} \cup \{(0,0)\}$.

Ignoring the technical terms that will be discussed in chapter 6, this theorem captures the shape of the positive elements of a rank 2 dimension group. Namely, the elements are the lattice points of a half-space. In [10], Effros and Shen deduced that the positive elements of limit groups of finite rank are either the lattice points of a half-space or of the intersection of half-spaces.

In chapter 6, we will look at examples of rank 3 dimension groups. The theorems and proofs of Effros and Shen guided most of the prose therein, even though we do not provide a treatment of Riesz groups explicitly.

Chapter 3

Bratteli Diagrams

In this chapter we introduce the notion of a Bratteli diagram and its features. Our goal is to define a transformation on the set of infinite paths in the diagram. To do so, we will develop the necessary topological structure. We begin with a list of preliminary definitions and facts.

3.1 The Basics of Bratteli Diagrams

Definition 3.1.

A **Bratteli diagram** is an infinite graph $B = (V, E)$ such that the vertex set V and the edge set E can be decomposed into a countable union of disjoint finite sets $V = \cup_{i \geq 0} V_i$ and $E = \cup_{i \geq 0} E_i$ with the following properties:

(i) $V_0 = \{v_0\}$ is a single point set

(ii) there exist maps $r, s : E \to V$, called the range and source maps respectively, such that $r(E_i) = V_{i+1}$ and $s(E_i) = V_i$ for $i \geq 0$

(iii) for all $v \in V \setminus V_0$, $r^{-1}(v) \neq \emptyset$ and for all $v \in V$, $s^{-1}(v) \neq \emptyset$

The last condition means that given any vertex in V_i in the graph, there is an edge joining it to a vertex in V_{i+1}, and, with the exception of v_0, there is an edge joining it to a vertex in V_{i-1}. We call the set V_i the i-th **level** of a Bratteli diagram B. Below is an example of a Bratteli diagram drawn up to V_3 with $V_1 = \{a, b\}$, $V_2 = \{c, d, e\}$, and $V_3 = \{f, g\}$.

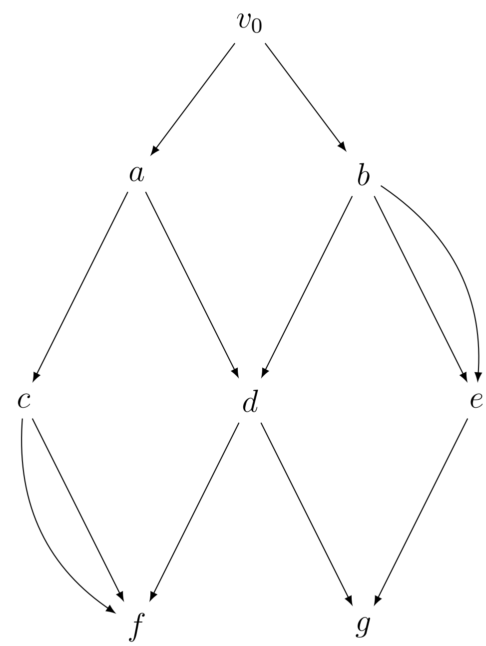


Figure 3.1: first few levels of a Bratteli diagram

We can extend the range and source maps to sets of finite paths. Given a Bratteli diagram B and non-negative integers k and l where $k < l$, let $P_{k,l}$ denote the set of all paths from the k-th level to the l-th level. In symbols,

$$P_{k,l} = \{(e_k, \dots, e_{l-1}) \mid e_i \in E_i \ \text{ for } \ k \leq i \leq l-1 \ \text{ and } \ r(e_i) = s(e_{i+1}) \ \text{ for } \ k \leq i \leq l-2\}.$$

We define $r : P_{k,l} \to V_l$ and $s : P_{k,l} \to V_k$ by $r((e_k, \ldots, e_{l-1})) = r(e_{l-1})$ and $s((e_k, \ldots, e_{l-1})) = s(e_k)$ respectively. The set $P_{k,k+1}$ can be identified with E_k.

For a fixed k and l the number of paths in $P_{k,l}$ can be expressed using matrices. First we encode the number of edges between adjacent levels in a single matrix. The i-th **incidence matrix** $F_i = (f^{(i)}_{v,w})$, $i \geq 0$, is a $|V_{i+1}| \times |V_i|$ matrix such that $f^{(i)}_{v,w} = \#\{e \in E_i \mid r(e) = v, s(e) = w\}$ for $v \in V_{i+1}$ and $w \in V_i$. The entries of this matrix are the number of edges between pairs of vertices $(v, w) \in V_{i+1} \times V_i$. The paths in $P_{k,l}$ can be sorted according to their range and source images. The number of finite paths between a pair of vertices, $(v, w) \in V_l \times V_k$, are then the entries in the product $F_{l-1}F_{l-2} \cdots F_k$.

Given $w \in V_i$, of particular interest is the number of paths from v_0 to w. We will refer to that number as the **height** of w and denote it $h^{(i)}_w$. For all $i \geq 0$

$$h^{(i+1)}_v = \sum_{w \in V_i} f^n_{v,w} h^{(i)}_w \text{ and } h^{(i+1)} = F_i h^{(i)} \text{ where } h^{(i)} = (h^{(i)}_w)_{w \in V_i}.$$

Definition 3.2.

Given a Bratteli Diagram B and a sequence of integers $m_0 = 0 < m_1 < m_2 < \cdots$, the **telescoping** of B to $(m_i)_{i \geq 0}$ is the Bratteli Diagram $B' = (V', E')$ where for $i \geq 0$, we have $V'_i = V_{m_i}$, $E'_i = P_{m_i, m_{i+1}}$, and r, s are the extended range and source maps described above.

The i-th incidence matrix is then the product $F_{m_{i+1}-1} \cdots F_{m_i}$. It can be verified that a telescoping is itself a Bratteli diagram. The two graphs below are telescopings of the Bratteli diagram in Figure 3.1. They are drawn up to the second level. The diagram on the left corresponds to the subsequence $(0, 2, 3, \ldots)$ and the diagram on the right corresponds to the subsequence $(0, 1, 3, \ldots)$. Going forward, we will abuse terminology and use the term telescoping interchangeably to mean either a Bratteli diagram or its subsequence.

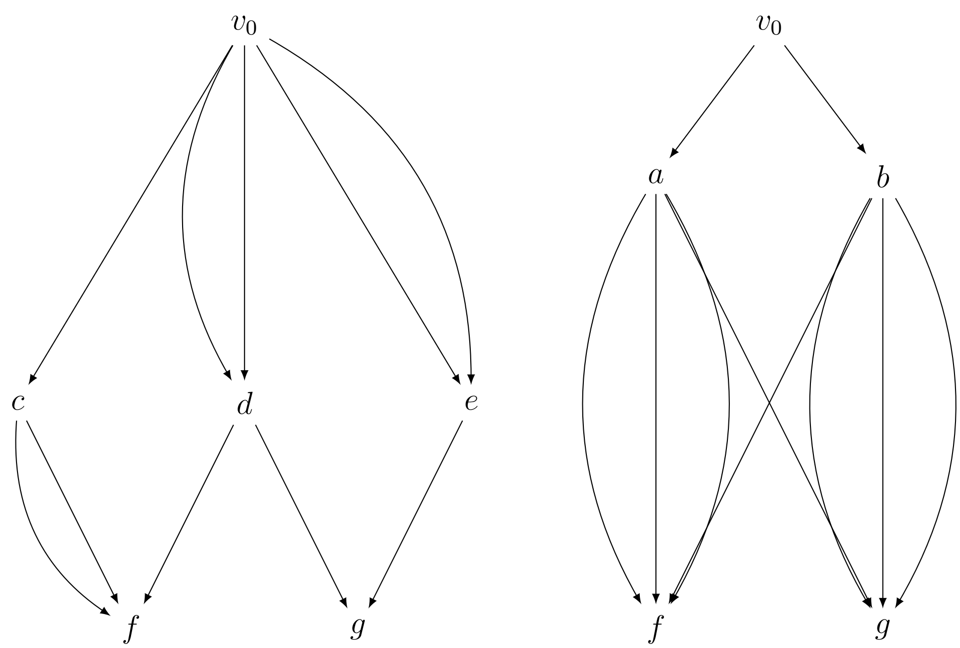


Figure 3.2: telescopings $(0, 2, 3, \ldots)$ (left) and $(0, 1, 3, \ldots)$ (right)

The matrix

$$F_0^{'} = F_1 F_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

is the zeroth incidence matrix for the Bratteli diagram corresponding to the telescoping $(0, 2, 3, \ldots)$. The matrix

$$F_2^{'} = F_2 F_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

is the second incidence matrix of the telescoping $(0, 1, 3, \ldots)$.

Definition 3.3.

A Bratteli diagram B is said to be **simple** if there exists a telescoping $B^{'}$ such that all the entries of the incidence matrices at each level of $B^{'}$ are non-zero.

This means that for any pair of vertices $(v, w) \in V_{i+1}^{'} \times V_i^{'}$ there exists at least one edge between them. Simple Bratteli diagrams are of interest because, once topologized, the path space of such diagrams has no isolated points. This is a key feature of Cantor spaces and will be verified once the topology is established.

Definition 3.4.

A Bratteli diagram B is said to be **ordered** if there is a partial ordering $>$ on E such that edges $e, e^{'} \in E$ are comparable if and only if $r(e) = r(e^{'})$.

In other words, for all $v \in V \setminus V_0$, we have an ordering on $r^{-1}(v)$. By an **ordered Bratteli diagram** we mean a Bratteli diagram B together with an ordering $>$ and we denote it by $(B, >)$. Technically the ordering $>$ is a collection of orderings $>_v$ for each $v \in V \setminus V_0$.

Figure 3.3 illustrates an ordering on the edges between the first few levels of a Bratteli diagram. The non-negative integer label denotes the placement of an edge among the edges that share its range. An edge e with label i is less than an edge f with label j exactly when $i < j$ and $r(e) = r(f)$. We will return to this example once we have defined the Vershik map.

Given an ordered Bratteli diagram $(B, >)$ and non-negative integers $k < l$ we can define an order on $P_{k,l}$ by stating $(e_k, \ldots, e_{l-1}) > (f_k, \ldots, f_{l-1})$ whenever, for some i with $k \leq i \leq l-1$, we have $e_j = f_j$ for $i < j \leq l-1$ (this condition holds vacuously

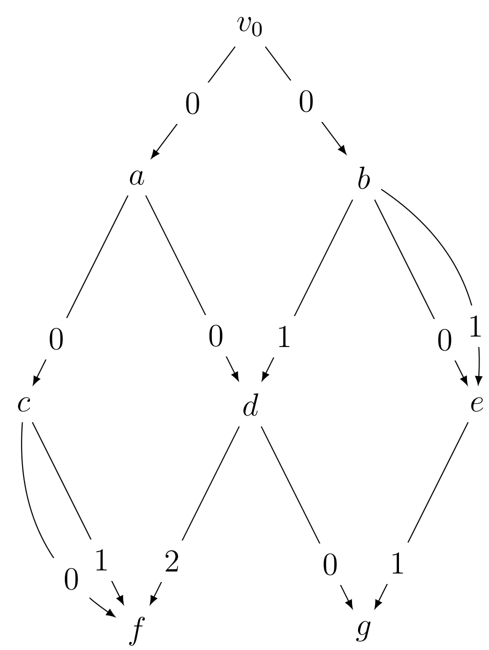


Figure 3.3: example of an ordered Bratteli diagram

if $i = l - 1$) and $e_i > f_i$. This is a reverse-lexicographic ordering on finite paths that share the same range.

This ordering can be used to compare two infinite paths that agree from some level i onwards by applying the rule to the finite segments from V_0 to V_i.

Given an ordered Bratteli diagram $(B, >)$ we can consider the following sets. Let $E_{\max}$ and $E_{\min}$ denote the maximal and minimal edges of E respectively. An infinite path $(e_1, e_2, \ldots)$ is maximal (resp. minimal) if for all $i \geq 0$, we have $e_i \in E_{\max}$ (resp. $E_{\min}$) and $r(e_i) = s(e_{i+1})$ (it is indeed a path).

Given an ordering $>$, we will denote $X_{\max}(>)$ and $X_{\min}(>)$ to be the sets of maximal and minimal paths respectively.

Definition 3.5.

An ordered Bratteli diagram $(B, >)$ is said to be **properly ordered** if, among infinite paths, there is a unique maximal path and a unique minimal path.

Remark 3.1. The sets $X_{\max}(>)$ and $X_{\min}(>)$ are always non-empty. This relies on the fact that for any vertex v at any level, there is a unique finite path that only takes maximal (resp. minimal) edges from v_0 to v. This path can be constructed by working back from v, since there is a unique maximal (resp. minimal) edge with range v by definition. We will argue for the existence of a maximal path (resp. minimal path) after a topology is placed on the set of infinite paths in B.

3.2 Topology of the Path Space

Our goal is to describe the action of a transformation, called a Vershik map, on the path space of a properly ordered Bratteli diagram. By the path space we mean the set of infinite paths that starts at v_0 and can be traced through the levels of the diagram together with a natural topology. What proceeds is a description of this topology and its properties.

Let $X_B = \{(e_0, e_1, \ldots) \mid e_i \in E_i, r(e_i) = s(e_{i+1}), \text{ for all } i \geq 0\}$ denote the set of infinite paths of a Bratteli diragram B. We will often refer to X_B as the path space. To form the topology that we are interested in we choose the basis to be the family of cylinder sets of the form

$$U(e_0, e_1, \ldots, e_k) = \{(f_0, f_1, \ldots) \in X_B \mid f_i = e_i, 0 \leq i \leq k\}.$$

A subset of the power set of X_B is a basis for a topology on X_B if it satisfies two properties:

(1) for each $(e_i)_{i\geq 0} \in X_B$, there is at least one element in the basis which contains $(e_i)_{i\geq 0}$

(2) if $(e_i)_{i\geq 0}$ belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing $(e_i)_{i\geq 0}$ such that $B_3 \subset B_1 \cap B_2$

The first condition follows by choosing $U(e_0)$. For the second condition observe that if two of these cylinder sets are distinct yet have a non-empty intersection, it must be that one of these sets specifies more edges than the other. Otherwise they would disagree at an edge at the same level and the intersection would be empty. A cylinder set B_2 which specifies all the edges that B_1 does and then some is a subset of B_1. Therefore the second condition follows by choosing $B_3 = B_2 = B_1 \cap B_2$.

The ultimate goal of chapter 4 is to relate dynamics on X_B to dynamics on a Cantor space. We will establish in this chapter that the path space can be a Cantor space under some conditions. To achieve this, we assume that X_B contains infinitely many points and that B is simple. Under these assumptions, the topology $\mathcal{T}_U$ generated by the basis of cylinder sets results in a Cantor space. That is, $(X_B, \mathcal{T}_U)$ is a metrizable, totally disconnected, compact space with no isolated points. We verify these properties in that order.

Proposition 3.2.

$(X_B, \mathcal{T}_U)$ is metrizable.

Proof. A metric can be given explicitly. We define $d : X_B \times X_B \to \mathbb{R}$ by

$$d((e_0, e_1, \ldots), (f_0, f_1, \ldots)) = \begin{cases} 0, \text{ if } (e_0, e_1, \ldots) = (f_0, f_1, \ldots), \\ \frac{1}{2^{k+1}} \text{ where } k = \inf\{i \mid e_i \neq f_i\}, \text{ otherwise.} \end{cases}$$

The symmetry requirement and the zero distance requirement follow from the definition. We establish the triangle inequality, let $x = (x_i)_{i\geq 0}$, $y = (y_i)_{i\geq 0}$, and

$z = (z_i)_{i\geq 0}$ be elements of X_B. Observe that if $d(x,z) > d(x,y)$ then the path x agrees with y along its sequence of edges for longer than it agrees with z. In fact, the path y will have its first disagreement with z in the same coordinate that x does, implying $d(x,z) = d(y,z)$. Therefore $d(x,z) \leq d(x,y) + d(y,z)$. This inequality clearly holds when $d(x,z) \leq d(x,y)$.

We will show that every metric ball with radius $r \leq 1/2$ can be expressed as a cylinder set and vice-versa. Let us first deal with $r > 1/2$. By definition, $d(e,f) \leq 1/2$. Therefore the metric ball of radius $r > 1/2$ centered at $e \in X_B$ is $B_r(e) = X_B$. The path space is of course open in the clopen topology $\mathcal{T}_U$. Suppose $r \leq 1/2$. There exists some $k \in \mathbb{N} \cup \{0\}$ such that $r \in (1/2^{k+2}, 1/2^{k+1}]$. Notice that the distance between some pair of paths in X_B could be either of the endpoints of this interval, but never the interior. Therefore we see that $f \in B_r(e)$ if and only if $d(e,f) < 1/2^{k+1}$. This occurs precisely when $f_i = e_i$ for $i = 0, \ldots, k$. In other words, precisely when $f \in U(e_0, \ldots, e_k)$. We conclude that $B_r(e) = U(e_0, \ldots, e_k)$.

The respective topologies generated by the metric balls and the cylinder sets must be the same topology. □

Remark 3.3. The metric proposed in the proof of Proposition 3.2 is in fact an ultrametric.

To show that a topological space is totally disconnected it is sufficient to show that the space satisfies the zeroth separation axiom and has a basis comprised of elements that are both closed and open. We say that such a space has a clopen basis.

Lemma 3.4.

If a topological space $(X, \mathcal{T})$ is a T_0 space and has a basis $\mathcal{B}$ of clopen sets, then the space is totally disconnected.

Proof. Recall that a space is totally disconnected if the only non-empty connected subspaces are singletons. Suppose A is a subset of X with more than one element. Let $a_1, a_2 \in A$. Since the space satisfies the T_0 axiom, we can choose a clopen basis element $B \in \mathcal{B}$ containing a_1 and not a_2. The set B is clopen if and only if $X \setminus B$ is clopen. This is a separation of X. If we consider A with the subspace topology we have that $\{A \cap B, A \setminus B\}$ is a separation of A. Therefore A is a disconnected subspace. □

Proposition 3.5.

The path space is totally disconnected.

Proof. The elements of the basis of the path space are clopen. This is because the complement of a basis element can be written as the union of other basis elements and therefore the basis element is closed. We have that for $\bar{e} = (e_0, e_1, \ldots, e_k) \in P_{0,k+1}$,

$$X_B \setminus U(\bar{e}) = \bigcup_{\substack{\bar{f} \in P_{0,k+1} \\ \bar{f} \neq \bar{e}}} U(\bar{f}).$$

It was shown that the path space is metrizable and therefore it must be a T_0 space. The result follows from the previous lemma. □

To achieve compactness we show that $(X_B, \mathcal{T}_U)$ is a closed subset of a larger compact space, namely the product space $\prod_{i=0}^{\infty} E_i$. If each E_i is endowed with the discrete topology then, by Tychonoff's Theorem, the product space is compact with respect to the product topology.

Observe that the basis elements of X_B with respect to the relative topology are precisely the clopen basis elements. That is,

$$X_B \cap \bigcap_{i=0}^{k} \pi_i^{-1}(\{e_i\}) = U(e_0, e_1, \ldots, e_k)$$

where $\pi_j : \prod_{i=0}^{\infty} E_i \to E_j$ is the canonical projection onto the jth coordinate. Therefore the relative topology and the proposed topology are the same.

Proposition 3.6.

$(X_B, \mathcal{T}_U)$ is a compact space.

Proof. We will show that the path space is closed by showing that its complement is open.

Let $y = (y_0, y_1, \ldots) \in \prod_{i=0}^{\infty} E_i \setminus X_B$. There exists k such that $r(y_k) \neq s(y_{k+1})$. Therefore $(\bigcap_{i=0}^{k+1} \pi_i^{-1}(\{y_i\})) \cap X_B = \emptyset$. This cylinder set both contains y and does not intersect X_B, implying that the complement is open and thus X_B is closed. A closed subset of a compact space is compact. $\square$

It should be noted that we have yet to use the assumption that B is simple. Thus, given any Bratteli diagram B with infinitely many paths, the pair $(X_B, \mathcal{T}_U)$ is a metrizable, totally disconnected, compact space. To be a Cantor space, the space must also be free of isolated points. The path space achieves this when the underlying Bratteli diagram is simple.

To demonstrate this, it helps to translate simplicity into something more concrete.

Lemma 3.7.

A Bratelli diagram $B = (V, E)$ is simple if and only if for every i there exists $j > i$ such that for every pair $(u, v) \in V_i \times V_j$, the set of paths with source u and range v, denoted $P(u, v)$, is non-empty.

Proof. Recall the definition of simplicity: There exists a telescoping B' whose incidence matrices have non-zero entries. This means that there exists a telescoping such that for any pair of vertices $(u, v) \in V_i' \times V_{i+1}'$ there exists $e_i \in E_i'$ where $s(e_i) = u$ and $r(e_i) = v$.

if $\Leftarrow$

We can describe any telescoping by specifying a subsequence of levels. Let $m_0 = 0$ and $m_1 = 1$. By the assumption there exists a level $j > 1 = m_1$ such that for all $(u, v) \in V_{m_1} \times V_j$, $P(u, v) \neq \emptyset$. Choose $m_2 = j$. Continuing in this way we generate a subsequence which prescribes the telescoping with the desired property.

only if $\Rightarrow$

Suppose the telescoping B' is described by the subsequence $(m_k)_{k \geq 0}$ and i is some level in the original diagram B. There exists l such that $m_l > i$. Now choose any $(u, w) \in V_i \times V_{m_{l+1}}$. This is the pair of vertices we will show is linked by a path.

Simplicity tells us that for every $v \in V_{m_l}$, $P(v, w) \neq \emptyset$. Otherwise there would not be an edge between them in B'. On the other hand, by the properties of a Bratteli diagram, there must be at least one edge $v_u \in V_{m_l}$ such that $P(u, v_u) \neq \emptyset$. Combining these facts, we have that $P(u, w) \neq \emptyset$. A path from u to w is formed by extending any path from u to v_u by any path from v_u to w.

This proves the lemma. $\square$

Proposition 3.8.

If $B = (V, E)$ is simple, then X_B contains no isolated points.

Proof. Suppose $x = (x_0, x_1, \ldots) \in X_B$ is an isolated point. This means there exists $k \geq 0$ such that $U(x_0, x_1, \ldots, x_k) = \{x\}$. This implies that $|s^{-1}(r(x_m))| = 1$ for every $m \geq k$. Otherwise the clopen neighbourhood suggested would contain more paths than just x.

If all paths eventually went through $r(x_m)$ for some $m \geq k$ there would only be finitely many paths. Therefore $|V_m| > 1$ for every $m \geq k+1$. It follows that for every $m > k+1$ there exists $v \in V_m$ such that $P(r(x_k), v) = \emptyset$.

This contradicts simplicity in the form established by the previous lemma. □

We have verified that the path space is indeed a Cantor space. As promised, we include an argument for the existence of maximal and minimal paths. The following lemma will be useful here and in other parts of this thesis.

Lemma 3.9.

Let S be a topological space and suppose $(C_i)_{i\geq 1}$ is a decreasing ($C_i \supset C_{i+1}$) sequence of non-empty, closed, compact subsets. The intersection $\bigcap_{i\geq 1} C_i \neq \emptyset$.

Proof. By way of contradiction, we assume that $\bigcap_{i\geq 1} C_i = \emptyset$. Choose $j \geq 1$. Consider C_j with the subspace topology. Each set $C_j \setminus C_i$ $(i > j)$ is open relative to C_j. It follows from our assumption that $\{C_j \setminus C_i \mid i > j\}$ is an open cover of C_j. By the compactness of C_j, we can extract a finite subcover $C_j \setminus C_{i_1}, \dots, C_j \setminus C_{i_n}$.

The decreasing property $C_i \supset C_{i+1}$ implies that if $m = \max\{i_1, \dots, i_n\}$, then $C_j \setminus C_m$ is an open cover of C_j. This is a contradiction because both C_j and C_m are non-empty. □

Remark 3.10. The traditional result, which states that the intersection of a decreasing sequence of non-empty compact subsets of euclidean space is non-empty, is sometimes called Cantor's Intersection Theorem ([5]). Another analysis text asserts this result in a general metric space ([21]). The proofs are all similar. The result here can also be seen as a special case of Theorem 26.9 in [19].

Proposition 3.11.

Given any ordered Bratteli diagram $B = (V, E, >)$, the sets $X_{max}(>)$ and $X_{min}(>)$ are non-empty.

Proof. We argue for the existence of an infinite maximal path.

For all $i \geq 1$ and for each vertex in V_i there exists a unique finite path from v_0 to v which only takes maximal edges. At a fixed level i, let us denote the set of these unique paths M_i. Let C_i denote the set of paths in X_B which take maximal edges from the root to level i. Observe that

$$C_i = \bigcup_{\bar{e} \in M_i} U(\bar{e}).$$

The union is finite because $|M_i| = |V_i|$. This implies that for each i, the set C_i is a clopen subset of X_B and is therefore compact. By definition $C_i \supset C_{i+1}$. The sequence $(C_i)_{i \geq 1}$ is a decreasing sequence of non-empty, closed, compact sets. By Lemma 3.9, the intersection $\bigcap_{i \geq 1} C_i \neq \emptyset$. It is clear that $X_{\max}(>) = \bigcap_{i \geq 1} C_i$. By swapping the word maximal for minimal it follows that $X_{\min}(>) \neq \emptyset$. $\square$

The stage is set to introduce a transformation on X_B.

3.3 Bratteli-Vershik Systems

The simplest class of Bratteli diagrams are those for which there is only a single vertex at each level of the diagram. They can be modeled by products of finite cyclic groups.

Let $b = (b_i)_{i \geq 0}$ be a sequence of non-negative integers where b_i is the number of edges between the ith and $(i+1)$st levels of a diagram with only one vertex per level.

Every path in this diagram can be identified with a sequence in $X = \{(x_i)_{i\geq 0} | x_i \in \mathbb{Z}_{b_i}\}$ where the edges inherit the natural order given by the cyclic group.

If $(x_i)_{i\geq 0} \neq (b_i - 1)_{i\geq 0}$ then there exists a smallest index j for which $x_j \neq b_j - 1$. We can define the transformation $T : X \to X$ by

$$T((x_i)_{i\geq 0}) = (x_i^{'})_{i\geq 0}$$

where

$$x_i^{'} = \begin{cases} 0 & \text{if } i < j, \\ x_i + 1 & \text{if } i = j, \\ x_i & \text{if } i > j. \end{cases}$$

We set $T((b_i - 1)_{i\geq 0}) = (0, 0, 0, \ldots)$.

This transformation is the $+1$ carry-forward map and (X, T) is an odometer system. When we remind ourselves that these sequences correspond to paths on a diagram with one vertex per level we see that T is a transformation from infinite paths to infinite paths.

The odometer system is a special case of a Bratteli-Vershik system. These systems are described by a generalized $+1$ carry-forward map on the path space of an ordered Bratteli diagram.

Definition 3.6.

Let $B = (V, E, >)$.

The function $\varphi : X_B \to X_B$ is called a **Bratteli-Vershik map** if

(i) φ is a homeomorphism

(ii) $\varphi(X_{\max}(>)) = X_{\min}(>)$

(iii) for $x = (x_i)_{i \geq 0} \in X_B \setminus X_{\max}(>)$ we have

$$\varphi(x) = (x_i^{'})_{i \geq 0} = (x_0^{(0)}, \ldots, x_{j-1}^{(0)}, \bar{x_j}, x_{j+1}, x_{j+2}, \ldots)$$

where

(a) $j = \min\{i \mid x_i \notin E_{\max}\}$

(b) $\bar{x_j}$ is the immediate successor of x_j

(c) $(x_0^{(0)}, \ldots, x_{j-1}^{(0)})$ is the minimal path from v_0 to $s(\bar{x_j})$

We will refer to Bratteli-Vershik maps simply as Vershik maps. In Figure 3.4 we demonstrate a map φ that satisfies property (iii) on a finite segment of an ordered Bratteli diagram. The bold lines indicate the preimage (left) and image (right) respectively.

Given an ordered Bratteli diagram there always exists the unique map $\varphi = \varphi_> : X_B \setminus X_{\max}(>) \to X_B \setminus X_{\min}(>)$ which satisfies property (iii). This map is a homeomorphism. The operation given by (iii) can be inverted by replacing the first non-minimal edge with its immediate predecessor $\tilde{x}$ and replacing the previous edges with the unique finite maximal path with range equal to $s(\tilde{x})$. Continuity of φ and its inverse follow because we can apply this reasoning to finite paths as well.

Any neighbourhood of paths in $X_B \setminus X_{\min}(>)$ can be written as a union of cylinder sets defined by finite paths that take at least one non-minimal edge. Let $(e_0, e_1, \ldots, e_n)$ be such that e_j is non-minimal. Then

$$\varphi^{-1}(U(e_0, \ldots, e_n)) = U(e_0^{(1)}, \ldots, e_{j-1}^{(1)}, \tilde{e_j}, e_{j+1}, \ldots, e_n)$$

where $\tilde{e_j}$ is the immediate predecessor of e_j and $(e_0^{(1)}, \ldots, e_{j-1}^{(1)})$ is the unique maximal path from v_0 to $s(\tilde{e_j})$.

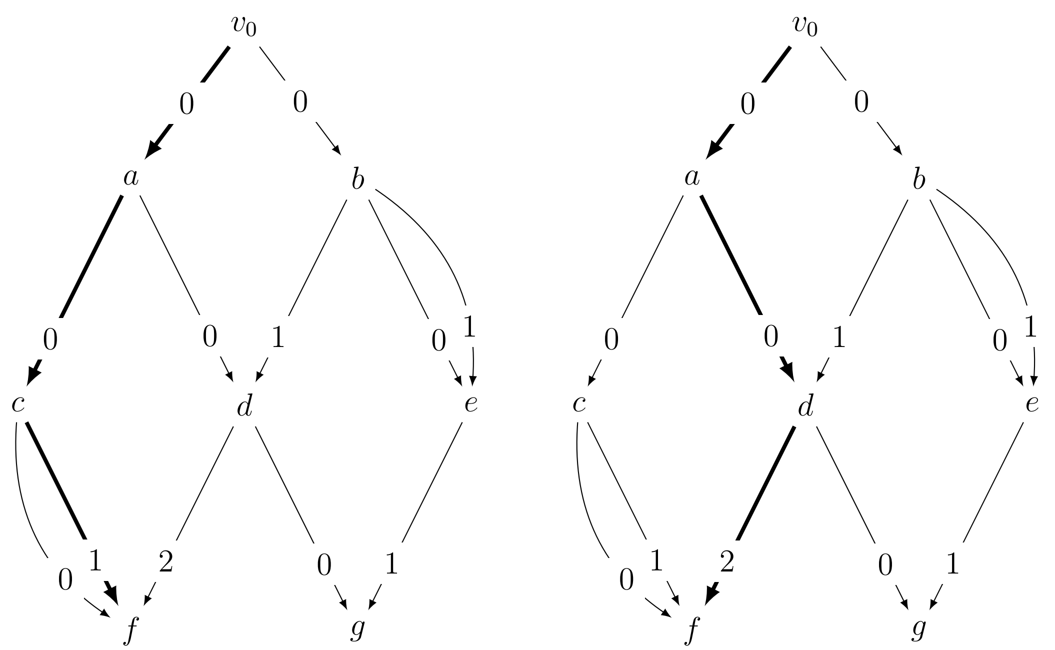


Figure 3.4: action of the Vershik map

Similarly, any neighbourhood of paths in $X_B \setminus X_{\max}(>)$ can be written as a union of cylinder sets defined by finite paths that take at least one non-maximal edge. Let $(f_0, f_1, \ldots, f_n)$ be such that f_j is non-maximal. Then

$$\varphi(U(f_0, \ldots, f_n)) = U(f_0^{(0)}, \ldots, f_{j-1}^{(0)}, \bar{f}_j, f_{j+1}, \ldots, f_n)$$

where $\bar{f}_j$ is the immediate successor of f_j and $(f_0^{(0)}, \ldots, f_{j-1}^{(0)})$ is the unique minimal path from v_0 to $s(\bar{f}_j)$.

It is not, however, the case that we can always define a Vershik map on X_B. It could be that $|X_{\max}(>)| \neq |X_{\min}(>)|$. We will see an example of this in chapter 5 (Example 5.12).

Fortunately, if B is properly ordered then the extension $\varphi : X_B \to X_B$, $\varphi(x_{\max}) = x_{\min}$ of the map given by (iii) on $X_B \setminus X_{\max}(>)$ is a homeomorphism. Furthermore, if $B = (V, E)$ is a simple Bratteli diagram, then there exists an order $>$ for which

$(V, E, >)$ is properly ordered ([17]).

In the next chapter, we connect Bratteli-vershik systems to Cantor minimal systems by building a properly ordered diagram out of a Cantor space.

Chapter 4

Cantor Minimal Systems

The goal of this chapter is to establish a topological conjugacy between Cantor minimal systems and Bratteli Vershik systems. We begin with the preliminaries of topological dynamical systems. Recall that a topological space is a Cantor space if it is a metrizable, totally disconnected, compact space with no isolated points.

4.1 Some Topological Dynamics

Typically, a topological dynamical system is a pair (X, T) where X is a topological space and T is a continuous transformation from X to itself. We will concern ourselves with a special kind of topological dynamical system.

Definition 4.1.

Suppose $T : X \to X$ is a homeomorphism. A subset $Z \subset X$ is **minimal** if Z is non-empty, closed, invariant under T, and contains no nontrivial subsets which satisfy the first three properties. A dynamical system (X, T) is **minimal** if X is itself a minimal set.

If (X, T) is a minimal system, then any closed T - invariant subset is either empty or the entire space. Also, we can now clarify that by Cantor minimal system we mean a minimal system (X, T) where X is a Cantor space.

Recall that the orbit of a point $x \in X$ under a bijective transformation T is the set $\mathrm{orb}_T(x) = \{T^n(x) \mid n \in \mathbb{Z}\}$. We have a useful variant of minimality involving the orbits of points in X. Both notions of minimality will be used in this chapter.

Proposition 4.1.

A dynamical system (X, T) is minimal if and only if for every point $x \in X$, $orb_T(x)$ is dense in X.

Proof. Let (X, T) be a minimal system. Given any point $x \in X$, $\overline{\mathrm{orb}_T(x)}$ is closed and non-empty. If we can show that the closure of the orbit of x is invariant under T then, by minimality, the closure must be the whole space. To simplify the notation we let $\mathcal{O}$ denote $\mathrm{orb}_T(x)$.

Observe that by definition $\mathcal{O} = T(\mathcal{O})$. Since $\mathcal{O} \subset \overline{\mathcal{O}}$, we have that $\mathcal{O} = T(\mathcal{O}) \subset T(\overline{\mathcal{O}})$. The image $T(\overline{\mathcal{O}})$ is closed because T is a homeomorphism. Recall that $\overline{\mathcal{O}}$ is the smallest closed set containing $\mathcal{O}$. It follows then that $\overline{\mathcal{O}} \subset T(\overline{\mathcal{O}})$. In the same way we can show that $\overline{\mathcal{O}} \subset T^{-1}(\overline{\mathcal{O}})$ and therefore $T(\overline{\mathcal{O}}) \subset T(T^{-1}(\overline{\mathcal{O}})) = \overline{\mathcal{O}}$. Together we have the equality $T(\overline{\mathcal{O}}) = \overline{\mathcal{O}}$.

Therefore the closure of the orbit of x is a closed, non-empty, T- invariant set. By minimality it is equal to X.

We will prove the converse using a contrapositive argument. Suppose there exists a proper subset $F \subsetneq X$ that is closed, T-invariant, and non-empty. Let $p \in F$. The complement $X \setminus F$ is a non-empty open subset of X. However, $\mathrm{orb}_T(p) \cap (X \setminus F) = \emptyset$ because F is invariant under T.

We have that the orbit of p does not intersect an open neighbourhood of some element of X and thus cannot be dense in X. □

Definition 4.2.

Dynamical systems (X, T) and (Y, S) are said to be **topologically conjugate** if there exists a homeomorphism $h : X \to Y$ such that $h \circ T = S \circ h$.

The homeomorphism h is called a **topological conjugacy**.

We are now ready to state the conjugacy result presented by Herman, Putnam, and Skau (HPS) in [17].

Main Result (Herman, Putnam, Skau 1992)**.**

Let (X, T) be a Cantor minimal system. There exists a properly ordered Bratteli diagram $B = (V, E, >)$ such that (X, T) is topologically conjugate to the Bratteli-Vershik system (X_B, φ_B).

Remark 4.2. The result in [17] is slightly stronger than what is presented here. We provide a more detailed comment at the end of this chapter in Remark 4.12.

The bulk of the work to prove this theorem is in constructing a Bratteli diagram from the Cantor minimal system. This is done in two parts: first the construction of the vertices, and then the drawing of the edges. The ordering will follow from the diagram's construction. We assume for the remainder of this chapter that X is part of a pair (X, T) and that this system is a Cantor minimal system.

Let us begin by describing a general construction for vertices at some level of the diagram.

Definition 4.3.

A **Kakutani-Rokhlin partition** (K-R partition) of the Cantor minimal system (X, T) is a clopen partition $\mathcal{P}$ of X of the form

$$\mathcal{P} = \{T^j Z_k \mid k \in A, 0 \leq j < h_k\}$$

where A is a finite set and h_k is a positive integer. The kth **tower** of $\mathcal{P}$ is the set $\{T^j Z_k \mid 0 \leq j < h_k\}$ and $T^j Z_k$ is the $(j+1)$st **floor** of the kth tower. The kth tower's **height** is h_k. The **base of** $\mathcal{P}$ is the set $Z = \bigcup_{k \in A} Z_k$.

By a **subtower** of a tower $\{T^j Z_k \mid 0 \leq j < h_k\}$ we mean a tower of the form $\{T^j W \mid 0 \leq j < h_k\}$ where $W \subset Z_k$. Notice that a subtower has the same height as the tower from which it is defined.

Remark 4.3. Notice that the number of elements in A and the heights of towers are all finite. This means that a K-R partition is a finite partition. We also want to make clear that this tower structure follows that of [7], not [17]. That is, the union of the first floors of all the towers is the base of the partition. In [17], the ground floors are the first images under T and the tower ranges all the way to the hth image and not the $(h-1)$st. In this way, the union of the top floors is the base of the partition.

There are a handful of useful phrases regarding K-R towers. By the **base of a tower** we mean the the tower's ground floor Z_k. By a **castle** we mean the collection of towers in the K-R partition. A castle can be viewed as a partition of the K-R partition where floors are contained in the same set exactly when they are part of the same tower. We denote castles by $\mathcal{C}$. Lastly, it is useful to speak of **traversing** a tower. This refers to the sequential movement from one floor of a tower to some floor above it. This coincides with a sequence of iterations of the transformation T. Typically this is some partial traversal and it can refer to the sequential images of the whole floor or only a subset. This will be made clear by the context. A **complete traversal** of a tower refers to the movement from the base of the tower to its top floor. Once again, the sequential images may form the entire tower or strictly a subtower.

We will use these objects to form the set of vertices. The idea is to define each level of the diagram by some K-R partition. At a given level, a correspondence is made between the vertices and the towers of the partition associated with that level. Since a Bratteli diagram has countably many levels we require countably many partitions.

But how do we construct just one?

It is a fact that a Cantor space has a clopen basis for its topology ([20]). We choose a clopen set as the base of a partition. We can generate a finite partition of a clopen set using the first return times of its elements. The following definition makes clear what we mean by first return time of a point.

Definition 4.4.

Suppose the forward orbit of every element x of $S \subset X$ intersects S (excluding the initial point x). Then the first return time map $r_S : S \to \mathbb{N}$ is given by

$$r_S(x) = \inf\{n \in \mathbb{N} \mid T^n(x) \in S\}$$

We will denote r_S by r unless the context demands greater specificity. We will argue that clopen sets satisfy the property $r(Z) \subset \mathbb{N}$ and that $r(Z)$ is a finite set.

Lemma 4.4.

Let Z be a clopen subset of X. There exists $n_1, \ldots, n_m$ such that if $Z_j = \{z \in Z \mid$

$r(z) = n_j\}$, *then* $\{Z_1, \ldots, Z_m\}$ *is a partition of* Z.

Proof. Let Z be a clopen subset of X. It follows from the dense orbit characterization of minimality that the first return time map r is well-defined on Z. Endow $\mathbb{N}$ with the discrete topology. The first return map is continuous under this setup (we show this below). Since Z is a closed subset of a compact set, namely X, it too is compact. Therefore its image under r must be a compact subset of the natural numbers. The singleton sets of $\mathbb{N}$ form an open cover $r(Z)$ and using compactness we may extract a finite subcover.

It follows that $r(Z) = \{n_1, \ldots, n_m\}$. Letting $Z_j = r^{-1}(\{n_j\})$ for $j = 1, \ldots, m$ produces the desired partition of Z. The disjoint property between any two distinct Z_j holds because the first return time map is well-defined. $\square$

Remark 4.5. To see the continuity of the first return time map observe that

$$\begin{aligned} r^{-1}(\{n\}) &= \{z \in Z \mid T^n(z) \in Z\} \cap \{z \in Z \mid T(z) \notin Z\} \cap \cdots \cap \{z \in Z \mid T^{n-1}(z) \notin Z\} \\ &= T^{-n}(Z) \cap \bigcap_{i=1}^{n-1} T^{-i}(X \setminus Z) \cap Z. \end{aligned}$$

We can now form towers. If $\{Z_1, \ldots, Z_m\}$ is the partition of Z into collections of points which share the same return time, then we choose the kth tower of the corresponding K-R partition to be $\{T^j Z_k \mid 0 \leq j < n_k\}$. The integer n_k is the return time which defines Z_k and is the height of the tower with base Z_k. Let us check that this choice is a partition of the Cantor space X.

Proposition 4.6.

Let Z be a non-empty clopen subset of X and let $r(Z) = \{n_1, \ldots, n_m\}$ where r is the first return time function on Z.

Then $\mathcal{P} = \{T^j Z_k \mid 1 \leq k \leq m, 0 \leq j < n_k\}$ is a partition of X where $Z_k = r^{-1}(\{n_k\})$.

Proof. Let $W = \bigcup_{k=1}^{m} \bigcup_{j=0}^{n_m-1} T^j Z_k$. We will argue that $\cap_{n\geq 0} T^n(W)$ is minimal and therefore $W = X$.

In Remark 4.5, it was established that r is continuous. Since singleton sets are clopen in the discrete topology, it follows that Z_k is clopen for each k. If we combine this with the facts that T is a homeomorphism and that the union defining W is finite, it follows that W must too be clopen. Now consider the image of any floor under T. If $j < n_k - 1$ then $T(T^j Z_k) = T^{j+1} Z_k$ is the next floor of the kth tower. Otherwise, by definition of the first return time, $T(T^{n_k-1} Z_k) = T^{n_k} Z_k \subset Z = \cup_{k=1}^{m} Z_k \subset W$. We have then that $T(W) \subset W$. We can extend this to demonstrate the T-invariance of $\cap_{n\geq 0} T^n(W)$ by observing that

$$T^{n+1}(W) \subset T^n(W) \Rightarrow T^k(\cap_{n\geq 0} T^n(W)) = \cap_{n\geq k} T^n(W) = \cap_{n\geq 0} T^n(W).$$

It can similarly be shown that $\cap_{n\geq 0} T^{-n}(X \setminus W)$ is invariant under T (preimages are taken because this is the complement). Because W is clopen it follows that both the intersections of the images of W and the preimages of $X \setminus W$ are respectively closed. Since X is minimal these intersections must either be empty or be X themselves. Lemma 3.9 states that the intersection of a decreasing sequence of non-empty, closed, compact subsets of a topological space is non-empty. The components of both the intersections in question are closed and compact. This is because both W and $X \setminus W$

are closed subsets of a compact space.

By applying Lemma 3.9 we see that for $\cap_{n\geq 0}T^n(W)$ to be empty $T^n(W) = \emptyset$ for some n, implying that W is empty. This is a contradiction because W contains Z. It follows that $\cap_{n\geq 0}T^{-n}(X \setminus W) = \emptyset$ and, by applying the same reasoning again, the complement $X \setminus W$ is empty. We conclude that $W = X$.

Now we use return times to show that any two floors are disjoint sets.

Suppose $x \in T^{j_1}Z_{k_1} \cap T^{j_2}Z_{k_2}$ where $j_1 \neq j_2$. Let $j = \min(j_1, j_2)$. Without loss of generality, let $y = T^{-j}(x) \in Z_{k_1} \cap T^{j_2-j}Z_{k_2}$. If y is contained in the intersection then it is in Z_{k_1} and we have that $T^{j-j_2}(y) \in Z_{k_2} \subset Z$. This means y returns to Z in $j_2 - j$ steps. However $j_2 - j < n_{k_2}$, the first return time of elements in Z_{k_2}. Therefore we have a contradiction. If $j_1 = j_2$ then $y \in Z_{k_1} \cap Z_{k_2}$ and this can only occur when $k_1 = k_2$. $\square$

Now we know how a single K-R partition is constructed from a clopen set. In order to form a partition for each level of the diagram we require a countable sequence of clopen sets. There are a number of requirements we would like our partitions to satisfy. Before we state them, recall the following definitions.

Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of the same set. We say that $\mathcal{Q}$ is **finer** than $\mathcal{P}$ if every element of $\mathcal{Q}$ is a subset of some element of $\mathcal{P}$. We denote this $\mathcal{P} \leq \mathcal{Q}$. A **refinement** of a partition $\mathcal{P}$ is any partition that is finer than $\mathcal{P}$. The **common refinement** of two partitions $\mathcal{P}$ and $\mathcal{Q}$ is the partition $\{P \cap Q \mid P \in \mathcal{P}, Q \in \mathcal{Q}\}$. Lastly, an **increasing sequence of partitions** is a sequence $(\mathcal{P}_i)_{i\geq 1}$ where $\mathcal{P}_1 \leq \mathcal{P}_2 \leq \cdots$.

We now state the list of requirements:

(i) the sequence of clopen sets $(Z^{(i)})$ are such that $Z^{(i)} \supset Z^{(i+1)}$

(ii) the partitions form an increasing sequence

(iii) the sequence of partitions generate the topology on X

Requirement (i) ensures that the base of a tower is contained in the base of any preceding partition. That is, no part of it is contained in a floor above the ground level. This means that as we traverse a tower at a particular point, there is a simultaneous traversal occurring in a tower at each of the preceding levels. We also wish for the intersection of the clopen sets to be a singleton, but this is a subtlety we will make use of later.

Requirement (ii) provides us with more information about the location of a floor. A floor in the partition of a later level is necessarily contained in some floor of the previous one, and by extension in some floor for each of the preceding partitions.

Requirement (iii) is a crucial step in forming a homeomorphism between X and the resulting path space. We will say more about this requirement soon but first let us begin by gathering a sequence of clopen sets.

Proposition 4.7.

There exists a decreasing sequence of clopen neighbourhoods $Z^{(1)} \supset Z^{(2)} \supset \cdots$ of some point $y \in X$ such that $\bigcap_{i \geq 1} Z^{(i)} = \{y\}$.

Proof. Let $y \in X$. Since X has a clopen basis, there exists a clopen neigbourhood N_1 of y. Choose $Z^{(1)} = N_1$. We will demonstrate how to choose $Z^{(2)}$. The space X is metrizable and therefore we can consider the ball of radius $r = 1/2$ centered at y. This ball is open so we can choose a clopen basis element N_2 that is both a neighbourhood of y and a subset of the ball. Choose $Z^{(2)} = N_2 \cap Z^{(1)}$.

Proceed inductively by choosing $r = 1/i$ to generate $Z^{(i)}$. The diameters along this decreasing sequence of compact neighbourhoods of y goes to zero. Therefore the intersection must be $\{y\}$. □

We will address requirements (ii) and (iii) in tandem. Let us say more about requirement (iii). We intend to draw a correspondence between the clopen floors and the clopen cylinders of the path space we construct. For there to be a topological bridge between X and the path space, the K-R construction must preserve the topology on X. We do this by refining the K-R partitions with a sequence of refining partitions which generate the topology on X. This requires some care because we want each refinement to also be a K-R partition.

We first argue that such a sequence of partitions exists. This procedure requires two observations. The first is that the diameter of any subset of X is finite. The space X is compact and so if we cover it with open balls of radius r we can extract a finite subcover $B_1(x_1), \ldots B_N(x_N)$. Let $M = \max\{d(x_j, x_k) \mid 1 \leq k, j \leq N\}$. Given any pair $x, y \in A \subset X$ we may assume without loss of generality that $x \in B_1(x_1)$ and $y \in B_2(x_2)$. Using the triangle inequality we see that

$$d(x, y) \leq d(x, x_1) + d(x_1, x_2) + d(x_2, y) < 2r + M < \infty$$

$$\Rightarrow \text{diam(A)} = \sup_{x,y \in A} d(x, y) < \infty.$$

The second observation we present as a technical lemma.

Lemma 4.8.

Given any clopen set $U \subset X$ there exists $\delta > 0$ such that whenever $x \in U$ and $y \in X \setminus U$ we have $d(x, y) > \delta$.

Proof. Let $f_U : X \to \mathbb{R}$ be the function $f_U(x) = \inf_{y \in U} d(x, y)$. Because $f_U(x) = 0$

if and only if $x \in U$ we restrict our interest to $g = f_U|_{X \setminus U}$ (the "only if" condition holds because U is closed). Let $a, b \in X \setminus U$. Given any $c \in U$ we can use the triangle inequality to see that

$$d(a,b) + d(b,c) \geq d(a,c) \geq g(a) \text{ and } d(a,b) + d(a,c) \geq d(b,c) \geq g(b).$$

Taking infimums over $c \in U$ yields

$$d(a,b) + g(b) \geq g(a) \text{ and } d(a,b) + g(a) \geq g(b).$$

Which is equivalent to

$$|g(a) - g(b)| \leq d(a,b).$$

Therefore g is Lipschitz and in particular is continuous. Observe that U is open and thus $X \setminus U$ is a closed subset of the compact space X. Therefore g is a continuous real-valued function on a compact domain. The Extreme Value Theorem tells us there exists a point p in $X \setminus U$ at which $g(p) = \inf_{a \in X \setminus U} g(a)$. Since $g(p) > 0$ we can choose $\delta \in (0, g(p))$ and have for all $x \in U$ and $y \in X \setminus U$, $d(x,y) \geq g(y) > \delta$. $\square$

Proposition 4.9.

There exists an increasing sequence of finite partitions of X, $\mathcal{Q}_1 \leq \mathcal{Q}_2 \leq \cdots$ that generate the topology on X. Furthermore, the elements of the partitions are clopen sets.

Proof. Let $r_i = 1/2^i$. The limit of this sequence indexed by the naturals decays to zero. Assume that for some $i \geq 0$ we have already constructed a partition $\mathcal{Q}_i$ of clopen sets such that $\operatorname{diam}(U) \leq r_i$ for all $U \in \mathcal{Q}_i$. We will construct $\mathcal{Q}_{i+1} \geq \mathcal{Q}_i$ such that $\operatorname{diam}(U) \leq r_{i+1}$ for all $U \in \mathcal{Q}_{i+1}$.

Given any member $U \in \mathcal{Q}_i$, let δ_U be the positive constant provided by the previous lemma. Let $\delta_i = \min\{\delta_U \mid U \in \mathcal{Q}_i\}$. The distance between any pair x and y which are contained in two different elements $\mathcal{Q}_i$ is bounded below by δ_i. The clopen basis of X allows us to choose neighbourhoods $U_x \subset B(x, \min\{r_{i+1}/2, \delta_i/2\})$ for every $x \in X$. This collection forms an open cover of X and so we extract the finite subcover $U_1, U_2, \ldots, U_k$. In particular they can be chosen such that every element of the cover contains at least one point that is not contained in any of the others.

By construction each set in this subcover has a diameter strictly less than both r_{i+1} and δ_i. Elements of $\mathcal{Q}_{i+1}$ will be chosen to be subsets of elements of the covering. This implies that their diameters will be less than r_{i+1} and so the diameters of partition elements will approach zero along the sequence of partitions. The diameters of the elements of $\mathcal{Q}_{i+1}$ will also be less than δ_i. This means that an element in $\mathcal{Q}_{i+1}$ is a subset of an element in $\mathcal{Q}_i$. This is because if $x, y \in V$ for $V \in \mathcal{Q}_{i+1}$, but were contained in different sets of $\mathcal{Q}_i$, their distance would be at least δ_i. The containment of elements of $\mathcal{Q}_{i+1}$ in elements of $\mathcal{Q}_i$ is precisely what it means to be a refinement.

We proceed by letting $V_1 = U_1$ and set $V_j = U_j \setminus (V_1 \cup \cdots \cup V_{j-1})$. The partition is $\mathcal{Q}_{i+1} = \{V_1, \ldots, V_k\}$. This partition of X is made up of subsets of elements of the subcover. Therefore the statements made about the diameters of the elements of partitions hold. We will now demonstrate that this construction generates the topology on X.

It is easy to see that $\bigcup_{i\geq 1} \mathcal{Q}_i$ is a basis of a topology. The intersection of a pair of its elements is either the smaller of the two or the empty set. Given any point $x \in X$ there is exactly one set in each $\mathcal{Q}_i$ that contains it because $\mathcal{Q}_i$ is a partition of X.

Since every set in this basis is open in the clopen topology on X we have that the topology it generates must be contained in the clopen topology.

To obtain the converse we consider the diameters of the basis elements. Let $\operatorname{diam}(\mathcal{Q}_i) := \max\{\operatorname{diam}(V) \mid V \in \mathcal{Q}_i\}$ and observe that since for all $V \in \mathcal{Q}_i$, $\operatorname{diam}(V) < r_i$, we have $\operatorname{diam}(\mathcal{Q}_i) \xrightarrow{i\to\infty} 0$. Let z be an element of a ball of some radius $r > 0$ centered at some point $x_0 \in X$. We can choose N sufficiently large enough that $r_N < r - d(z, x_0)$. We claim that that clopen set $V \in \mathcal{Q}_N$ which contains z is a subset of the ball of radius r centered at x_0.

Let $w \in V$. By the triangle inequality we obtain

$$\begin{aligned} d(w, x_0) &\leq d(w, z) + d(z, x_0) \\ &< r_N + d(x_0, z) \\ &< r. \end{aligned}$$

This proves that there exists a partition basis element contained in an arbitrary metric ball and means that the metrizable clopen topology is contained in the topology generated by the basis of partitions. They must be equal. □

Given a pair of partitions, one a K-R partition of X and the other a finite partition of X, we wish to produce a new K-R partition that is a refinement of the pair. Let us demonstrate this.

Proposition 4.10.

Let $\mathcal{P} = \{T^j Z_k \mid k \in A, 0 \leq j < h_k\}$ be a K-R partition of X constructed from Proposition 4.6 and $\mathcal{Q}$ be any finite partition of X. There exists a common refinement $\mathcal{R}$ such that $\mathcal{R}$ is also a K-R partition.

Proof. It suffices to demonstrate the procedure for $\{T^j Z \mid 0 \leq j < h\}$, an arbitrary tower of $\mathcal{P}$. Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ be the finite partition of X. For each j, $0 \leq j < h$, we can form the partition $\mathcal{Q}_j = \{T^{-j}Q_i \mid i = 1, \ldots, n\}$. This collection of partitions (which includes $\mathcal{Q}$ since $j = 0$ is an index) provides at most h ways of partitioning Z, the base of the tower. An element from any of these partitions has the form $Z \cap T^{-j}Q_i$. Let us take the common refinement among all these partitions.

Let us denote this refinement by $\mathcal{Z} = \{Z_1, \ldots, Z_K\}$. We can now form new towers by iterating our transformation $h-1$ times on each of the elements of $\mathcal{Z}$. In symbols, we obtain

$$\mathcal{P}_{\mathcal{Z}} = \{T^j Z_k \mid 0 \leq j < h, 1 \leq k \leq K\}.$$

The claim is that this forms a K-R partition of the original tower. Notice that all the new towers share the same height h. This procedure does not discard nor acquire any points and therefore the union of all the floors must equal the union of all the floors in the original tower. Distinct floors are disjoint because either they are at different heights and are therefore a different number of steps from the hth iteration (so a shared point would have two first return times) or they are at the same height and therefore we can pull back T to show that the preimages are distinct elements of $\mathcal{Z}$. The tower structure is bestowed by construction.

We can repeat this procedure for every tower and the same finite partition $\mathcal{Q}$. The collection of all floors across all the resulting towers is the desired K-R partition $\mathcal{R}$.

Observe that every element of $\mathcal{Z}$ has the form $Z \cap Q_{i_1} \cap T^{-1}Q_{i_2} \cdots \cap T^{-h+1}Q_{i_h}$. Since every floor is some T^j image of such a set, $0 \leq j < h$, each floor is contained in some Q_{i_j}. This property holds across all towers and thus all the new floors are contained in an element of $\mathcal{Q}$. Therefore the partition generated by this procedure is a refinement of $\mathcal{Q}$. $\square$

We can now form a sequence of K-R partitions conforming to each of the three requirements. We have

- the decreasing sequence of clopen sets $(Z^{(i)})_{i\geq 1}$, where $\bigcap_{i\geq 1} Z^{(i)} = \{y\}$
- the increasing sequence of partitions $(\mathcal{Q}_i)_{i\geq 1}$ which generate X's clopen topology

We produce the K-R partitions as follows:

Let $\mathcal{P}_1'$ be the K-R partition constructed as in Proposition 4.6 using $Z^{(1)}$ alone. Let $\mathcal{Q}_1' = \mathcal{Q}_1$. Then set $\mathcal{P}_1 = \mathcal{P}_1' \cap_T \mathcal{Q}_1'$ where $\cap_T$ denotes the refinement from Proposition 4.10 which preserves the tower structure. Now assume that we have generated $i-1$ K-R partitions in the desired way.

Let $\mathcal{P}_i'$ be the K-R partition constructed from $Z^{(i)}$ and let $\mathcal{Q}_i' = \mathcal{Q}_i \cap \mathcal{P}_{i-1}$, the common refinement between them. We set $\mathcal{P}_i = \mathcal{P}_i' \cap_T \mathcal{Q}_i'$. Notice that the elements of all the partitions in question are clopen sets. Therefore each K-R partition in the final sequence is still composed of clopen sets with respect to the clopen topology and the topology they collectively generate is contained in the clopen topology. The converse follows because for every i we can write each element of Q_i as a union of floors.

4.2 Drawing the Diagram

It was communicated near the beginning of this chapter that each of the levels of the Bratteli diagram are defined by a K-R partition. What we mean by this is that, at a given level, we define a unique vertex representative for each of the towers in the partition associated to that level.

We assume that we have the sequence $(\mathcal{P}_i)_{i\geq 1}$ produced by the refining procedure detailed just prior to the beginning of this section. We extend this sequence to $(\mathcal{P}_i)_{i\geq 0}$ by setting $\mathcal{P}_0 = \{X\}$. This K-R partition contains only the single floored tower $\{X\}$. Recall that $\mathcal{C}_i$ denotes the set of towers comprised of the floors in $\mathcal{P}_i$. The ith level of the diagram is therefore

$$V_i = \{v_Y \mid Y \in \mathcal{C}_i\}.$$

The only question remaining is: how are the edges drawn between the levels? It suffices to demonstrate how edges are chosen between two levels of the diagram. In fact, we need only define the set $r^{-1}(\{v_Y\})$ for $v_Y \in V_{i+1}$ where r is the range function.

Let $Y = \{T^j Z \mid 0 \leq j < h\}$ be a tower in $\mathcal{C}_{i+1}$. The base of this tower, Z, is a subset of the base of $\mathcal{P}_{i+1}$ and thus also of $\mathcal{P}_i$. This means that Z is a subset of the base of a tower in $\mathcal{C}_i$ because $\mathcal{P}_i \leq \mathcal{P}_{i+1}$. Let us denote it by $Y_1 = \{T^j Z_1 \mid 0 \leq j < h_1\}$. The return time for points in Z_1 to the base of $\mathcal{P}_i$ is equal to the return time for points in Z to the base of $\mathcal{P}_i$ because $Z \subset Z_1$. This return time is h_1. Since the base of P_{i+1} is a subset of the base of P_i, the return time for points in Z to the base of P_{i+1} is greater than or equal to h_1. Therefore $h_1 \leq h$. This means that the height of Y_1 is at most the height of Y.

Dynamically speaking, a traversal of the first h_1 floors of Y coincides with a complete traversal of a subtower of Y_1. This traversal produces an edge with range v_Y and source v_{Y_1}. We will develop a notation for the edges in E soon. In general, an edge is drawn with range v_Y and source v_{Y_k} if any partial traversal of Y coincides with a complete traversal of a subtower of Y_k. If $h_1 = h$ then there is only a single edge with range v_y. If $h_1 < h$, then $r^{-1}(\{v_Y\})$ will contain multiple edges. But currently it is unclear how to continue. To proceed we should answer the question: where does

$T^{h_1}Z$ reside in $\mathcal{P}_i$?

Schematically this image has gone beyond the top floor of Y_1. At the $(i+1)$st level, however, this has not yet occurred. It is still a floor of Y. This immediately tells us that it must be contained in a floor of some tower in $\mathcal{C}_i$. By the construction of Y_1, the image of a point in Z_1 must return to the base of the partition $\mathcal{P}_i$. Therefore $T^{h_1}Z$ is contained in the base of a tower $Y_2 \in \mathcal{C}_i$. We present the following lemma.

Lemma 4.11.

Let $Y \in \mathcal{C}_{i+1}$ and $Y' \in \mathcal{C}_i$. If a floor of Y is contained in the base of Y', then a complete traversal of the corresponding subtower of Y' will either coincide with or occur before reaching the top floor of Y.

We will postpone the proof of this lemma. The application of this lemma results in another edge being drawn, with range v_Y. The source is v_{Y_2}. If the process ended here then $h = h_1 + h_2$. There is nothing inherent to this process that prevents the two traversals from corresponding to two (different) subtowers of the same tower in $\mathcal{C}_i$. That is $Y_1 = Y_2$. This is precisely how multiple edges between vertices arise in the Bratteli diagram.

Suppose Y_1 and Y_2 be towers in $\mathcal{C}_i$ and Y_3 be a tower in $\mathcal{C}_{i+1}$. It is plausible that a complete traversal of Y_3 coincides with a traversal of a subtower of Y_1, followed by a traversal of a subtower of Y_2, and terminating with a traversal of another subtower of Y_1. This tower structure and the resulting assignment of vertices and edges are shown below in Figure 4.1.

In general, this process will terminate once we reach the full height of Y. We may think of Y as the tower that is built by stacking the subtowers it traverses on top of each other. This is done in the order they are traversed. This places an ordering on

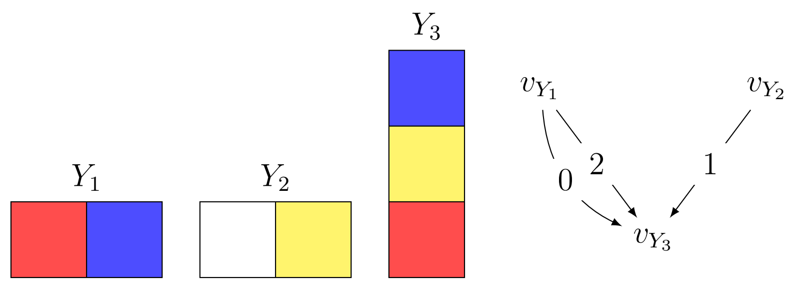


Figure 4.1: drawing edges from traversals

the edges in $r^{-1}(\{v_Y\})$.

If Y can be expressed by stacking subtowers of $Y_1, \ldots, Y_n$, with potential repeats, then

$$h = \sum_{k=1}^{n} h_k \text{ and } r^{-1}(\{v_Y\}) = \{e_k = (i, v_Y, v_{Y_k}, k-1) \mid k = 1, \ldots, n\} \subset E_i$$

where $e_k < e_{k+1}$. The edge corresponding to the first subtower traversed is the minimal edge and the edge corresponding to the last subtower traversed is the maximal edge.

In general, E_i is a collection of tuples $(i, v, v^{'}, k)$. The first entry i designates the level of the source vertex, v designates the range vertex, $v^{'}$ designates the source vertex, and k designates its placement in the order on $r^{-1}(\{v\})$. We claim that this collection $V = \bigcup_{i\geq 0} V_i$ and $E = \bigcup_{i\geq 0} E_i$ with the range and source maps described form a Bratteli diagram B.

The condition that $r^{-1}(\{v\}) \neq \emptyset$ for all $v \in V \setminus V_0$ is clear from the construction. We only need to argue that for all $v \in V$, $s^{-1}(\{v\}) \neq \emptyset$. Given any vertex v at any level i, the base of the tower represented by this vertex can be written as a union of floors in $\mathcal{P}_{i+1}$. If we apply Lemma 4.11 we see that the entire tower in $\mathcal{C}_i$ can be written as a union of subtowers which serve as building blocks of towers in $\mathcal{C}_{i+1}$. This necessitates the existence of at least one edge with source v.

Let us prove Lemma 4.11.

proof of Lemma 4.11. Let Z and $Z^{'}$ be bases of towers $Y \in \mathcal{C}_{i+1}$ and $Y^{'} \in \mathcal{C}_i$ respectively. Let us denote the heights of Y and $Y^{'}$ by h and $h^{'}$. Suppose that there exists $J < h$ such that $T^J Z \subset Z^{'}$. By way of contradiction, assume $h < J + h^{'}$. This is tantamount to stating that the traversal of the tower with base Z will terminate before completing the traversal of the subtower initiated by the Jth application of the transformation T.

By the construction of our towers we know that $T^h Z$ is a subset of the base of a tower in $\mathcal{C}_i$. Let us call this base W. We reach a contradiction because $T^h Z \subset T^{h-J} Z^{'}$. In particular $h - J < h^{'}$ and therefore we can be sure that $T^{h-J} Z^{'} \cap W = \emptyset$. It must be that $h \geq J + h^{'}$ and the complete traversal of a tower in $\mathcal{C}_{i+1}$ will never cut short the traversal of a subtower in a previous partition. $\square$

We have constructed an ordered Bratteli diagram B from the Cantor minimal system (X, T). In the following section we will define a map between X and the path space X_B which we claim is the topological conjugacy referred to in the theorem of HPS.

4.3 The Conjugacy

Definition 4.5.

Let $h : X \to X_B$ be defined by $h(x) = (e_i)_{i \geq 0}$, $e_i = (i, v_Y, v_{Y^{'}}, k)$ where

- the tower Y is the unique tower in $\mathcal{C}_{i+1}$ containing x

- the tower $Y^{'}$ is the unique tower in $\mathcal{C}_i$ containing x
- the integer k is the placement of the subtower of $Y^{'}$ containing x within Y

The function h maps x to the path which traces the location of x through the partitions at successive levels.

The condition $r(e_i) = s(e_{i+1})$ for all $i \geq 0$ holds because the tower in $\mathcal{C}_{i+1}$ which contains x is unique in this regard. In fact, this entails that for a given x there is only a single choice in E_i for each i that satisfies the rule given by h. Therefore h is well-defined.

This concludes the list of ingredients needed to prove the main theorem of this chapter. We repeat it here for convenience.

Main Result (Herman, Putnam, Skau 1992)**.**

Let (X, T) be a Cantor minimal system. There exists a properly ordered Bratteli diagram $B = (V, E, >)$ such that (X, T) is topologically conjugate to the Bratteli - Vershik system (X_B, φ_B).

Remark 4.12. In [17] a slightly stronger result is proved. Namely, there exists a pointed topological conjugacy between the essentially minimal system (X, T, y) and the Bratteli-Vershik system $(X_B, \varphi_B, x_{max})$. An essentially minimal system (X, T, y) is a system in which y is in the closure of every forward and backward orbit respectively, the collection of all the forward and backward images of any neigbourhood of y is an open cover of X, and X contains a unique minimal set Y containing y. Every Cantor minimal system is essentially minimal. In fact, we can choose any y in X.

The pointedness of the conjugacy h stipulates that $h(y) = x_{max}$ where x_{max} is the unique maximal path. Recall Remark 4.3: the role of our ground floors is that of the top floors in [17]. Therefore, in our context, $h(y) = x_{min}$ where x_{min} is the unique minimal path. In the case of the Cantor minimal system, the unique minimal path can be chosen to correspond to any point in X.

In order to prove the theorem we must establish the following facts:

(a) the function h is a bijection

(b) both h and h^{-1} are continuous

(c) the function h intertwines the dynamics (i.e. $h \circ T = \varphi_B \circ h$)

(d) the Bratteli diagram B is properly ordered

We begin with a lemma that does much of the work. Namely, every floor in $\mathcal{P}_i$ corresponds to a cylinder set which specifies edges from E_0 to E_{i-1} and vice-versa.

Lemma 4.13.

Let $(e_0, \ldots, e_{i-1}) \in P_{0,i}$ be a finite path in B. Then there exists a floor $F_i \in \mathcal{P}_i$ such that

$$h^{-1}(U(e_0, \ldots, e_{i-1})) = F_i.$$

Similarly, given any floor $G_i \in \mathcal{P}_i$ there exists a finite path $(f_0, \ldots, f_{i-1})$ such that

$$h(G_i) \subset U(f_0, \ldots, f_{i-1}).$$

Proof. Let $x \in X$ and suppose $h(x) \in U(e_0, \ldots, e_{i-1})$, where $e_j = (j, v_{Y_{j+1}}, v_{Y_j}, k_j)$ and $Y_j \in C_j$. Let π_j denote the projection from X_B onto the $(j+1)$st coordinate. The set $h^{-1}(\pi_j^{-1}(\{e_j\}))$ contains the points in Y_j which form the $k_j th$ subtower used to build Y_{j+1}. Therefore

$$h^{-1}(U(e_0, \ldots, e_{i-1})) = \cap_{j=0}^{j=i-1} h^{-1}((\pi_j^{-1}(e_j))$$

is the intersection of the subtowers that correspond to the edges $e_0, \ldots, e_{i-1}$. We may identify e_j with a unique subtower of the tower represented by $s(e_j)$. For the sake of cleanliness we abuse notation and treat e_j as though it is equal to $h^{-1}(\pi_j^{-1}(\{e_j\}))$ and vertices as though they are the towers they represent.

By construction e_0 is a subfloor of the single-floored tower $s(e_0) = \{X\}$ and a floor of $r(e_0)$. The edge e_1 is a subtower of $s(e_1) = r(e_0)$. Therefore $e_0 \cap e_1$ is a subfloor of $s(e_1) = r(e_0)$ and precisely a floor of $r(e_1)$. This argument can be used again to show that if $\bigcap_{j=0}^{i-1} e_j$ is a floor in $\mathcal{P}_{i-1}$, then $(\bigcap_{j=0}^{i-1} e_j) \cap e_i$ is a floor $F_i \in \mathcal{P}_i$. We see that

$$h^{-1}(U(e_0, \ldots, e_{i-1})) = F_i.$$

Conversely, if given a floor $G_i \in \mathcal{P}_i$, we can locate which tower in $\mathcal{C}_i$ it resides in. This is the range of an edge in E_{i-1}. We can break this tower into the subtowers used to build it and identify both the source and the order the corresponding edge should have. This identifies the edge $f_{i-1} \in E_{i-1}$ precisely. Since we know the exact floor of the subtower represented by f_{i-1}, and thus the exact floor of the tower in $\mathcal{C}_{i-1}$ containing it, we can conduct the same investigation to find f_{i-2}. This process will terminate after finding f_0. These choices satisfy the definition of h such that

$$h(G_i) \subset U(f_0, \ldots, f_{i-1}).$$

□

Remark 4.14. The number of floors in a tower is equal to the number of finite paths from the root to the vertex representative of that tower. In other words (using terminology from the previous chapter), the height of a tower is equal to the height of its vertex representative.

We will now make our way through the list (a) - (d).

Proof of HPS Conjugacy.

(a)

Let $(e_i)_{i\geq 0} \in X_B$. We will show that h is bijective. Observe that $\{(e_i)_{i\geq 0}\} = \bigcap_{k\geq 0} U(e_0, \ldots, e_k)$. Therefore

$$h^{-1}\Big(\{(e_i)_{i\geq 0})\}\Big) = h^{-1}\Big(\bigcap_{k\geq 0} U(e_0, \ldots, e_k)\Big) = \bigcap_{k\geq 0} h^{-1}\Big(U(e_0, \ldots, e_k)\Big) = \bigcap_{k\geq 0} F_{k+1}$$

where F_{k+1} is a floor in $\mathcal{P}_{k+1}$. Therefore the preimage is the intersection of a decreasing sequence of non-empty, closed, compact sets. By Lemma 3.9 it must be non-empty and thus h is surjective. Moreover, recall that $\operatorname{diam}(F_k) \xrightarrow{k\to\infty} 0$. Therefore the intersection is a singleton. This means that h is also injective.

(b)

The bijectivity of h strengthens Lemma 4.13. That is, the image of a floor under h is equal to a cylinder set. The continuity of both h and its inverse follow immediately because the collections of floors and the collection of cylinder sets are bases of the topologies on X and X_B respectively.

(c)

Let $g : X_B \to X_B$ be defined by

$$g = h \circ T \circ h^{-1}.$$

This function is clearly a homeomorphism. We will demonstrate that g is the Vershik map on X_B. Let $(e_i)_{i\geq 0}$ be a sub-maximal element of X_B. Let us examine the action of $h \circ T \circ h^{-1}$ on this path.

The inverse of h maps the path to a point $x \in \bigcap_{i\geq 0} F_i$ where $F_i \in \mathcal{P}_i$ for each i. Suppose e_J is the first sub-maximal edge that $(e_i)_{i\geq 0}$ takes. The floor F_{J+1} is contained in a tower $Y_{J+1} \in \mathcal{C}_{J+1}$. In particular, among the subtowers that are stacked to build Y_{J+1}, the subtower containing F_{J+1} is any but the last. This is what it means for the corresponding edge to be sub-maximal. We claim, however, that F_{J+1} is the top floor of its subtower.

To see this we establish the following fact: the floor associated with a finite path of maximal edges with source v_0 is the top floor of the tower represented by the range of that path. This holds for a single edged path because any maximal edge with source v_0 is precisely the top floor of its range. In general, suppose that the path $(f_0, \ldots, f_i)$ corresponds to the top floor of the tower represented by $r(f_i)$. Suppose further that the edge f_{i+1} corresponds to a subtower of $r(f_i)$. It follows that the path $(f_0, \ldots, f_{i+1})$ corresponds to the intersection of the subtower with the top floor of the tower it is cut from. This is the top floor of the subtower. If we assume that f_{i+1} is maximal then the subtower it represents is the last to be stacked among the subtowers which are used to build $r(f_{i+1})$. The path must then correspond to the top floor of the tower corresponding to $r(f_{i+1})$.

Our claim that F_{J+1} is the top floor of its subtower holds. In addition, if $k > J+1$, then F_k cannot be the top floor of its subtower. If this were true then all the previous floors F_k containing x would be the top floors of their respective towers and e_J would be maximal. Let us summarize the location of x across all partitions.

(i) for $i = 0, \ldots, J$, the point x is contained in the top floor of a tower in $\mathcal{C}_i$

(ii) for $i = J + 1$, the point x is not contained in the top floor of its tower in $\mathcal{C}_{i+1}$ but it is in the top floor of a subtower used in the stacking procedure

(iii) for $i > J + 1$, the point x is not contained in the top floor of any subtower containing it

The application of T in the composition $h \circ T \circ h^{-1}$ moves x up a floor across all the towers it is contained in. Condition (iii) tells us that $T(x)$ resides in the same subtower as x for all $i > J+1$ and therefore the edges taken by the path $h(T(x))$ are unchanged from $(e_i)_{i\geq 0}$ for all $i > J+1$. Condition (ii) indicates a change in subtower, but not in tower, at level $J + 1$. This results in $h(T(x))$ choosing the successor of the edge e_J. Lastly (i) tells us that for the partitions $\mathcal{P}_0, \ldots, \mathcal{P}_J$, the point $T(x)$ is contained in the base of a tower. This means $h(T(x))$ takes the minimal path from v_0 to $s^{-1}(\bar{e}_J)$, where $\bar{e}_J$ is the successor of e_J.

We see that g behaves precisely the same way as the Vershik map does on $X_B \setminus X_{\max}(>)$. It is not difficult to see that $g(X_{\max}(>)) = X_{\min}(>)$. A maximal path corresponds to a point in the top floor of every tower containing it. The point $T(x)$ is therefore in the base of every tower containing it and thus corresponds to a minimal path in X_B. We have that $g = h \circ T \circ h^{-1}$ is the Vershik map on X_B.

(d)

Recall that a Bratteli diagram is properly ordered if there exists only one minimal path and one maximal path. Observe that the preimage of a minimal path under h is a point that is contained in the base of some tower at every level. This means that the preimage is an element of the base of the K-R partition at every level. Namely, $Z^{(i)}$ for all $i \geq 1$. Recall that these clopen sets were chosen such that $\bigcap_{i \geq 1} Z^{(i)} = \{y\}$. This means that y is the only point that satisfies the condition to be the preimage of a minimal path under h and so there must only be a single minimal path.

By construction $T^{-1}(y)$ corresponds to a maximal path in the diagram. If there were more than one point in X contained in the top floor of every tower containing it then there would be multiple elements whose image under T is y. This is a contradiction because T is injective. We may conclude that the Bratteli diagram constructed from (X, T) is properly ordered. $\square$

This concludes the proof of the theorem of HPS. The next chapter presents an analogue to the sequence of K-R partitions found here, but in a concrete setting.

Chapter 5

A Model for Irrational Rotations

In [4], a temporal central limit theorem is proven for the pair (R_α, f_β), where $R_\alpha : [0,1) \to [0,1)$ is the rotation by an irrational angle $\alpha \in (0,1)$ and f_β is the indicator function on the interval $[0,\beta)$ shifted by $-\beta$. In symbols,

$$R_\alpha(x) = (x+\alpha) \bmod 1, \text{ and } f_\beta(x) = 1_{[0,\beta)} - \beta.$$

Their strategy involves building what are essentially Kakutani-Rokhlin partitions in a non-Cantor setting, which they call skyscrapers, using a decreasing sequence of intervals as bases. The construction provides us with a concrete realization of the relationships shared between towers in the Cantor setting. Many of the ideas presented in the first part of this chapter can be traced back to lecture 9 of [22].

The main goal of this chapter is to provide a detailed description of the Bratteli diagram formed by the towers in [4]. We will discuss properties of the diagram and its path space. We wish to be clear that we are not considering some Cantor minimal sub-system of the rotation system. Instead, what is built in this chapter can be viewed as an adic model associated with the irrational rotation system when viewed as a measure preserving transformation on a probability space. We will introduce a function called the coding map (Definition 5.4) which is a measurable version of the conjugacy h presented in the previous chapter. In the next chapter, we will explore the algebraic properties of Bratteli diagrams drawn from this construction.

5.1 Preliminaries

Let $\alpha \in (0,1)$ be an irrational number. Any irrational number can be expressed using a **continued fraction** ([9]). That is,

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}.$$

We call the nonnegative integers a_i the **continued fraction entries** of α. Since they completely determine α, we may express α by $[a_0; a_1, a_2, \ldots]$. We call this the **continued fraction expansion** of α. In our case, $a_0 = 0$.

An important function that should be discussed alongside continued fractions is called the **Gauss map**. It is defined by

$$\mathcal{G} : [0,1) \to [0,1), \quad \mathcal{G}(x) = \{1/x\},$$

where $\{1/x\}$ denotes the fractional part of $1/x$. If $[y]$ denotes the greatest integer less than or equal to y then the fractional part of y can be expressed as $\{y\} = y - [y]$. We highlight two relationships between continued fractions and the Gauss map:

(i) $\alpha = [0; a_1, a_2, \ldots] \Rightarrow \mathcal{G}^i(\alpha) = [0; a_{i+1}, a_{i+2}, \ldots]$ for all $i \geq 1$

(ii) $a_i = [1/\mathcal{G}^{i-1}(\alpha)]$

For every $n \geq 1$, one can consider the truncation of a continued fraction,

$$r_n = \frac{p_n}{q_n} := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

where $\gcd(p_n, q_n) = 1$. The denominator q_n is called the **nth denominator convergent** of α. The convergents satisfy the recurrence relation

$$q_n = a_n q_{n-1} + q_{n-2} \text{ for all } n \geq 2, \;\; q_0 := 1.$$

These convergents will arise as the heights of the towers. For convenience, unless otherwise stated, we will refer to q_n as the nth convergent.

Instead of building our towers on the unit interval we move $([0,1), R_\alpha)$, to a conjugate system $([-1, \alpha_1), T_{\alpha_1})$ where

$$T_{\alpha_1} = \begin{cases} x + \alpha_1, & \text{if } x \in [-1, 0), \\ x - 1, & \text{if } x \in [0, \alpha_1). \end{cases}$$

Transformations of this kind are called interval exchanges ([4]). Observe that the short segment $[0, \alpha_0)$ is mapped into the long segment $[-1, 0)$, and the long segment is mapped onto an interval that covers the short segment and some of the long segment. This is accomplished by adding either the "long endpoint" or the "short endpoint". The parameter $\alpha_1 := \alpha/(1-\alpha)$. In order for α_1 to be in $(0,1)$ we restrict $\alpha \in (0, 1/2)$. Therefore $\alpha_1 = [0; a_1 - 1, a_2, a_3, \ldots]$. The interval $[-1, \alpha_1)$ will act as the base of the first non-trivial "K-R" partition. Admittedly, this is not a partition of a Cantor set, but a partition of an interval using dynamical towers. We will make precise what we mean by towers in this setting in Section 5.3, but intuitively there is no difference from the towers in chapter 4. Subsequent bases will also have disjoint long and short segments. Partitioning into long and short segments partitions the set of towers that we will build upon them. The towers whose bases are subsets of the short segment will share one height and the towers whose base are subsets of the long segment will share a different height.

Proposition 5.1.

The function

$$\psi : [0,1) \to [-1,\alpha_1) \quad \psi(x) = (\alpha_1 + 1)x - 1$$

is a topological conjugacy between the systems $([0,1), R_\alpha)$ *and* $([-1,\alpha_1), T_{\alpha_1})$.

Proof. The function ψ is a homeomorphism between these two spaces because it is a monomial (assume the standard topology). We will verify the intertwining of the dynamics.

The rotation by α can be viewed as a piecewise function.

$$R_\alpha(x) = (x+\alpha) \bmod 1 = \begin{cases} x+\alpha, & \text{if } 0 \le x < 1-\alpha, \\ x+\alpha-1, & \text{if } 1-\alpha \le x < 1. \end{cases}$$

Therefore the composition $\psi \circ R_\alpha$ has a piecewise representation. First recall that by definition $\alpha_1 = \alpha/(1-\alpha)$. Rearranging for α produces the identity $\alpha = \alpha_1/(\alpha_1+1)$.

Using this identity it can be shown that

$$(\psi \circ R_\alpha)(x) = \begin{cases} \psi(x) + \alpha_1, & \text{if } 0 \le x < 1 - \alpha, \\ \psi(x) - 1, & \text{if } 1 - \alpha \le x < 1. \end{cases}$$

These are precisely the pieces which define composition $T_{\alpha_1} \circ \psi$.

$$(T_{\alpha_1} \circ \psi)(x) = \begin{cases} \psi(x) + \alpha_1, & \text{if } (\alpha_1 + 1)x - 1 \in [-1, 0), \\ \psi(x) - 1, & \text{if } (\alpha_1 + 1)x - 1 \in [0, \alpha_1). \end{cases}$$

We need only demonstrate that the identical expressions in each of the compositions are applied to the same points. Observe that

$$(\alpha_1 + 1)x - 1 \in [-1, 0) \quad \text{if and only if} \quad 0 \le x < \frac{1}{1 + \alpha_1} = 1 - \alpha$$

and

$$(\alpha_1 + 1)x - 1 \in [0, \alpha_1) \quad \text{if and only if} \quad \frac{1}{1 + \alpha_1} = 1 - \alpha \le x < 1.$$

This completes the proof. $\square$

The location of β (the parameter defining f_β at the beginning of this chapter) in the conjugate system plays an important role in defining towers. We denote its image, $\psi(\beta)$, by β_1.

In [4], the assumption that β be badly approximable with respect to α and that α be of bounded type are leveraged in order to prove the temporal CLT. They are unnecessary for the construction of the Bratteli diagram. The only assumption we place on β is that it not be in the forward orbit of $1 - \alpha$ under the rotation by α.

This assumption is a consequence of the badly approximable assumption in [4].

5.2 Bases of Partitions

In chapter 4 we formed a sequence of collections of towers that served as the levels of a Bratteli diagram. This required a decreasing sequence of clopen sets where each clopen set acted as the base of a K-R partition. Here we choose a decreasing sequence of subintervals of $[-1, \alpha_1)$. We present them now. Let

$$I_n = \begin{cases} [-\alpha_{n-1}, \alpha_n) & \text{if } n \text{ is odd,} \\ [-\alpha_n, \alpha_{n-1}) & \text{if } n \text{ is even,} \end{cases}$$

where $\alpha_n = \alpha_{n-2} - a_{n-1}\alpha_{n-1}$ for all $n \geq 3$, and $\alpha_0 = 1$. For $n = 2$ we use $a_1 - 1$ instead of a_1 due to the change from α to α_1 (see page 52). This correction is implicit in many of the statements we make about the convergents of α_1. The elements of the sequence $(\alpha_n)_{n\geq 1}$ are products along the forward orbit of α_1 under the Gauss map. We demonstrate this using an inductive argument. Assume that $\alpha_k = \prod_{i=0}^{k-1} \mathcal{G}^i(\alpha_1)$, which clearly holds for $k = 1$. We have that

$$\begin{aligned} \alpha_{k+1} &= \alpha_{k-1} - a_k\alpha_k \\ &= \alpha_{k-1} - [1/\mathcal{G}^{k-1}(\alpha_1)]\alpha_k \\ &= \alpha_k(\alpha_{k-1}/\alpha_k - [1/\mathcal{G}^{k-1}(\alpha_1)]) \\ &= \alpha_k(1/\mathcal{G}^{k-1}(\alpha_1) - [1/\mathcal{G}^{k-1}(\alpha_1)]) \end{aligned}$$

$$
\begin{aligned}
&= \alpha_k\{1/\mathcal{G}^{k-1}(\alpha_1)\} \\
&= \alpha_k \mathcal{G}^k(\alpha_1) \\
&= \prod_{i=0}^{k} \mathcal{G}^i(\alpha_1).
\end{aligned}
$$

From this we see that in general $a_n = [\alpha_{n-1}/\alpha_n]$. This frames the continued fraction entry a_n as the number of times an interval of length α_n fits into one of length α_{n-1}. The recursive definition of α_n can be interpreted as defining α_{n+1} as the length that remains after cutting away a_n strands of length α_n from an interval of length α_{n-1}.

This perspective tells us that successive movement along the sequence $(I_n)_{n\geq 1}$ involves alternating between cutting away the appropriate amount of length from the left and right sides of the current interval. The construction of towers can be described by a cutting and stacking procedure. The cutting component is encapsulated by the idea presented here. The stacking component will have to do with the placement of what is cut away.

On each of these subintervals we define an interval exchange system (I_n, T_{α_n}). For n odd

$$
T_{\alpha_n}(x) = \begin{cases} x + \alpha_n, & \text{if } x \in [-\alpha_{n-1}, 0), \\ x - \alpha_{n-1}, & \text{if } x \in [0, \alpha_n), \end{cases}
$$

and for n even

$$
T_{\alpha_n}(x) = \begin{cases} x - \alpha_n, & \text{if } x \in [0, \alpha_{n-1}), \\ x + \alpha_{n-1}, & \text{if } x \in [-\alpha_n, 0). \end{cases}
$$

We will show that these systems are the induced systems of T_{α_1} on I_n. Let us be precise about what we mean.

Definition 5.1.

Let (X, T) be a dynamical system and suppose $A \subset X$ is such that the first return time to A is finite for every element of A. We denote the first return time map by r_A (Definition 4.4 from chapter 4). The transformation $T_A(x) = T^{r_A(x)}(x)$ is then a well-defined function from A to A. We call (A, T_A) the **induced system** of A with respect to (X, T).

The property $r_A(x) < \infty$ is sometimes expressed by stating that the points of A are T-recurrent ([1]). We will make use of this terminology.

It is a fact that the irrational rotation system $([0,1), R_\alpha)$ is minimal (has dense orbits). It follows that (I_1, T_{α_1}) is minimal and thus the points of each of the subintervals I_n are T_{α_1}-recurrent. We claim that for each n, the system (I_n, T_{α_n}) is the induced system on I_n with respect to (I_1, T_{α_1}). The following lemma lays the groundwork for the proof.

Lemma 5.2.

Let (X, T) be a dynamical system and suppose $B \subset A \subset X$ are such that the points of A and B are both T-recurrent and the points of B are T_A-recurrent where T_A is the induced transformation on A from Definition 5.1. Let (B, S_B) be the induced system on B with respect to (A, T_A) and let (B, T_B) be the induced system on B with respect to (X, T). Then (B, S_B) and (B, T_B) are the same systems.

Moreover,

$$r_B(x) = \sum_{i=0}^{\rho_B(x)-1} r_A(T_A^i(x))$$

where $r_B(x)$ is the first return time of $x \in B$ with respect to T and $\rho_B(x)$ is the first return time of x with respect to T_A.

Proof. In this set-up there are three first return time functions:

$$\rho_B : B \to \mathbb{N}, \ \ \rho_B(x) = \inf\{n \in \mathbb{N} \mid T_A^n(x) \in B\},$$

$$r_B : B \to \mathbb{N}, \ \ r_B(x) = \inf\{n \in \mathbb{N} \mid T^n(x) \in B\},$$

$$\text{and } \ r_A : A \to \mathbb{N}, \ \ r_A(x) = \inf\{n \in \mathbb{N} \mid T^n(x) \in A\}.$$

Observe that

$$S_B(x) = T_A^{\rho_B(x)}(x) = \overbrace{(T_A \circ \cdots \circ T_A)}^{\rho_B \ \text{ times}}(x) = T^{R(x)}(x)$$

for all $x \in B$ where

$$R(x) = \sum_{i=0}^{\rho_B(x)-1} r_A(T_A^i(x)).$$

Since S_B is a map from B to B it follows from the definition of r_B that $r_B(x) \leq R(x)$ for all $x \in B$. Suppose there exists $x_0 \in B$ such that $r_B(x_0) < R(x_0)$.

Therefore $r_B(x_0)$ is of the form $\sum_{i=0}^{j} r_A(T_A^i(x_0)) + k$ where $0 \leq j < \rho_B(x_0) - 1$ and $0 \leq k < r_A(T_A^{j+1}(x_0))$.

If $k = 0$ then $T_A^{j+1}(x_0) = T^{r_B(x_0)}(x_0) \in B$. This is a contradiction because $j+1 < \rho_B(x_0)$. On the other hand, if $k \neq 0$, we have $T^k(T_A^{j+1}(x_0)) = T^{r_B}(x_0) \in B \subset A$. This is a contradiction because $T_A^{j+1}(x_0) \in A$ and $T^k((T_A^{j+1}(x_0)) \in A$ but $k < r_A(T_A^{j+1}(x_0))$.

By way of contradiction $r_B(x) = R(x)$ for all $x \in B$. Thus $S_B(x) = T^{R(x)}(x) = T^{r_B(x)}(x)$ for all $x \in B$. $\square$

We can now establish the following assertion.

Proposition 5.3.

The interval exchange system (I_n, T_{α_n}) is the induced system of I_n with respect to (I_1, T_{α_1}). Moreover, the first return time function r_{I_n} is constant on the long and short segments respectively. In particular and regardless of the parity of n:

$$r_{I_n}(x) = q_{n-1} \textit{ for every } x \textit{ in the long segment of } I_n, \textit{and}$$

$$r_{I_n}(x) = q_{n-2} \textit{ for every } x \textit{ in the short segment of } I_n,$$

where q_{n-1} and q_{n-2} are convergents of α (not α_1) and $q_{-1} = q_0 := 1$

Proof. First we will demonstrate the statement holds for $n = 2$. This case will make clear that the convergents in the claim are of α. We partition I_2 into $[-\alpha_2, 0) \cup [0, \alpha_1)$. Let $x \in [-\alpha_2, 0)$, the short segment of I_2. The point x is in the long segment of I_1 and therefore $T_{\alpha_1}(x) = x + \alpha_1 \in [0, \alpha_1) \subset I_2$. This implies $r_{I_2}(x) = 1 = q_0$. Since x is in the short segment of I_2 we have that $T_{\alpha_2}(x) = x + \alpha_1 = T_{\alpha_1}$.

If $x \in [0, \alpha_1)$ then it will take a_1 applications of T_{α_1} to return to I_2. This is because $T^j_{\alpha_1}(x) \in [-1, -\alpha_2)$ for $1 \leq j < a_1$. At $j = a_1$ we obtain

$$\begin{aligned} T^{a_1}_{\alpha_1}(x) &= x - 1 + (a_1 - 1)\alpha_1 \\ &= x - \alpha_2 \\ &= T_{\alpha_2}(x). \end{aligned}$$

Therefore $r_{I_2}(x) = a_1 = q_0$ for all $x \in [0, \alpha_1)$.

Let us assume that the statement holds at the kth stage. We will assume k is even and thus $k+1$ is odd. Therefore $I_{k+1} = [-\alpha_k, \alpha_{k+1})$. The case where k is odd is analogous. Following the notation from the previous lemma, we let $\rho_{I_{k+1}}(x)$ denote the first return time of $x \in I_{k+1}$ with respect to T_{α_k}.

For $x \in [0, \alpha_{k+1}) \subset [0, \alpha_{k-1})$ (the long segment of I_k) we have that $T_{\alpha_k}(x) = x - \alpha_k$. Therefore $\rho_{I_{k+1}}(x) = 1$ for x in the short segment of I_{k+1}.

Now let $x \in [-\alpha_k, 0)$.

By construction it will take $a_k + 1$ iterations of T_{α_k} before an image of x is in I_{k+1}. This is true for all $k > 1$. In the case $k = 1$ there was a discrepancy caused by the difference between α and α_1. We have that

$$\begin{aligned} T_{\alpha_k}^{a_k+1}(x) &= x + \alpha_{k-1} - a_k \alpha_k \\ &= x + \alpha_{k+1} \\ &= T_{\alpha_{k+1}}(x). \end{aligned}$$

Therefore $\rho_{I_{k+1}}(x) = a_k + 1$. We conclude that $(I_{k+1}, T_{\alpha_{k+1}})$ is the induced system on I_{k+1} with respect to (I_k, T_{α_k}).

Since we assumed (I_k, T_{α_k}) was the induced system on I_k with respect to (I_1, T_{α_1}), it follows by Lemma 5.2 that $(I_{k+1}, T_{\alpha_{k+1}})$ is the induced system on I_{k+1} with respect to (I_1, T_{α_1}). Lemma 5.2 also takes care of the return times. For $x \in [0, \alpha_{k+1})$, the short segment of I_{k+1}, we have $r_{I_k}(x) = q_{k-1}$ because $[0, \alpha_{k+1}) \subset [0, \alpha_{k-1})$, the long segment of I_k. It follows from the formula for return times from Lemma 5.2 that $r_{I_{k+1}}(x) = r_{I_k}(x) = q_{k-1}$ for all x in the short segment of I_{k+1}.

For $x \in [-\alpha_k, 0)$ the formula implies

$$r_{I_{k+1}}(x) = \sum_{i=0}^{a_k} r_{I_k}(T_{\alpha_k}^i(x)).$$

At $i = 0$ we have that $r_{I_k}(T_{\alpha_k}^i(x)) = r_{I_k}(x) = q_{n-2}$ because $[-\alpha_k, 0)$ is the short segment of I_k. The remaining return times in the sum are all of elements in the long segment of I_k. Therefore

$$r_{I_{k+1}}(x) = q_{k-2} + a_k q_{k-1} = q_k$$

for all x in the long segment of I_{k+1}. $\square$

In order to track β_1 through the towers, it is convenient to define a new sequence $(\beta_n)_{n\geq 1}$ where $\beta_n \in I_n$ for all $n \geq 1$. We suppose that n is odd. The steps for the even procedure are analogous. Under this assumption $I_n = [-\alpha_{n-1}, \alpha_n)$. Let us assume that $\beta_n \in I_n$. This is true of β_1 by definition.

The long segment $[-\alpha_{n-1}, 0)$ can be divided into a_n bins of length α_n plus some remainder (recall for this discussion that for $n = 1$ we make the correction $a_1 - 1$). We initiate the cutting from the left endpoint to produce the intervals

$$[-\alpha_{n-1}, -\alpha_{n-1}+\alpha_n), \dots, [-\alpha_{n-1}+(a_n-1)\alpha_n, -\alpha_{n-1}+a_n\alpha_n) \text{ with remainder } [-\alpha_{n+1}, 0).$$

We distinguish between $a_n + 1$ locations that β_n could be in: the a_n segments of size α_n and the interval $[-\alpha_{n+1}, \alpha_n)$. If $\beta_n \in [-\alpha_{n-1} + (b_n - 1)\alpha_n, -\alpha_{n-1} + b_n\alpha_n)$ for $1 \leq b_n \leq a_n$, then we let x_n be the left endpoint of the interval β_n is contained in. If $\beta_n \in [-\alpha_{n+1}, \alpha_n) = I_{n+1}$ then we set $x_n = b_n = 0$. We define $\beta_{n+1} = \beta_n - x_n$.

For a fixed β_1 the integers $(b_n)_{n\geq 1}$ are called the **Ostrowski entries** of β_1 ([4]). The primary difference in the even case is that the cutting occurs from the right because the long segment is a subset of the non-negative reals. Therefore the numbering of the bins is also done from the right. In addition, if β_n is an element of one of the bins then x_n is chosen to be the right endpoint of the bin, instead of the left. We still set $b_n = x_n = 0$ if β_n is not in any of the bins.

This procedure has a dynamical description. That is, regardless of the whether n is even or odd, the a_n bins are the first a_n images of the short segment of I_n under the induced map T_{α_n}. In the odd case worked out above, if $b_n > 0$, then $\beta_n \in T_{\alpha_n}^{b_n}([0, \alpha_n))$.

For this sequence to be of any use we must argue that $\beta_{n+1} \in I_{n+1}$. Let us suppose β_n is not in a bin and therefore $\beta_{n+1} = \beta_n$. Observe that β_n is not in a bin exactly when $\beta_n \in [-\alpha_{n+1}, \alpha_n) = I_{n+1}$. It follows that $\beta_{n+1} \in I_{n+1}$. Suppose now that $\beta_n \in [-\alpha_{n-1} + (b_n - 1)\alpha_n, -\alpha_{n-1} + b_n\alpha_n)$.The size of the interval is α_n. Therefore the distance between any two elements in this interval cannot be more than α_n. If we combine this with the fact that $\beta_n \geq x_n$ we get that $\beta_{n+1} \in [0, \alpha_n) \subset I_{n+1}$. An analogous argument can be applied to the even case.

Let us discuss some properties of the sequence $(\beta_n)_{n\geq 1}$.

Proposition 5.4.

For every $n \geq 1$, $\beta_n = T_{\alpha_n}^{b_n}(\beta_{n+1})$.

Proof. We will demonstrate the case when n is odd. If $x_n = b_n = 0$ the claim holds immediately. We assume that $1 \leq b_n \leq a_n$. Observe that $x_n = -\alpha_{n-1} + (b_n - 1)\alpha_n$. If we substitute this expression into the definition of β_{n+1} and rearrange for β_n we get $\beta_n = \beta_{n+1} - \alpha_{n-1} + (b_n - 1)\alpha_n$.

The assumption $b_n > 0$ implies $\beta_n < 0$ and thus $\beta_{n+1} \in [0, \alpha_n)$. It follows from the definition of T_{α_n} that $\beta_{n+1} - \alpha_{n-1} + (b_n - 1)\alpha_n = T_{\alpha_n}^{b_n}(\beta_{n+1})$.

A similar computation demonstrates the case when n is even. □

Once we build the towers, this observation allows us to identify the tower and floor containing β_{n+1} using the knowledge of the tower and floor containing β_n.

Extending this proposition yields the equation $\beta_1 = T_{\alpha_1}^{b_1} \circ T_{\alpha_2}^{b_2} \circ \cdots \circ T_{\alpha_{n-1}}^{b_{n-1}}(\beta_n)$. Recall that for each n the transformation T_{α_n} is an induced map of T_{α_1} and therefore the image of a point in its domain is equivalent to the image under some number of compositions of T_{α_1} with itself. It follows that β_1 is in the image of β_n under T_{α_1}.

Corollary 5.5.

Let $\beta \neq R_\alpha^t(1-\alpha)$ for $t \geq 0$. Then, for every $n \geq 1$, β_n does not equal the endpoint of a bin.

Proof. Suppose n is odd. The endpoints of bins are all of the form $T_{\alpha_n}^j(0)$ where $1 \leq j \leq a_n$. If $\beta_n = T_{\alpha_n}^j(0)$ then, given the expression we found for β_1 using Proposition 5.4, it follows that β_1 is in the forward orbit of 0 under T_{α_1}. Using the conjugacy between the interval exchange system and rotations on the circle (Proposition 5.1) we have that $\psi^{-1}(0) = 1-\alpha$. It follows that β is in the forward orbit of $1-\alpha$ under R_α. This is a contradiction.

For the case when n is even, we assume that $\beta_n = T_{\alpha_n}^{j+1}(0)$ where $1 \leq j \leq a_n$. This leads to the same contradiction as in the odd case. Notice that the list of endpoints captured by the assumption does not include α_{n-1}. This is fine because $\beta_n \in I_n$ and $\alpha_{n-1} \notin I_n$. They cannot be equal from the outset. $\square$

The bases of towers will be cut at the point β_n. The cutting and stacking procedure will cut along the endpoints of bins. This result ensures that the two kinds of cuts do not coincide, otherwise the definition of the towers to come falls apart. We discuss this collapse in chapter 6. In particular, under our assumptions on β, we have $\beta_n \neq x_n$ for all $n \geq 1$.

The next proposition provides us with another way of reclaiming β_1.

Proposition 5.6.

The parameter $\beta_1 = \sum_{n=1}^{\infty} x_n$.

In fact, the series $\sum_{n=1}^{\infty} x_n$ *is a finite sum if and only if* $\beta = R_\alpha^t(1-\alpha)$ *for* $t \geq 0$.

Proof. We first establish the limit.

For every $N \geq 1$ let $S_N = \sum_{n=1}^{N} x_n$. Since $x_n = \beta_n - \beta_{n+1}$, it is not difficult to see that S_N is a telescoping series such that $S_N = \beta_1 - \beta_{N+1}$. It was shown that for every $N \geq 1$, $\beta_{N+1} \in I_{N+1}$. It is clear that $\bigcap_{N\geq 1} I_N = \{0\}$. Therefore β_{N+1} tends to zero as N tends to infinity. We may conclude that $\lim_{N\to\infty} S_N = \beta_1$.

For the "if and only if" statement we prove the "only if" portion. The "if" statement will become clear once we have explored the structure of the towers we are building (Remark 5.7). As such, the "if" statement is not used in the construction of the towers.

Suppose there exists M such that $\sum_{n=1}^{\infty} x_n = S_M$. This implies that $x_{M+k} = 0$ for all $k \geq 1$. Again we use the fact that $\beta_{M+k} \in I_{M+k}$ for every $k \geq 1$. It follows that $\beta_{M+1} = \beta_{M+2} = \cdots = 0 \in \bigcap_{k\geq 1} I_{M+k}$. By Proposition 5.4 there exists some $j \geq 0$ such that $\beta_1 = T_{\alpha_1}^j(\beta_{M+1})$. The rest of the claim can be established by applying the conjugacy ψ in the same way as in the previous proof. □

In [4] the sum of the terms in $(x_n)_{n\geq 1}$ is called the **Ostrowski expansion** of β_1.

We wish to classify potential positions of β_n in I_n. The interval I_n can be partitioned into three regions where the location of β_n in one region as opposed to another has meaningful consequences for the structure of towers built on the foundation of I_{n+1}.

Definition 5.2.

Let $\{G, B_1, B_2\}$ be a three letter alphabet.

Given $\alpha, \beta \in (0,1)$ (α is irrational and $\beta \neq R_\alpha^t(1-\alpha)$ for $t \geq 0$) we may derive the sequence $(\beta_n)_{n\geq 1}$. We call the sequence $(s_n)_{n\geq 1}$ the **itinerary** of β with respect to α where if n is odd, then

$$s_n = \begin{cases} G, & \text{if } \beta_n \in [-\alpha_{n-1}, -\alpha_{n+1}), \\ B_1, & \text{if } \beta_n \in [-\alpha_{n+1}, 0), \\ B_2, & \text{if } \beta_n \in [0, \alpha_n), \end{cases}$$

and if n is even, then

$$s_n = \begin{cases} G, & \text{if } \beta_n \in [\alpha_{n+1}, \alpha_{n-1}), \\ B_1, & \text{if } \beta_n \in [0, \alpha_{n+1}), \\ B_2, & \text{if } \beta_n \in [-\alpha_n, 0). \end{cases}$$

Well call s_n the nth **scenario** of the itinerary.

Essentially, an itinerary is a sequence of letters indicating the location of β_n for each n.

Not all countable sequences consisting of the symbols G, B_1, and B_2 are admissible itineraries. There are exclusions regarding which symbols may follow others. Let us analyze the case when n is odd. If $s_n = G$ then $\beta_n \in [-\alpha_{n-1}, \alpha_{n+1})$. This places $\beta_{n+1} \in [0, \alpha_n)$ and implies that $s_{n+1} = G$ or B_1. If $s_n = B_1$ then $\beta_{n+1} = \beta_n \in [-\alpha_{n+1}, 0)$ and $s_{n+1} = B_2$. Lastly, if $s_n = B_2$, then $\beta_{n+1} = \beta_n \in [0, \alpha_n)$ and $s_{n+1} = G$ or B_1. We capture these relationships in the following graph.

$$\begin{tikzcd} G \arrow[loop left] \arrow[dr] & B_2 \arrow[l] \arrow[d, bend left] \\ & B_1 \arrow[u] \end{tikzcd}$$

Figure 5.1: the rules governing itineraries - s_{n+1} must be the range of an edge with source s_n

5.3 The Towers Modeling Rotations

Using the sequence of bases $(I_n)_{n\geq 1}$, and the return times found for the long and short segments of each interval, we will define towers in way similar to those defined in the Cantor setting. We claim that this produces a sequence of collections of towers where those built on the foundation of I_{n+1} are the result of cutting and stacking towers built on I_n. We claim further that we can explicitly describe the structure of the cutting and stacking given the itinerary. First let us be clear about what we mean by towers and subtowers in this chapter.

Definition 5.3.

If a set $B \subset [-1, \alpha_1)$ and a positive integer h are such that $Z = \{T^i_{\alpha_1} B \mid 0 \leq i < h\}$ is a disjoint collection of sets, we call Z a **dynamical tower** with base B and height h.

This definition is essentially the same as the definition of a Kakutani-Rokhlin tower with the exception that it specifically entails that the towers model iterations of T_{α_1} and I_1 is not a Cantor space. For the rest of this chapter, when we speak of or call anything a tower we mean a dynamical tower.

Since a partition of $[-1, \alpha_1)$ formed by the floors of dynamical towers is essentially a K-R partition, the definitions of subtower and castle are applicable here. It it is convenient to adopt looser language regarding the towers in this chapter. We will often abuse terminology and notation in the following ways. When we say that a point is contained in a tower we mean it is contained in the floor of some tower. Similarly, a tower Y is a subset of a tower Z, denoted $Y \subset Z$, if the union over the floors of Y are a subset of the union over the floors in Z. The intersection of Y and Z, denoted $Y \cap Z$, refers to the intersection of the respective unions over the floors of each tower. The union of Y and Z, denoted $Y \cup Z$, is the union over all floors in both towers.

Observe that the action of T_{α_1} on an interval I strictly contained in either the long segment or the short segment of I_1 is a translation. Therefore the image, $T_{\alpha_1}(I)$, is an interval of the same length. Moreover, the image of some point $x \in I$ is the same distance from the endpoints of $T_{\alpha_1}(I)$ as x is from the endpoints of I. As long as the height is not too high, all the floors of a dynamical tower with a base strictly in the long or short segment of I_1 will share these properties. Therefore, if the floors are stacked on top of each other, T_{α_1} will take an element in a floor to the point in the interval directly above it.

Let us describe the castle $\mathcal{C}_n$ using the base I_n. We will assume that n is odd and therefore $I_n = [-\alpha_{n-1}, \alpha_n)$. The parameter β_n is either in the long segment $[-\alpha_{n-1}, 0)$ or the short segment $[0, \alpha_n)$. If $\beta_n \in [-\alpha_{n-1}, 0)$, we form the partition $\{I_n^{(L)}, I_n^{(M)}, I_n^{(S)}\}$ of I_n, where

$$I_n^{(L)} = [-\alpha_{n-1}, \beta_n), \ \ I_n^{(M)} = [\beta_n, 0), \ \text{ and } \ I_n^{(S)} = [0, \alpha_n).$$

Otherwise, if $\beta_n \in [0, \alpha_n)$, we choose

$$I_n^{(L)} = [-\alpha_{n-1}, 0), \quad I_n^{(M)} = [0, \beta_n), \quad \text{and} \quad I_n^{(S)} = [\beta_n, \alpha_n).$$

The labels L, M, and S refer to long, middle, and short respectively. In either case, the three intervals are the bases of three towers which comprise the castle $\mathcal{C}_n$. In Proposition 5.3 it was established that the first return time to I_n under the action of T_{α_1} is constant on the segments $[-\alpha_{n-1}, 0)$ and $[0, \alpha_n)$. In particular the first return time on $I_n^{(L)}$ is q_{n-1} and q_{n-2} on $I_n^{(S)}$ (we define $q_{-1} = q_0 = 1$). The return time of points in $I_n^{(M)}$ depends on the location of β_n since β_n decided whether $I_n^{(M)}$ was contained in the long segment of I_n or the short segment.

For $J \in \{L, M, S\}$, let us denote the first return time of $I_n^{(J)}$ to I_n under T_{α_1} by $h_n^{(J)}$. Then we can form the tower $Z_n^{(J)} = \{T_{\alpha_1}^i I_n^{(J)} \mid 0 \leq i < h_n^{(J)}\}$. The castle $\mathcal{C}_n$ is the set $\{Z_n^{(L)}, Z_n^{(M)}, Z_n^{(S)}\}$. Each of these towers has one of two heights, q_{n-1} or q_{n-2}. This will be reflected in the Bratteli diagram by three vertices at the nth level. In symbols, $V_n = \{L, M, S\}$ for all $n \geq 1$. The root $v_0 \in V_0$ will represent the single floored tower $\{I_1\}$. Notice that all the towers represented by the vertices in V_1 all have height one as well.

We will now demonstrate that towers in $\mathcal{C}_{n+1}$ can be expressed by cutting and stacking towers in $\mathcal{C}_n$. This automatically means that a complete traversal of a tower in $\mathcal{C}_{n+1}$ coincides with a number of successive traversals of subtowers of towers in $\mathcal{C}_n$. We will see that the number of subtowers used to build $Z_n^{(J)}$ is either 1 or $a_n + 1$.

We will proceed under the assumption that n is odd and that the scenario $s_n = G$. This means that, among the bins that partition $[-\alpha_{n-1}, \alpha_{n+1})$, β_n is contained in the b_nth. We can write the bins which partition $[-\alpha_{n-1}, \alpha_{n+1})$ as $T_{\alpha_n}^i(I_n^{(S)})$ for

$i = 1, \ldots, a_n$. We present the images for reference:

$$T_{\alpha_n}^i(I_n^{(S)}) = [-\alpha_{n-1} + (i-1)\alpha_n, -\alpha_{n-1} + i\alpha_n).$$

The convergent q_{n-2} is the first return time of elements in $I_n^{(S)} \subset I_n$ to I_n under T_{α_1}. Let us instead consider the first return time of elements in $I_n^{(S)}$ back to $I_{n+1} = [-\alpha_{n+1}, \alpha_n)$ under, not T_{α_1}, but T_{α_n}. Using the bins, we see that this occurs after a_n+1 iterations of T_{α_n}. In particular the first iterate is on $I_n^{(S)}$ and the next a_n are images of points in either $I_n^{(L)}$ or $I_n^{(M)}$. Recall that on $I_n^{(S)}$, the transformation T_{α_n} corresponds to q_{n-2} iterations of T_{α_1}. Each of the next a_n iterates of the induced map correspond to q_{n-1} iterations of T_{α_1}. The total number of iterations is $q_{n-2} + a_n q_{n-1} = q_n$. It is no coincidence that this number is the height of the tower $Z_{n+1}^{(L)}$.

Let Y denote the tower with base $[-\alpha_{n-1}, 0)$ with height $h_Y = q_{n-1}$. This is the tower that is formed if we attach $Z_n^{(L)}$ to $Z_n^{(M)}$ along the shared endpoints of their floors. For $i = 1, \ldots, a_n$, the interval $T_{\alpha_n}^i(I_n^{(S)})$ is the base of a subtower of Y. These bases all have length α_n and therefore the subtowers fit perfectly on top of each other. In fact, the base $T_{\alpha_n}^i(I_n(S))$ is the image of the top floor of the subtower with base $T_{\alpha_n}^{i-1}(I_n(S))$. This holds true because T_{α_n} represents $q_{n-1} = h_Y$ iterations of T_{α_1}. Moreover, they fit on top of $Z_n^{(S)}$ whose base is $I_n^{(S)} = [0, \alpha_n)$. The floors of the ith subtower to be stacked are

$$T_{\alpha_n}^i(I_n^{(S)}), T_{\alpha_1}T_{\alpha_n}^i(I_n^{(S)}), \ldots, T_{\alpha_1}^{q_{n-1}-1}T_{\alpha_n}^i(I_n^{(S)}), \text{ and } T_{\alpha_1}^{q_{n-1}}T_{\alpha_n}^i(I_n^{(S)}).$$

Reading this list from left to right and running i from 1 to a_n after each cycle of q_{n-1} floors is the precise order in which the floors are stacked on the pre-existing tower

$Z_n^{(S)}$. The orientation of each interval is preserved. This forms a new tower structure that contains β_n. Specifically, β_n is contained in the base of the b_nth subtower that was stacked. In Proposition 5.4 it was established that $\beta_{n+1} = T_{\alpha_n}^{-b_n}(\beta_n)$. This places β_{n+1} in the base of this new tower. Pictorially this is the point in $[0, \alpha_n)$ that is aligned with $\beta_n \in [-\alpha_{n-1} + (b_n - 1)\alpha_n, -\alpha_{n-1} + b_n\alpha_n)$ when one is placed above the other. They are the same distance from the endpoints of their respective intervals and thus we can project β_n down vertically to find its preimage β_{n+1}. The parameter β_{n+1} is guaranteed to be neither of the endpoints of $[0, \alpha_n)$ because β_n is neither of the endpoints of $T_{\alpha_n}^{b_n}(I_n(S))$.

We can form two towers of the same height by splitting the base $I_n^{(S)}$ into the intervals $[0, \beta_{n+1})$ and $[\beta_{n+1}, \alpha_n)$. These are the bases of $Z_{n+1}^{(M)}$ and $Z_{n+1}^{(L)}$ respectively. Since $Z_n^{(S)}$ has height q_{n-2} and each of the a_n subtowers have height q_{n-1} it follows that the towers built by stacking them have height q_n. The respective bases and shared height tell us that these towers are indeed $Z_{n+1}^{(M)}$ and $Z_{n+1}^{(L)}$.

The unused portion of Y is a subtower of $Z_n^{(M)}$ with base $[-\alpha_{n+1}, 0)$ and height q_{n-1}. This is precisely $Z_{n+1}^{(S)}$. This ends the procedure. The even case is symmetric.

Remark 5.7. We can now address the "if" direction of Proposition 5.6. Suppose $\beta_1 = T_{\alpha_1}^j(0)$ for $j \geq 0$. If there are infinitely many non-zero elements in the sequence $(\beta_n)_{n\geq 1}$ then there exists N sufficiently large that $\beta_1 = T_{\alpha_1}^i(\beta_N)$ where $i > j$ and $\beta_N \neq 0$. This implies that $T_{\alpha_1}^{i-j}(\beta_N) = 0 \in I_N$. For some $J_{n+1} \in \{L, M, S\}$, the element β_n is contained in a floor of $Z_{n+1}^{(J_{n+1})}$ for all $n \geq 1$. Since the cutting and stacking procedure never cuts short the height of any subtower that is stacked (we will show this to be the case in general), it follows that β_1 is contained in some floor of $Z_N^{(J_N)}$. Therefore $h_N^{(J_N)} \geq i > i - j$ and β_N "returned" sooner than it should have. It must be that the tail of the sequence $(\beta_n)_{n\geq 1}$ is a sequence of zeros and in particular there exists t such that $x_n = 0$ for all $n \geq t$. The series $\sum_{n=1}^{\infty} x_n$ is then a finite sum.

We have that towers built upon some interval base in the sequence of bases can be expressed as the union of subtowers of towers built upon a base earlier in the sequence. The specificity of this example, in contrast to the abstract Cantor setting, allows us to do better. In fact, we are able to identify the towers from which the subtowers are cut. Ultimately, we claim that we can describe the composition of towers in $\mathcal{C}_{n+1}$ in terms of those in $\mathcal{C}_n$ using only the knowledge of which scenario we are in.

Although $Z_{n+1}^{(L)}$ and $Z_{n+1}^{(M)}$ have the same height, their composition is different. For all $i = 1, \ldots, a_n$, except $i = b_n$, the floor $T_{\alpha_n}^{i}(I_n^{(S)})$ is the base of a subtower either of $Z_n^{(L)}$ or $Z_n^{(M)}$. The images where $1 \leq i \leq b_n - 1$ are bases of subtowers of $Z_n^{(L)}$. The images where $b_n + 1 \leq i \leq a_n$ are the bases of subtowers of $Z_n^{(M)}$. The image $T_{\alpha_n}^{b_n}(I_n^{(S)})$ which contains β_n straddles the bases of both because β_n is strictly between the two endpoints. When we cut the floors along the line of images and preimages of β_n under T_{α_1} we see that each subtower is cut into two smaller subtowers. In those cases that $i \neq b_n$, both pieces are still subtowers of either the long or middle tower, as dictated by i. When $i = b_n$ the sub-subtower with base $[-\alpha_{n-1} + (b_n - 1)\alpha_n, \beta_n)$ is a subtower of $Z_n^{(L)}$ and the sub-subtower with base $[\beta_n, -\alpha_{n-1} + b_n\alpha_n)$ is a subtower of $Z_n^{(M)}$. The composition of the long, middle, and short towers in $\mathcal{C}_{n+1}$ can be described as follows.

- The tower $Z_{n+1}^{(L)}$ is built by stacking $b_n - 1$ subtowers of $Z_n^{(L)}$ on top of a subtower of $Z_n^{(S)}$ followed by $a_n - b_n + 1$ subtowers of $Z_n^{(M)}$.

- The tower $Z_{n+1}^{(M)}$ is built by stacking b_n subtowers of $Z_n^{(L)}$ on top of a subtower of $Z_n^{(S)}$ followed by $a_n - b_n$ subtowers of $Z_n^{(M)}$.

- The tower $Z_{n+1}^{(S)}$ is a single subtower of $Z_n^{(M)}$.

It is convenient to encode these relationships in substitutions. A substitution is a function on an alphabet of letters $\mathcal{A}$ into the set of finite words spelled using the letters of $\mathcal{A}$ (which we will denote $\mathcal{L}(\mathcal{A})$).

We first provide a general coding scheme for any scenario. We define τ_n to be a substitution on the alphabet $\{L, M, S\}$. For each $J \in \{L, M, S\}$, $\tau_n(J) = J_0 \cdots J_l$, where $J_i \in \{L, M, S\}$ for every $0 \leq i \leq l$.

The symbol J represents the tower $Z_{n+1}^{(J)}$ and the letter J_i, where $0 \leq i \leq l$, represents the subtower of $Z_n^{(J_i)}$ that is the $(i+1)$st among those used to build $Z_{n+1}^{(J)}$. This enumeration is from the bottom most subtower to the topmost subtower. In every scenario, G, B_1, and B_2, we have $l = 0$ or $l = a_n$. We will refer to the word $\tau_n(J)$ as the **composition** of $Z_{n+1}^{(J)}$.

Suppose the nth scenario is G. The images of L, M, and S under τ_n are $\tau_n(L) = SL^{b_n-1}M^{a_n-b_n+1}, \tau_n(M) = SL^{b_n}M^{a_n-b_n}$, and $\tau_n(S) = M$.

The following proposition summarizes the substitutions tied to each of the three scenarios.

Proposition 5.8.

Let $\alpha \in (0, 1/2)$ be an irrational number. Let $\beta \in (0, 1)$ be such that $\beta \neq R_\alpha^t(1-\alpha)$ for $t \geq 0$. Let $(s_n)_{n\geq 1}$ be the itinerary of β with respect to α. There exists a sequence of substitutions $(\tau_n)_{n\geq 1}$ where for all $n \geq 2$ (use $a_1 - 1$ when $n = 1$) and $J \in \{L, M, S\}$, the word $\tau_n(J)$ is the composition of $Z_{n+1}^{(J)}$ such that

if $s_n = G$, then

$$\tau_n(L) = SL^{b_n-1}M^{a_n-b_n+1},$$

$$\tau_n(M) = SL^{b_n}M^{a_n-b_n},$$

$$\tau_n(S) = M,$$

if $s_n = B_1$*, then*

$$\tau_n(L) = SL^{a_n},$$

$$\tau_n(M) = M,$$

$$\tau_n(S) = L,$$

and if $s_n = B_2$*, then*

$$\tau_n(L) = SL^{a_n},$$

$$\tau_n(M) = ML^{a_n},$$

$$\tau_n(S) = L.$$

Proof. The case of $s_n = G$ has already been detailed (following Remark 5.7). Let us assume $s_n = B_1$ and that n is odd. We proceed in the same way as the G scenario. That is, as we enumerate $i = 1, \ldots, a_n$, we stack the subtower with base $T^i_{\alpha_n}(I^{(S)}_n)$ on top of $Z^{(S)}_n$. Since the procedure has not deviated from before, the height of this tower is q_n. The scenario $s_n = B_1$ is defined by $\beta_n = \beta_{n+1} \in [-\alpha_{n+1}, 0)$. This means that the base of $Z^{(n+1)}_L$ is $[0, \alpha_n)$. It follows that the tower built by stacking subtowers onto $Z^{(S)}_n$ is $Z^{(L)}_{n+1}$. The other consequence of the location of β_n is that all the bases of the subtowers are strictly contained in $[-\alpha_{n-1}, \beta_n)$. This interval is the base of $Z^{(L)}_n$. We conclude that $Z^{(L)}_{n+1}$ is built by stacking a_n subtowers of $Z^{(L)}_n$ onto the trivial subtower of $Z^{(S)}_n$ (the entirety of the short tower). Therefore $\tau_n(L) = SL^{a_n}$.

What remains is a tower with base $[-\alpha_{n+1}, 0)$ with height q_{n-1}. The subtowers with bases $[-\alpha_{n+1}, \beta_{n+1})$ and $[\beta_{n+1}, 0)$ are respectively $Z^{(S)}_{n+1}$ and $Z^{(M)}_{n+1}$. Since $\beta_{n+1} =$

β_n, we see that $Z_{n+1}^{(S)}$ is a subtower of $Z_n^{(L)}$ and $Z_{n+1}^{(M)} = Z_n^{(M)}$. Therefore $\tau_n(S) = L$ and $\tau_n(M) = M$.

Now we assume that $s_n = B_2$ and n is odd. The intervals $[-\alpha_{n-1}, 0)$, $[0, \beta_n)$, and $[\beta_n, \alpha_n)$ are the respective bases for the long, middle, and short towers in $\mathcal{C}_n$. Let Y be the tower with base $[0, \alpha_n)$ formed by attaching $Z_n^{(M)}$ to $Z_n^{(S)}$. Again, for each $i = 1, \ldots, a_n$, we stack the subtower of $Z_n^{(L)}$ with base $T_{\alpha_n}^i([0, \alpha_n))$ on top of Y. This tower has height q_n. This scenario also dictates $\beta_{n+1} = \beta_n$ and thus the subtowers with base $[0, \beta_n)$ and $[\beta_n, \alpha_n)$ with height q_n are $Z_{n+1}^{(M)}$ and $Z_{n+1}^{(L)}$. We see that $Z_{n+1}^{(M)}$ is built by stacking a_n subtowers of $Z_n^{(L)}$ onto $Z_n^{(M)}$ and $Z_{n+1}^{(L)}$ is built by stacking a_n subtowers of $Z_n^{(L)}$ onto $Z_n^{(S)}$. Therefore $\tau_n(M) = ML^{a_n}$ and $\tau_n(L) = SL^{a_n}$. The remaining subtower of $Z_n^{(L)}$ with base $[-\alpha_{n+1}, 0)$ has height q_{n-1} and must be $Z_{n+1}^{(S)}$. Therefore $\tau_n(S) = L$.

The even cases can be derived in essentially the same way. Since the positions of the long and short segments of I_n are swapped, we instead cut from the right end and stack the subtowers onto the towers whose bases lie on the negative real line. □

We can use the sequence of substitutions $(\tau_n)_{n\geq 1}$ to define an ordered Bratteli diagram $(V, E, >)$. What follows is the construction of a Bratteli diagram from a more general sequence of substitutions ([7]).

Let $(V_n)_{n\geq 0}$ be a sequence of finite alphabets and $\mathcal{L}(V_n)$ be the set of finite words spelled with letters of V_n. We denote the $(j+1)$st letter of a word w by w_j. Suppose $(\sigma_n)_{n\geq 0}$ is a sequence of substitutions where $\sigma_n : V_{n+1} \to \mathcal{L}(V_n)$. We assume that V_0 is a singleton, otherwise we may extend both sequences by shifting the index up by one for all existing elements and setting $V_0 = \{v_0\}$ and $\sigma_0(J) = v_0$ for all $J \in V_1$.

Define $V := \bigcup_{n\geq 0} V_n$. This is a disjoint union. That is, letters in $V_n \cap V_m$ for

$n \neq m$ are treated as distinct elements corresponding to a distinct "level" (n versus m). This will serve as the vertex set of the Bratteli diagram. We define the edge set by giving the preimage of a singleton from $V \setminus V_0$ under the range map. Let $n \geq 0$ and suppose $J \in V_{n+1}$. If $\sigma_n(J) = J_0 \cdots J_l$ then we define the incoming edges by

$$r^{-1}(\{J\}) := \{(J, j) \mid 0 \leq j \leq l\}.$$

Then $E_n := \bigcup_{J \in V_{n+1}} r^{-1}(\{J\})$ and $E := \bigcup_{n \geq 0} E_n$. The source map is defined by $s((J, j)) = (\sigma_n(J))_j = J_j$. We can place an order on $r^{-1}(\{J\})$ by setting $(J, j_1) < (J, j_2)$ whenever $j_1 < j_2$. This provides us with an order $>$ for every set of edges that share the same range. We call the Bratteli diagram formed by $(V, E, >)$ the **Bratteli diagram read off of the sequence of substitutions** $(\sigma_n)_{n \geq 0}$.

Let $B_{\alpha,\beta}$ denote the ordered Bratteli diagram read off of $(\tau_n)_{n \geq 0}$ where we extend the sequence of substitutions and vertex sets in the way described above. In particular, identify v_0 with the single-floored tower $\{I_1\}$. With the exception of the zeroth level, $V_n = \{L, M, S\}$. We have an identification between towers in $\mathcal{C}_n$ and vertices in V_n. Since the image of $J \in V_{n+1}$ represents a word specifying both the order in which subtowers are traversed and the towers from which they came, we see that the edge assignment given by reading the substitutions matches the assignment given by subtower traversals. This is even true of E_0 since the towers in $\mathcal{C}_1$ all have height one and represent one full traversal of a subtower of $\{I_1\} = v_0$. Hence this construction is an explicit realization of the model for Cantor minimal systems (besides the clopen nature of the floors). We can even write down the incidence matrices for any level $n \geq 1$. The zeroth incidence matrix is merely the column vector $(1, 1, 1)^T$.

Recall that the nth incidence matrix is the $|V_{n+1}| \times |V_n|$ matrix where each element of V_{n+1} is identified with a row and each element of V_n is identified with a column. The (i,j)th entry is the number of edges that connect the jth vertex in V_n to the ith vertex in V_{n+1}. Therefore the incidence matrices of $B_{\alpha,\beta}$, for $n \geq 1$ are 3 by 3 matrices where the (i,j)th entry is the number of times the jth vertex, a letter in $\{L, M, S\}$, appears in the composition of the tower corresponding to the ith vertex. We see that the incidence matrices are scenario dependent.

We will denote the incidence matrices of $B_{\alpha,\beta}$ by $(A_n)_{n\geq 1}$ and since $|V_{n+1}| = |V_n|$ for all n we will identify the vertices L, M, and S in V_{n+1} (resp. V_n) to the row numbers (resp. column numbers) 1, 2, and 3 for every n.

If $s_n = G$, then $A_n = \begin{pmatrix} b_n - 1 & a_n - b_n + 1 & 1 \\ b_n & a_n - b_n & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

If $s_n = B_1$, then $A_n = \begin{pmatrix} a_n & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Lastly, if $s_n = B_2$, then $A_n = \begin{pmatrix} a_n & 0 & 1 \\ a_n & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Before proceeding, let us work out an example in which we draw E_1 given a pair α and β.

Example 5.9. Let $\alpha = [0; 3, 1, 3, 1, \ldots] = \frac{-3+\sqrt{21}}{6}$ and $\beta = 1/2$. Then $\alpha_1 = \frac{\alpha}{1-\alpha} = 0.35825..$ and $\beta_1 = \frac{\alpha_1 - 1}{2} = -0.32087..$ are the parameters that will define the towers at the root and the first level.

We have $I_1 = [-1, \alpha_1) = [-1, 0.3526..)$ and therefore the towers are the first level are $Z_1^{(L)} = \{[-1, \beta_1)\} = \{[-1, -0.32087..)\}$, $Z_1^{(M)} = \{[\beta_1, 0)\} = \{[-0.32087.., 0)\}$, and $Z_1^{(S)} = \{[0, \alpha_1)\} = \{[0, 0.35825..)\}$. The interval $[-1, 0)$ can be partitioned into 2 bins of size α_1 with a remainder. These are $[-1, -1+\alpha_1) = [-1, -0.64174..)$ and $[-1+\alpha_1, -1+2\alpha_1) = [-0.64174.., -0.28348..)$ with remainder $[-\alpha_2, 0) = [-0.28348.., 0)$. We see that β_1 is contained in the second bin. This corresponds with scenario $s_1 = G$ and $b_1 = 2$. We could derive the substitution and thus draw the edges in E_1 now, but let us pretend we do not have this knowledge. Instead we continue with the cutting and stacking procedure. The bins $[-1, -0.64174..)$ and $[-0.64174.., -0.28348..)$ are placed on top of the interval $[0, 0.35825..)$ in the order they are listed.

The next beta is $\beta_2 = \beta_1 - x_1 = -0.32087.. - -0.64174.. = 0.32087..$. This tells us to split the tower $Y = \{[0, \alpha_1), [-1, -1+\alpha_1), [-1+\alpha_1, -\alpha_2)\}$ along 2 steps of β_2's orbit under T_{α_1} to obtain $Z_2^{(M)}$ and $Z_2^{(L)}$. At the second level the towers are

$$Z_2^{(M)} = \{[0, 0.32087..), [-1, -0.67912..), [-0.64174.., -0.32087..)\}$$

and

$$Z_2^{(L)} = \{[0.32087.., 0.35825..), [-0.67173.., -0.64174..), [-0.32087.., -0.28348..)\}.$$

We see that $T_{\alpha_1}^{b_1}(\beta_2) = T_{\alpha_1}^{2}(0.32087..) = -0.32087.. = \beta_1$. The remaining tower is $Z_2^{(S)} = \{[-0.28348.., 0)\}$. We have $h_2^{(L)} = h_2^{(M)} = q_1 = 3$ and $h_2^{(S)} = q_0 = 1$. By examining the floors we can classify the subtowers (which all have only a single floor) that were stacked.

The tower $Z_2^{(L)}$'s first floor is the single floor of a subtower of $Z_1^{(S)}$ since the interval $[0.32087.., 0.35825..)$ is a subset of $[0, 0.35825..)$. The second floor

$[-0.67173.., -0.64174..)$ is a subset of $[-1, -0.32087..)$ and thus is contained in a subtower of $Z_1^{(L)}$ and the top floor $[-0.32087.., -0.28348..)$ is a subset of $[-0.32087.., 0)$ and thus is contained in a subtower of $Z_1^{(M)}$. The vertices $S, L, M \in V_1$ represent the towers which contain the subtowers that are traversed by a complete traversal of $Z_2^{(L)}$. The word SLM is also the order in which they were traversed and thus the sources of the edges with range $L \in V_2$ are $s((L,0)) = S$, $s((L,1)) = L$, and $s((L,2)) = M$. This corresponds with $\tau_1(L) = SL^{b_1-1}M^{a_1-1-b_1+1} = SLM$. The correction $a_1 - 1$ is made at this stage because of the discrepancy between α and α_1.

The other images $\tau_1(S) = M$ and $\tau_1(M) = SL^2$ can be verified in the same way. The diagram up to V_2 is

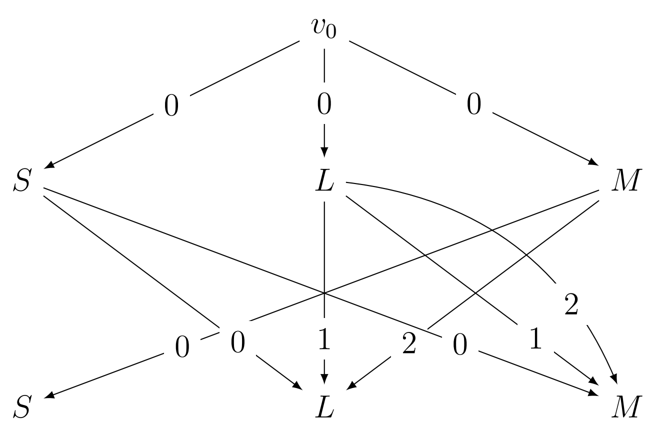


Figure 5.2: a diagram derived from quadratic irrational alpha and rational beta

In general, the Bratteli diagram $B_{\alpha,\beta}$ is simple. The proof of this claim can be found in [4] but it is done outside of the context of $B_{\alpha,\beta}$. We begin with a technical lemma.

Lemma 5.10.

Suppose for some $M \geq 3$, $s_M = G$. The product of incidence matrices

$$A_{M+2}A_{M+1}A_M A_{M-1}A_{M-2}$$

has strictly positive entries.

Proof. If we do not fix α and β then there are infinitely many choices for the incidence matrices in the product. There is, however, a way of getting around this problem by checking only a finite number of products among a related collection of matrices. We define the matrices $F(s_n)$, where

$$F(G) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; F(B_1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } F(B_2) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $F(s_n) = (f_{ij}^{(n)})$ and $A_n = (a_{ij}^{(n)})$.

The matrix $F(s_n)$ has been defined such that the entry $f_{ij}^{(n)} = 1$ if $a_{ij}^{(n)}$ is guaranteed to be greater than zero. That is, $a_{ij}^{(n)} > 0$ regardless of the choice of α and β. The entry is zero otherwise. Therefore $f_{ij}^{(n)} \leq a_{ij}^{(n)}$ for all n. It follows that for any n and any m

$$(F(s_m)F(s_n))_{ij} = \sum_{k=1}^{3} f_{ik}^{(m)} f_{kj}^{(n)} \leq \sum_{k=1}^{3} a_{ik}^{(m)} a_{kj}^{(n)} = (A_m A_n)_{ij}.$$

This can be extended inductively to show that if the product $F(s_{M+2}) \cdots F(s_{M-2})$ has positive entries then so does $A_{M+2} \cdots A_{M-2}$. Using the rules in Figure 5.1 that govern itineraries, we can limit our investigation to the products corresponding to the possible subwords of the itinerary which are of length 5 and whose middle letter is G. There are 8 such subwords.

A subword $s_{M-2} \cdots s_{M+2}$ corresponds to the product $F(s_{M+2}) \cdots F(s_{M-2})$. We include all possible subwords and their products below.

$$G^5 \Rightarrow \begin{pmatrix} 3 & 5 & 5 \\ 4 & 4 & 5 \\ 1 & 4 & 3 \end{pmatrix}$$

$$G^4 B_1 \Rightarrow \begin{pmatrix} 4 & 4 & 5 \\ 1 & 4 & 3 \\ 2 & 3 & 3 \end{pmatrix} \quad B_2 G^4 \Rightarrow \begin{pmatrix} 8 & 3 & 2 \\ 8 & 4 & 1 \\ 5 & 1 & 2 \end{pmatrix}$$

$$G^3 B_1 B_2 \Rightarrow \begin{pmatrix} 6 & 7 & 8 \\ 5 & 8 & 8 \\ 4 & 4 & 5 \end{pmatrix} \quad B_1 B_2 G^3 \Rightarrow \begin{pmatrix} 6 & 2 & 5 \\ 7 & 1 & 5 \\ 3 & 2 & 3 \end{pmatrix}$$

$$B_2 G^2 B_1 B_2 \Rightarrow \begin{pmatrix} 8 & 2 & 3 \\ 8 & 3 & 2 \\ 5 & 1 & 2 \end{pmatrix} \quad B_1 B_2 G^2 B_1 \Rightarrow \begin{pmatrix} 7 & 1 & 5 \\ 3 & 2 & 3 \\ 4 & 1 & 3 \end{pmatrix}$$

$$B_1 B_2 G B_1 B_2 \Rightarrow \begin{pmatrix} 5 & 3 & 5 \\ 6 & 2 & 5 \\ 3 & 2 & 3 \end{pmatrix}$$

□

Theorem 5.11.

Let $\alpha \in (0, 1/2)$ be irrational. Let $\beta \in (0,1)$ be such that $\beta \neq R_\alpha^t(1-\alpha)$ for $t \geq 0$. The Bratteli diagram $B_{\alpha,\beta}$ is simple.

Proof. We first claim that $(s_n)_{n\geq 1}$ contains infinitely many G's. If there exists N such that $s_n = B_1$ or B_2 for all $n \geq N$, then $\beta_n = \beta_{N+1}$ for all $n \geq N+1$. It follows that $\beta_{N+1} \in \bigcap_{n\geq N+1} I_n = \{0\}$. This contradicts the fact that $\beta \neq R_\alpha^t(1-\alpha)$ for $t \geq 0$. Therefore for any K there exists $M > K$ such that $s_M = G$.

Let V_n be the nth level of $B_{\alpha,\beta}$. We may choose $M > n+2$ such that $s_M = G$. The Bratteli diagram is simple if there exists a level V_l such that given any vertex in V_n and any vertex in V_l, there is at least one path between them. In particular we show that $l = M+3$.

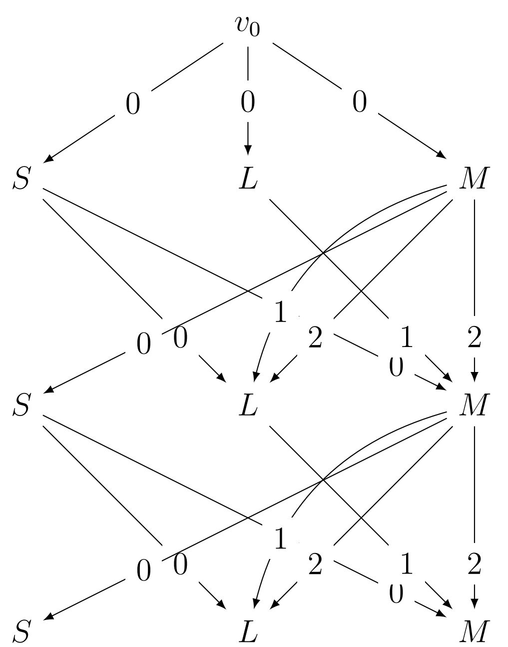
Lemma 5.10 tells us that $A_{M+2} \cdots A_{M-2}$ has strictly positive entries. Let $F(s_n)$ be defined as it is in the proof of Lemma 5.10. It can be checked explicitly that given any matrix P with positive entries and any scenario s_n, the product $PF(s_n)$ has positive entries. Verifying this requires the computation of three products of pairs as opposed to eight products of five matrices each. We leave this to the reader. It follows that the product $A_{M+2} \cdots A_n$ has positive entries. This competes the proof because the number of paths between any two vertices, in V_n and V_{M+3} respectively, are precisely the entries of this product. $\square$

It is not true, in general, that the diagram given by this construction is properly ordered. We provide a counterexample.

Example 5.12. Suppose $\alpha = [0; 3, 2, 2, \ldots]$. Therefore $\alpha_1 = [0; 2, 2, \ldots]$. We choose β such that the Ostrowski sequence reads $(1, 1, 1, \ldots)$ (we compute β below). In other words $s_n = G$ for all $n \geq 1$.

This implies the edge assignment given by $\tau_n(S) = M$, $\tau_n(L) = SM^2$, and $\tau_n(M) = SLM$ for all $n \geq 1$. We include the orders as well. The Bratteli diagram is shown below in Figure 5.3.

The periodicity inherent in this example makes it possible to compute α and β. Based on the setup, α_1 is a quadratic irrational. This provides us with the equation $\alpha_1 = \frac{1}{2+\alpha_1}$ and thus $\alpha_1^2 + 2\alpha_1 - 1 = 0$. Solving the quadratic equation (and using the



This pattern of edges continues throughout the diagram.
The path that runs straight down the right hand side, $((M,0),(M,2),(M,2),\ldots)$, is the maximal path.
There are two minimal paths, $((M,0),(S,0),(M,0),\ldots)$ and $((S,0),(M,0),(S,0),\ldots)$, which criss-cross between M and S.

Figure 5.3: a diagram derived from the cutting and stacking procedure that is not properly ordered

fact that $\alpha_1 > 0$) tells us that $\alpha_1 = \sqrt{2} - 1$. It follows that $\alpha = \frac{1}{2+\sqrt{2}}$. To find β we will use the Ostrowski expansion of β_1.

Since $b_n = 1$ for every n it follows that $x_n = (-1)^n(\alpha_1)^{n-1}$. This turns the Ostrowski expansion into the geometric series

$$\sum_{i=1}^{\infty}(-1)^n(\alpha_1)^{n-1} = -\sum_{i=0}^{\infty}(-\alpha_1)^n = \frac{-1}{1+\alpha_1}.$$

This implies $\beta = \psi^{-1}(\beta_1) = \frac{\alpha_1}{1+\alpha_1^2} = \frac{\sqrt{2}-1}{2}$.

5.4 The Adic Coding

The structure of the towers allows us to define a map similar to the conjugacy between a Cantor minimal system and its Bratteli-Vershik model. Recall that the edges bridging V_n and V_{n+1} are elements of the set

$$E_n = \{(J, j) \mid J \in \{L, M, S\}, 0 \leq j < |\tau_n(J)|\}.$$

The path space of $B_{\alpha,\beta}$ is then captured by the set

$$\Sigma_{\alpha,\beta} = \{((J_i, j_i))_{i\geq 1} \mid (\tau_i(J_{i+1}))_{j_{i+1}} = J_i\}.$$

Given $x \in [-1, \alpha_1)$, for each $n \geq 1$, the point x is contained in a unique tower $Z_n^{(J_n(x))}$ where $J_n(x) \in \{L, M, S\}$. Furthermore x is contained in a unique subtower among those stacked to build $Z_n^{(J_n(x))}$. Suppose that $\tau_{n-1}(J_n(x)) = J_0 \cdots J_l$. Then there exists $j_n(x) \in \{0, \ldots, l\}$ such that the $j_n(x)$th subtower used to build $Z_n^{(J_n(x))}$ contains x and is a subtower of the tower in $\mathcal{C}_{n-1}$ with vertex representative $J_{j_n(x)}$. This manifests the edge $(J_n(x), j_n(x)) \in E_{n-1}$.

Definition 5.4.

Let $\Psi : I_1 \to \Sigma_{\alpha,\beta}$ be the function given by

$$\Psi(x) = ((J_1(x), j_1(x)), \ldots, (J_n(x), j_n(x)), \ldots).$$

This function is called the **coding map** and the image $\Psi(I_1)$ is called the **adic coding** ([4]).

We comment that $j_1(x) = 0$ for every x regardless of $J_1(x)$ because $Z_1^{J_1(x)}$ is a single-floored subtower of $\{I_1\}$. The coding map "behaves" the same way as the conjugacy h from chapter 4 (see Definition 4.5) in so far that it maps points to paths using the same methodology. However, it is not surprising that the properties of the coding map differ from h because the domain is clearly not homeomorphic to a Cantor space.

Unlike the conjugacy h from the previous chapter, this function is not surjective. To see an example, observe that like h, the image $\Psi(x)$ can only be a minimal path if x is in the bottom floor of every tower that contains it. This means that $x \in I_n$ for every n. Therefore $x = 0$. Example 5.12 has two minimal paths. For reference, in that example we have $\Psi(0) = ((S,0),(M,0),(S,0),\ldots)$. It is the minimal path $((M,0),(S,0),(M,0),\ldots)$ that has no preimage in I_1. As an aside, since T_{α_1} is injective, the only element in the top floor of every tower is $-\alpha_1$. Therefore $\Psi(-\alpha_1) = ((M,0),(M,2),(M,2),\ldots)$.

We will postpone the argument that Ψ is not surjective for any parameters α and β and instead argue that Ψ is injective. To show this, we first demonstrate that the preimage of a cylinder set of $\Sigma_{\alpha,\beta}$ is the floor of some tower. This is another property Ψ shares with h.

Proposition 5.13.

The preimage of any cylinder

$$[(J_1,j_1),\ldots,(J_n,j_n)] := \{\omega \in \Sigma_{\alpha,\beta} \mid \omega_i = (J_i,j_i), i = 1,\ldots n\}$$

is a floor of the tower $Z_n^{(J_n)}$. In symbols,

$$\Psi^{-1}([(J_1,j_1),\ldots,(J_n,j_n)]) = T_{\alpha_1}^i(I_n^{(J_n)}),\ \ 0 \le i < h_n^{(J_n)}.$$

Proof. Observe that $\Psi^{-1}([(J_1,j_1)]) = \Psi^{-1}([(J_1,0)]) = \{x \in I_1 \mid (J_1(x), j_1(x)) = (J_1,0)\}$ is $I_1^{J_1}$. This is the single floor that comprises $Z_1^{(J_1)}$. Suppose now that

$\Psi^{-1}([(J_1, j_1), \ldots, (J_k, j_k)])$ is a single floor of $Z_k^{(J_k)}$.

The preimage $\Psi^{-1}([(J_1, j_1), \ldots, (J_{k+1}, j_{k+1})])$ is equal to

$$\Psi^{-1}([(J_1, j_1), \ldots, (J_k, j_k)]) \cap \{x \in I_1 \mid (J_{k+1}(x), j_{k+1}(x)) = (J_{k+1}, j_{k+1})\}.$$

The set $\{x \in I_1 \mid (J_{k+1}(x), j_{k+1}(x)) = (J_{k+1}, j_{k+1})\}$ is the collection of points that forms the j_{k+1}th subtower that is stacked to build $Z_{k+1}^{(J_{k+1})}$. Since $(\tau_k(J_{k+1}))_{j_{k+1}} = J_k$, we have that the subtower is of $Z_k^{(J_k)}$.

It follows from the definition of a subtower that if Y is a subtower of Z then the intersection of the union of floors of Y with a single floor of Z is a single floor of Y. It follows that the intersection above is a single floor of the j_{k+1}th subtower that is stacked to build $Z_{k+1}^{(J_{k+1})}$ and therefore a single floor of $Z_{k+1}^{(J_{k+1})}$. □

We can now demonstrate that Ψ is injective. If $\Psi(x_1) = \Psi(x_2) = ((J_n, j_n))_{n \geq 1}$, then $x_1, x_2 \in \Psi^{-1}([(J_1, j_1), \ldots (J_m, j_m)])$ for every $m \geq 1$. Proposition 5.13 states that for each $m \geq 1$ there exists i_m $(0 \leq i_m < h_m^{(J_m)})$ such that $x_1, x_2 \in T_{\alpha_1}^{i_m}(I_m^{(J_m)})$. Since the length of the interval $I_m^{(J_m)}$ goes to zero as m grows, it must be that $x_1 = x_2$.

Proposition 5.13 is invertible. That is, given a floor F of a tower $Z_n^{(J)}$ for $J \in \{L, M, S\}$ we can choose a unique cylinder set $[(J_1, j_1), \ldots, (J_n, j_n)]$ such that $\Psi^{-1}([(J_1, j_1), \ldots, (J_n, j_n)]) = F$. This is because we know the composition of $Z_n^{(J)}$. The location of F tells us that it is contained in the jth subtower that is stacked to build $Z_n^{(J)}$. This subtower corresponds to the edge (J, j) and thus we choose $(J_n, j_n) = (J, j)$. We set $J_{n-1} = (\tau_{n-1}(J))_j$ and repeat until the process terminates with the choice of (J_1, j_1). Although Ψ is not bijective, we do obtain a bijective correspondence between floors of towers and cylinder sets. This is not to be confused with a bijective correspondence between the elements of the floor and the paths in the corresponding cylinder set.

There are occasionally towers, and thus floors, that are unaffected by the cutting and stacking procedure. That is, there exists a floor of a tower in $\mathcal{C}_n$ that is also the floor of a tower in $\mathcal{C}_{n+1}$. This occurs for the towers $Z_{n+1}^{(S)}$ and $Z_{n+1}^{(M)}$ when $s_n = B_1$ or B_2. The last edge of a cylinder set specifying n edges that corresponds to a floor in either tower must have range S or M. Fixing either scenario and either range, the associated substitution dictates that there is only one edge in E_{n+1} with source equal to that range. Therefore the "new" cylinder set specifying $n+1$ edges for that same floor, now as a floor of a tower in $\mathcal{C}_{n+1}$, is actually the same collection of paths as the cylinder set that only specified n edges. In other words, there are multiple labels, but the correspondence between sets is not broken.

5.5 From 2 Vertices to 3

Let $\Lambda_n^{(L)}$ and $\Lambda_n^{(S)}$ denote the long and short segments of I_n respectively. The middle tower in $\mathcal{C}_n$ is formed by cutting along a finite orbit of β_n. This results in either splitting the tower with height q_{n-1} and base $\Lambda_n^{(L)}$ or the tower with height q_{n-2} and base $\Lambda_n^{(S)}$. Suppose, however, that we forgot about the sequence $(\beta_n)_{n\geq 1}$ and instead defined the nth castle to be $\{Y_n^{(L)}, Y_n^{(S)}\}$ for every $n \geq 1$ where $Y_n^{(L)} = \{T_{\alpha_1}^i \Lambda_n^{(L)} \mid 0 \leq i < h_n^{(L)}\}$ and $Y_n^{(S)} = \{T_{\alpha_1}^i \Lambda_n^{(S)} \mid 0 \leq i < h_n^{(S)}\}$. We will denote the nth castle by $\mathcal{K}_n$. Beyond the root, this describes a Bratteli diagram with two vertices per level as opposed to three.

Proposition 5.14.

Let $\alpha \in (0, 1/2)$ be an irrational number. There exists a sequence of substitutions

$(\sigma_n)_{n\geq 1}$ *(*$\sigma_n : \{L, S\} \to \mathcal{L}(\{L, S\})$*) where for every* $n \geq 1$ *and* $K \in \{L, S\}$, $\sigma_n(K)$ *is the composition of* $Y_{n+1}^{(K)}$ *such that*

$$\sigma_n(L) = SL^{a_n} \text{ and } \sigma_n(S) = L$$

With the correction $a_1 - 1$ *for* $n = 1$.

Proof. For a fixed n, the substitution can be verified by following the same cutting and stacking procedure for the case with three towers with the exception that we do not distinguish any portion of a tower as a middle tower. □

We can extend the sequence of substitutions to $(\sigma_n)_{n\geq 0}$ in order to construct the ordered Bratteli diagram read off of $(\sigma_n)_{n\geq 0}$. Let us denote it by B_α. We use the substitutions given by Proposition 5.14 to produce Figure 5.4 where $a_1 = a_2 = 3$.

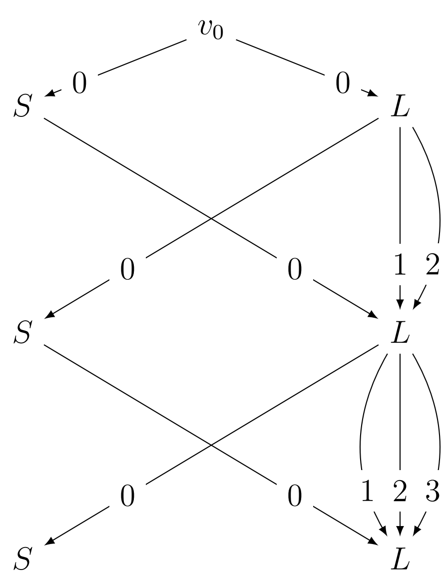


Figure 5.4: a diagram derived from irrational alpha alone

We can encode the path space in a way that is analogous to the case with three vertices:

$$\Sigma_\alpha := \{((K_i, k_i))_{i\geq 1} \mid (\sigma_i(J_{i+1}))_{j_{i+1}} = K_i\}.$$

We also have a coding map $\Gamma : I_1 \to \Sigma_\alpha$ given by

$$\Gamma(x) = ((K_1(x), k_1(x)), \ldots, (K_n(x), k_n(x)), \ldots)$$

where $Y_n^{K_n(x)}$ is the unique tower in $\mathcal{K}_n$ containing x and $k_n(x)$ is the non-negative integer such that $(\sigma_{n-1}(K_n(x)))_{k_n(x)} = K_{n-1}(x)$.

Remark 5.15. Like Ψ (Definition 5.4), the coding map Γ is not surjective. For any α, the path space Σ_α contains two minimal paths and one maximal path. These are the paths $((L,0),(S,0),(L,0),\ldots)$, $((S,0),(L,0),(S,0),\ldots)$ and $((L,a_1-1),(L,a_2),(L,a_3),\ldots)$ respectively. Since zero is the only element in the base of some tower for every castle, we see that $\Gamma(I_1)$ cannot contain both minimal paths. In particular, $\Gamma(0) = ((S,0),(L,0),(S,0),\ldots)$.

Proposition 5.16.

The preimage of any cylinder

$$[(K_1,k_1),\ldots,(K_n,k_n)] := \{\omega \in \Sigma_\alpha \mid \omega_i = (K_i,k_i), i = 1,\ldots n\}$$

is a floor of the tower $Y_n^{(K_n)}$. In symbols

$$\Gamma^{-1}([(K_1,k_1),\ldots,(K_n,k_n)]) = T^i_{\alpha_1}(\Lambda_n^{(K_n)}),\ \ 0 \leq i < h_n^{(K_n)}$$

Proof. The proof is no different from the proof of Proposition 5.13. ∎

This proposition can be used to show that Γ is injective in the same way that Proposition 5.13 was used to show Ψ was injective. More importantly, this proposition is also invertible such that we have a bijective correspondence between the floors of towers and cylinder sets of Σ_α.

We will make use of these properties in order to relate Σ_α and $\Sigma_{\alpha,\beta}$. We begin with an observation.

The Bratteli diagram B_α provides us with a way of generating a diagram $B_{\alpha,\beta}$. Rather than specifying a point in $\beta \in (0,1)$ we choose a path $p = \Gamma(\psi(\beta)) \in \Sigma_\alpha$. At each level p will take one of $a_n + 2$ edges in E_n. Let $E_n^{'}$ denote the collection of edges between the nth level and $(n+1)$st level of $B_{\alpha,\beta}$. We can identify the choice of edge in E_n with a substitution that dictates the assignment of edges and the partial ordering on $E_n^{'}$. What follows are the instructions for generating a diagram. Afterwards, we will explain why it works.

Let $p = (e_i)_{i \geq 0}$ be an element of the path space of B_α.

(i) $s(e_n) = L$ and $r(e_n) = S$ implies $\tau_n(S) = L$, $\tau_n(L) = SL^{a_n}$, and $\tau_n(M) = M$.

(ii) $s(e_n) = S$ and $r(e_n) = L$ implies $\tau_n(S) = L$, $\tau_n(L) = SL^{a_n}$, and $\tau_n(M) = ML^{a_n}$.

(iii) $s(e_n) = L$ and $r(e_n) = L$. This case contains a_n choices. We will denote a choice by b_n ($1 \leq b_n \leq a_n$). This implies $\tau_n(S) = M$, $\tau_n(L) = SL^{b_n - 1}M^{a_n - b_n + 1}$, and $\tau_n(M) = SL^{b_n}M^{a_n - b_n}$.

We include examples where $a_n = 2$. The bold line indicates the edge taken by p and the graph to the right is the corresponding arrangement of edges in $E_n^{'}$ for $B_{\alpha,\beta}$. The last two cases correspond to $b_n = 1$ and $b_n = 2$ respectively. This works because choosing a path $p \in \Sigma_\alpha$ is essentially the same as choosing a sequence of bin numbers

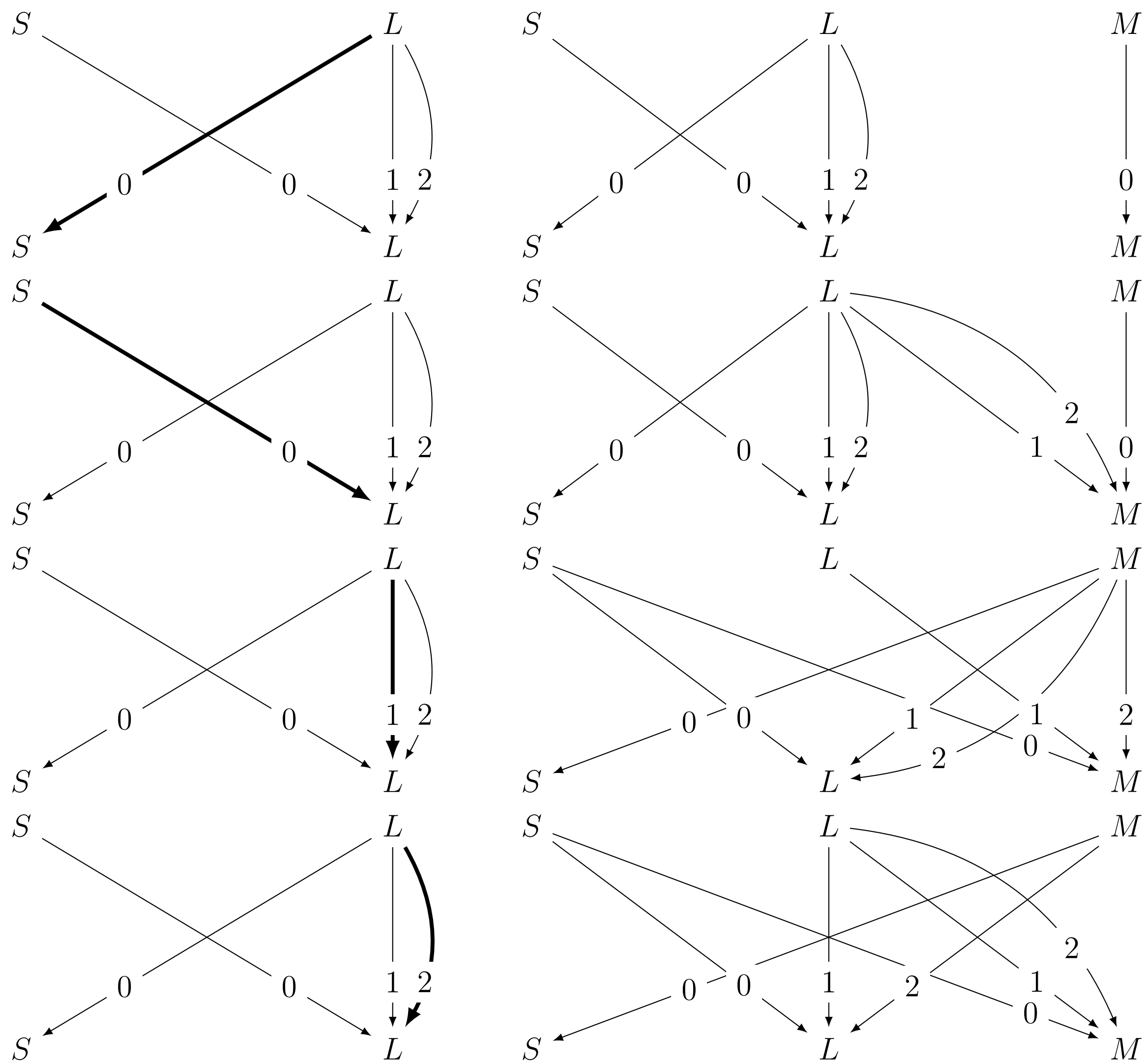


Figure 5.5: cutting along the path of beta

$(b_n)_{\geq 1}$ and scenarios $(s_n)_{\geq 1}$ corresponding to some $\beta_1 \in I_1$. An edge in E_n corresponds to a subtower containing a potential β_1 and, among the subtowers of towers in $\mathcal{K}_n$ which are stacked to build towers in $\mathcal{K}_{n+1}$, we can show that the subtower corresponds to a unique pair (s_n, b_n) if β_1 was contained in that subtower.

Let $e \in E_n$. Suppose that $r(e) = S$ and $s(e) = L$. This corresponds to the subtower of $Y_n^{(L)}$ which matches $Y_{n+1}^{(S)}$. This is the portion of $Y_n^{(L)}$ that remains after the cutting and stacking procedure. The parameter β_1 is in this subtower exactly when $|\beta_n| < \alpha_{n+1}$. This corresponds to the pair of scenario and bin $(B_1, 0)$.

Suppose that $r(e) = L$ and $s(e) = S$. This corresponds to $Y_n^{(S)}$ exactly. The parameter β_1 is in this subtower exactly when β_n is too. This corresponds to the pair of scenario and bin $(B_2, 0)$.

Lastly, suppose that $r(e) = L$ and $s(e) = L$. The ith edge among the a_n edges with this property corresponds to precisely to the subtower of $Y_n^{(L)}$ whose base is the ith bin. Recall that if n is odd, this is the interval $[-\alpha_{n-1} + (i-1)\alpha_n, -\alpha_{n-1} + i\alpha_n)$. The parameter β_1 is in this subtower exactly when β_n is in this bin. This corresponds to the pair of scenario and bin (G, i).

Naturally, according to this identification, the sequence of edges of p adheres to the same restrictions that the sequence of scenarios abides by (see Figure 5.1).

It follows that if we choose $p = \Gamma(\beta_1)$ then, using the instructions above, we will generate $B_{\alpha,\beta}$ where $\beta = \psi^{-1}(\beta_1)$. We know, however, that the coding map Γ is not surjective (Remark 5.15). Therefore the instructions above can generate diagrams that do not correspond to any choice of parameter $\beta_1 \in I_1$.

We draw attention to the fact that elements of I_1 which are in the forward orbit of 0 under T_{α_1} (i.e. $R_\alpha^t(1-\alpha)$ for $t \geq 0$) are mapped by the coding map Γ into a subset of the paths which criss-cross between the vertices S and L (the other paths with this property do not have preimages). We will see in the next chapter that a diagram generated by one of these paths (according to the instructions given here) does not agree with the cutting and stacking procedure.

For the remainder of this chapter we will explore the relationship between Σ_α and

$\Sigma_{\alpha,\beta}$. It is implicit in the remaining propositions and lemmas that $\alpha \in (0, 1/2)$ is irrational and β is not in the forward orbit of $1 - \alpha$ under R_α.

We begin by examining the difference between the towers in $\mathcal{K}_n = \{Y_n^{(L)}, Y_n^{(S)}\}$ and $\mathcal{C}_n = \{Z_n^{(L)}, Z_n^{(M)}, Z_n^{(S)}\}$. Let us assume that the scenario $s_n = B_2$ or G. This would mean that $Z_n^{(L)}$ and $Z_n^{(M)}$ are both subtowers of $Y_n^{(L)}$. This is because $\beta_{n+1} \in \Lambda_n^{(L)} = I_n^{(L)} \cup I_n^{(M)}$. The ith floor of $Y_n^{(L)}$ is the union of the ith floors of $Z_n^{(L)}$ and $Z_n^{(M)}$. We also have $Z_n^{(S)} = Y_n^{(S)}$.

In general, a tower in $\mathcal{C}_n$ is a subtower of a tower in $\mathcal{K}_n$. The cutting and stacking procedure tells us that the number of subtowers stacked to build a subtower Z of a tower $Y \in \mathcal{K}_n$ is the same as the number of subtowers that were stacked to build Y. In fact, because the splitting of $Y \in \mathcal{K}_n$ into towers in $\mathcal{C}_n$ is done after the stacking, the subtowers that are stacked to build $Z \in \mathcal{C}_n$ are sub-subtowers of the subtowers that are stacked to build the tower $Y \in \mathcal{K}_n$ that Z is cut from. This provides us with a way of associating the edges of $B_{\alpha,\beta}$ with edges in B_α.

We define a map $q : \Sigma_{\alpha,\beta} \to \Sigma_\alpha$ that undoes the procedure above that generates $B_{\alpha,\beta}$ from B_α. For $((J_n, j_n))_{n\geq 1} \in \Sigma_{\alpha,\beta}$ we set

$$q(((J_n, j_n))_{n\geq 1}) = ((K_n, k_n))_{n\geq 1}$$

where K_n is the vertex representative of the unique tower $Y_n^{(K_n)} \in \mathcal{K}_n$ for which $Z_n^{(J_n)}$ is a subtower of and $k_n = j_n$.

The discussion that preceded the definition of q establishes that for any edge (J_n, j_n) there exists only one edge (K_n, k_n) which satisfies the rule given by q. We remark that it also follows from the discussion above that the subtower associated with (J_n, j_n) is a sub-subtower of the subtower associated with (K_n, k_n). We now argue that the sequence of (K_n, k_n) forms a path in Σ_α.

Suppose the image $q(((J_n, j_n)_{n\geq 1})) = ((K_n, k_n))_{n\geq 1}$ is not in Σ_α. There are three ways this can happen for elements of $\prod_{n\geq 0} E_n$, where E_n are edges of B_α. The first is that there exists $N \geq 1$ such that $(K_N, k_N) = (L, m)$ and $(K_{N+1}, k_{N+1}) = (L, 0)$. This is not a path because the range of (L, m) is L while the source of $(L, 0)$ is S. By definition $Z_N^{(J_N)}$ would have to be a subtower of $Y_N^{(L)}$. However the subtower corresponding to (J_{N+1}, j_{N+1}) is a sub-subtower of the subtower represented by $(L, 0)$ and therefore a subtower of $Y_N^{(S)}$. Since $(\tau_N(J_{N+1}))_{j_{N+1}} = J_N$ it follows that $Z_N^{(J_N)}$ contains a subtower of $Y_N^{(S)}$. This is a contradiction because $Y_N^{(S)}$ and $Y_N^{(L)}$ are disjoint. Therefore $((J_n, j_n))_{n\geq 1}$ must fail to be a path.

The second way that $((K_n, k_n))_{n\geq 1}$ can fail to be a path is that there exists $N \geq 1$ such that $(K_N, k_N) = (S, 0)$ and $(K_{N+1}, k_{N+1}) \neq (L, 0)$. The range of $(S, 0)$ is S while $(L, 0)$ is the only edge with source S. Now $Z_N^{(J_N)}$ is a subtower of $Y_N^{(S)}$ while the subtower corresponding to (J_{N+1}, j_{N+1}) is a subtower of $Y_N^{(L)}$. Once again, $((J_n, j_n))_{n\geq 1}$ must fail to be a path. The third way assumes that the range of $(K_1, 0)$ does not match that source of (K_2, k_2). The same kind of argument can be applied to see that $((J_n, j_n))_{n\geq 1}$ must fail to be a path.

We conclude that if $((J_n, j_n))_{n\geq 1} \in \Sigma_{\alpha,\beta}$ it must be that $((K_n, k_n))_{n\geq 1} \in \Sigma_\alpha$.

What the function q does is glue back together the towers and subtowers that were split by β_1. We claim that q is a quotient map from $\Sigma_{\alpha,\beta}$ to Σ_α.

Definition 5.5.

Let X and Y be topological spaces and $f : X \to Y$ be a continuous function. The function f is called a **quotient map** if f is surjective and $U \subset Y$ is open if and only if $f^{-1}(U)$ is open in X.

We proceed under the assumption that Σ_α and $\Sigma_{\alpha,\beta}$ are endowed with the clopen topologies generated by their respective cylinder sets as in Section 3.2 in chapter 3. In particular, recall that these spaces are compact.

Proposition 5.17.

The function $q : \Sigma_{\alpha,\beta} \to \Sigma_\alpha$ is a quotient map.

Proof. We begin with continuity. Suppose $U = [(K_1, k_1), \ldots, (K_m, k_m)]$ is a cylinder set of paths in Σ_α. For $((J_n, j_n))_{n\geq 1}$ to be in $q^{-1}(U)$ it must be that for every n, $Z_n^{(J_n)}$ is a subtower of $Y_n^{(k_n)}$ and $j_n = k_n$ for $1 \leq n \leq m$. We know that this coincides with the statement: the subtower represented by (J_n, j_n) is a sub-subtower of (K_n, k_n) for $1 \leq n \leq m$. This occurs exactly when the floor of the tower $Z_m^{(J_m)}$ corresponding to the cylinder $[(J_1, j_1), \ldots, (J_m, j_m)]$ is a subfloor of the floor corresponding to $[(K_1, k_1), \ldots, (K_m, k_m)]$.

Therefore $q^{-1}(U) = \bigcup_{i\geq 1}[(J_1^{(i)}, j_1^{(i)}), \ldots, (J_m^{(i)}, j_m^{(i)})]$ where the union of the floors corresponding to the cylinders $[(J_1^{(i)}, j_1^{(i)}), \ldots, (J_m^{(i)}, j_m^{(i)})]$ is the floor corresponding to U. Therefore q is continuous.

For the proof of surjectivity recall that the cylinder sets are clopen in a compact space and are thus both closed and compact. For any path $p = ((K_n, k_n))_{n\geq 1} \in \Sigma_\alpha$, we have $q^{-1}(\{p\}) = \bigcap_{m\geq 1} q^{-1}([(K_1, k_1), \ldots (K_m, k_m)])$. The intersection is over a decreasing sequence of sets that are non-empty, closed, and compact. Lemma 3.9 from chapter 4 asserts that the intersection is non-empty. Therefore p has a non-empty preimage.

Lastly, observe that $q([(J_1, j_1), \ldots, (J_m, j_m)]) = [(K_1, k_1), \ldots, (K_m, j_m)]$ where $Z_n^{(J_n)}$ is a subtower of $Y_n^{(K_n)}$ for $1 \leq n \leq m$. Therefore the image of a cylinder set of $\Sigma_{\alpha,\beta}$ is a cylinder set of Σ_α. We will use this fact to show that q satisfies the open set condition of a quotient map. Suppose that $q^{-1}(U)$ is open in $\Sigma_{\alpha,\beta}$. Therefore it can be expressed as a union of cylinder sets of $\Sigma_{\alpha,\beta}$. Since q is surjective, we obtain that $U = q(q^{-1}(U))$ is a union of cylinder sets of Σ_α. Therefore U is open. The converse is achieved by the continuity of q. $\square$

Remark 5.18. We remark that the preimage of a cylinder set of Σ_α is either the union of two cylinder sets of $\Sigma_{\alpha,\beta}$ or a single cylinder set. Let $U = [(K_1, k_1), \ldots, (K_m, k_m)] \subset \Sigma_\alpha$ correspond to the ith floor of $Y_m^{(K_m)}$. If $Y_m^{(K_m)}$ does not contain β_1 then $Y_m^{(K_m)} = Z_m^{(K_m)}$ and $q^{-1}(U) = [(J_1, k_1), \ldots, (J_{m-1}, k_{m-1}), (K_m, k_m)]$, where this last cylinder is the address of the same floor, but with respect to $\Sigma_{\alpha,\beta}$. Otherwise $Y_m^{(K_m)}$ is split between the two towers $Z_m^{(K_m)}$ and $Z_m^{(M)}$ and the ith floor of $Y_m^{(K_m)}$ is the union of the ith floors of $Z_m^{(K_m)}$ and $Z_m^{(M)}$. In this case $q^{-1}(U) = [(J_1, k_1), \ldots, (J_{m-1}, k_{m-1}), (K_m, k_m)] \cup [(J_1', k_1), \ldots, (J_{m-1}', k_{m-1}), (M, k_m)]$ where M represents the middle tower.

The quotient map q relates the coding maps Γ and Ψ.

Proposition 5.19.

For every $x \in I_1$, we have $(q \circ \Psi)(x) = \Gamma(x)$.

Proof. Let (Q_n, q_n) be the nth edge of $(q \circ \Psi)(x)$, (J_n, j_n) be the nth edge of $\Psi(x)$, and (K_n, k_n) be the nth edge of $\Gamma(x)$.

By the definition of q, the tower $Z_n^{(J_n)} \in \mathcal{C}_n$ containing x is a subtower of $Y_n^{(Q_n)} \in \mathcal{K}_n$. Therefore $Y_n^{(Q_n)}$ contains x. It follows from the definition of Γ that $Y_n^{(K_n)}$ contains x. Since the tower in $\mathcal{K}_n$ containing x is unique we have $K_n = Q_n$.

Furthermore, $Z_n^{(J_n)}$ is then a subtower of $Y_n^{(K_n)}$ and thus the j_nth subtower that is stacked to build $Z_n^{(J_n)}$ is a sub-subtower of the j_nth subtower that is stacked to build $Y_n^{(K_n)}$. It is evident from the cutting and stacking procedure that the subtowers that are stacked are disjoint. Therefore the j_nth subtower that is stacked to build $Y_n^{(K_n)}$ is the unique subtower that contains x. It follows that $k_n = j_n = q_n$. □

We are now able to prove the following assertion.

Corollary 5.20.

The coding map $\Psi : I_1 \to \Sigma_{\alpha,\beta}$ *is not surjective.*

Proof. Recall that the coding map $\Gamma : I_1 \to \Sigma_\alpha$ is never surjective for any parameter α (5.15). If Ψ was surjective onto $\Sigma_{\alpha,\beta}$ for some parameters α and β then the surjectivity of $q : \Sigma_{\alpha,\beta} \to \Sigma_\alpha$ implies that $q \circ \Psi : I_1 \to \Sigma_\alpha$ is surjective. Due to the proposition we just proved, this is equivalent to stating that Γ is surjective, which is a contradiction. It must be that Ψ is never surjective for any choice of parameters. □

We conclude this chapter by discussing Bratteli-Vershik systems on the path spaces Σ_α and $\Sigma_{\alpha,\beta}$.

The ordering on B_α given by Proposition 5.14 implies that Σ_α contains two minimal paths and one maximal path. We cannot place a Vershik map on this path space. A similar phenomenon was seen for a particular instance of $B_{\alpha,\beta}$. Let $\Sigma_\alpha(\max)$ and $\Sigma_\alpha(\min)$ denote the sets of maximal and minimal paths in Σ_α respectively. We can adopt similar notation for $\Sigma_{\alpha,\beta}$ and define "near-Vershik" maps

$$\varphi_\alpha : \Sigma_\alpha \setminus \Sigma_\alpha(\max) \to \Sigma_\alpha \setminus \Sigma_\alpha(\min)$$

and

$$\varphi_{\alpha,\beta} : \Sigma_{\alpha,\beta} \setminus \Sigma_{\alpha,\beta}(\max) \to \Sigma_{\alpha,\beta} \setminus \Sigma_{\alpha,\beta}(\min)$$

such that φ_α and $\varphi_{\alpha,\beta}$ satisfy property (iii) from Definition 3.6 in chapter 3. These functions intertwine with the quotient map q.

Proposition 5.21.

Let $p \in \Sigma_{\alpha,\beta} \setminus \Sigma_{\alpha,\beta}(max)$. Then $(q \circ \varphi_{\alpha,\beta})(p) = (\varphi_\alpha \circ q)(p)$.

Proof. Observe that the image $q(p)$ cannot be maximal if p is not maximal. Therefore $\varphi_\alpha \circ q$ is well-defined. Let $p = ((J_n, j_n))_{n\geq 1}$ and $q(p) = ((K_n, k_n))_{n\geq 1}$. Suppose (J_m, j_m) is the first sub-maximal edge of p. We have that (J_n, j_n) is maximal if and only if (K_n, k_n) is maximal because $j_n = k_n$. It follows that (K_m, k_m) is the first sub-maximal edge of $q(p)$.

By definition $\varphi_{\alpha,\beta}(p)$ has all the same edges as p for $n > m$ and the mth edge is $(J_m, j_m + 1)$. Therefore the edges of $q(\varphi_{\alpha,\beta}(p))$ for $n > m$ are the same as $q(p)$ and the mth edge is $(K_m, j_m + 1) = (K_m, k_m)$. We have

$$q(\varphi_{\alpha,\beta}(p)) = ((Q_1, 0), \ldots, (Q_{m-1}, 0), (K_m, k_m + 1), (K_{m+1}, k_{m+1}), (K_{m+2}, k_{m+2}), \ldots)$$

where $((Q_1, 0), \ldots, (Q_{m-1}, 0))$ is the minimal path to the source of $(K_m, k_m + 1)$. This holds because the first $m-1$ edges of $\varphi_{\alpha,\beta}(p)$ form the minimal path from the root to the source of $(J_m, j_m + 1)$.

By definition

$$\varphi_\alpha(q(p)) = ((K_1^{(0)}, 0), \ldots, (K_{m-1}^{(0)}, 0), (K_m, k_m + 1), (K_{m+1}, k_{m+1}), (K_{m+2}, k_{m+2}), \ldots)$$

where $((K_1^{(0)}, 0), \ldots, (K_{m-1}^{(0)}, 0))$ is the minimal path from the root to the source of $(K_m, k_m + 1)$. Since this finite path is unique we have $((K_1^{(0)}, 0), \ldots, (K_{m-1}^{(0)}, 0)) = ((Q_1, 0), \ldots, (Q_{m-1}, 0))$. Therefore $\varphi_\alpha(q(p)) = q(\varphi_{\alpha,\beta}(p))$. $\square$

This concludes the dynamical portion of the thesis.

Chapter 6

Dimension Groups

Bratteli diagrams originated in the field of operator algebras. They were introduced by Ola Bratteli in [3] as a way of modeling approximately finite dimensional C*-algebras (AF algebras). It was later discovered by George Elliott in [11] that the incidence matrices of a Bratteli diagram could be used to form an algebraic invariant that classified AF-algebras up to isomorphism. Elliott called these objects dimension groups. In [13] it was shown that the dimension group is a dynamical invariant that classifies Bratteli-Vershik systems derived from Cantor minimal systems (see Definition 4.1 in chapter 4) up to strong orbit equivalence.

Definition 6.1.

Topological dynamical systems (X, T) and (Y, S) are **orbit equivalent** if there exists a homeomorphism $h : X \to Y$ such that for every $x \in X$, we have $h(\mathrm{orb}_T(x)) = \mathrm{orb}_S(h(x))$.

For this to be true, there must exist functions $p, q : X \to \mathbb{N}$ such that $(h \circ T)(x) = (S^{p(x)} \circ h)(x)$ and $(h \circ T^{q(x)})(x) = (S \circ h)(x)$. The functions are called cocycles ([13]). If both cocycles are continuous everywhere with the exception of at most one point, then the systems (X, T) and (Y, S) are **strong orbit equivalent**. Topological conjugacy (Definition 4.2) is a special case (and stronger form) of strong orbit equivalence where both of the cocycles are the constant function 1.

This chapter will explore the dimension groups of $B_{\alpha,\beta}$. This is the Bratteli diagram derived from the cutting and stacking procedure from chapter 5. See Proposition 5.8 to recall the substitutions that define these diagrams.

6.1 The Basics of Dimension Groups

We begin with the notion of an ordered group. An **ordered group** is an abelian group G with a subset G^+ such that $G^+ + G^+ \subset G^+$, $G^+ - G^+ = G$, and $G^+ \cap (-G^+) = \{0\}$. We call G^+ the **positive cone**. It places the following partial ordering on G: given elements $a, b \in G$, we have $a \leq b$ whenever $b - a \in G^+$. An element u in an ordered group G is called an **order unit** if for every $a \in G^+$ there exists $n \geq 1$ such that $a \leq nu$.

Definition 6.2.

Let $(G_i)_{i\geq 1}$ be a countable sequence of abelian groups (not necessarily ordered) and suppose that $(\varphi_{ij})_{1\leq i<j}$ is a countable collection of group homomorphisms such that $\varphi_{ij} : G_i \to G_j$ and for any $i < j < k$ we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. We call the sequence of groups and the maps between them a **directed system** of groups and denote it by (G_i, φ_{ij}).

When helpful, we will alternatively represent a directed system (G_i, φ_{ij}) by the chain

$$G_1 \xrightarrow{\varphi_{12}} G_2 \xrightarrow{\varphi_{23}} G_3 \xrightarrow{\varphi_{34}} \cdots .$$

Dimension groups are formed by taking the inductive limit of a directed system of ordered groups. In order to define the inductive limit of a directed system we introduce two intermediate constructions.

We first form the set $\Sigma_i G_i := \{(a,i) \mid a \in G_i, i \geq 1\}$. This is different from a union because if there exists $a \in G_i \cap G_j$ for $i \neq j$, then $(a,i) \neq (a,j)$. Next, we place the following equivalence relation $\sim$ on $\Sigma_i G_i$.

We define $(a,i) \sim (b,j)$ whenever there exists $k > \max\{i,j\}$ such that $\varphi_{ik}(a) = \varphi_{jk}(b)$. In other words, two elements are equivalent if their images coincide after some number of steps along the directed system. The relation $\sim$ is clearly reflexive and symmetric. We argue that $\sim$ is transitive. Suppose $(a,i) \sim (b,j)$ and $(b,j) \sim (c,l)$. Let k_1 and k_2 be such that $\varphi_{ik_1}(a) = \varphi_{jk_1}(b)$ and $\varphi_{jk_2}(b) = \varphi_{lk_2}(c)$. If we choose $k > \max\{k_1,k_2\}$ then

$$\begin{aligned}
\varphi_{ik}(a) &= \varphi_{k_1k}(\varphi_{ik_1}(a)) \\
&= \varphi_{k_1k}(\varphi_{jk_1}(b)) \\
&= \varphi_{jk}(b) \\
&= \varphi_{k_2k}(\varphi_{jk_2}(b)) \\
&= \varphi_{k_2k}(\varphi_{lk_2}(c)) = \varphi_{lk}(c).
\end{aligned}$$

Therefore $(a,i) \sim (c,l)$.

Definition 6.3.

Let (G_i, φ_{ij}) be a directed system of abelian groups. The **inductive limit** of the directed system is the set $\varinjlim(G_i, \varphi_{ij}) := \Sigma_i G_i / \sim$.

In order to simplify the notation we will denote the equivalence class with representative (a, i) by $[a, i]$ as opposed to $[(a, i)]$.

Inductive limits can be defined more generally for uncountable directed systems of arbitrary sets as long as there is an ordering on the index set and the maps between the sets satisfy the composition property. The inductive limit of a directed system of abelian groups can be made into an abelian group by endowing it with the following binary operation.

Let $[a, i], [b, j] \in \overrightarrow{\lim}(G_i, \varphi_{ij})$. Without loss of generality $i \leq j$. Then $[a, i]+[b, j] :=$ $[\varphi_{ij}(a) + b, j]$ if $i < j$ and $[a + b, j]$ if $i = j$. We check that this operation is well-defined. Suppose $[a_1, i_1] = [a_2, i_2]$ and $[b_1, j_1] = [b_2, j_2]$ such that $i_1 < j_1$ and $i_2 < j_2$. The other cases follow similarly. There exists k and l such that $\varphi_{i_1k}(a_1) = \varphi_{i_2k}(a_2)$ and $\varphi_{j_1k}(b_1) = \varphi_{j_2k}(b_2)$. Let $m > \max\{l, k\}$ and we obtain

$$\begin{aligned}
[a_1, i_1] + [b_1, j_1] &= [\varphi_{i_1j_1}(a_1) + b_1, j_1] \\
&= [\varphi_{i_1m}(a_1) + \varphi_{j_1m}(b_1), m] \\
&= [\varphi_{km}(\varphi_{i_1k}(a_1)) + \varphi_{lm}(\varphi_{j_1l}(b_1)), m] \\
&= [\varphi_{km}(\varphi_{i_2k}(a_2)) + \varphi_{lm}(\varphi_{j_2l}(b_2)), m] \\
&= [\varphi_{i_2m}(a_2) + \varphi_{j_2m}(b_2), m] \\
&= [\varphi_{i_2j_2}(a_2) + b_2, j_2] = [a_2, i_2] + [b_2, j_2]
\end{aligned}$$

The commutativity of the operation follows from the fact that G_i is an abelian group for each i. Let 0_i denote the identity element of G_i. The identity element of the inductive limit is the equivalence class $[0_1, 1]$. To see this, observe that $(0_i, i) \sim (0_j, j)$ for all $i \leq j$ because φ_{ij} is a homomorphism. It follows that the inverse of $[a, i]$ is $[-a, i]$. We demonstrate the associativity of this operation. Suppose $i < k < j$. The cases where any of these indices are equal follow similarly. We see that

$$\begin{aligned}([a,i]+[b,j])+[c,k] &= [\varphi_{ij}(a)+b,j]+[c,k]\\ &= [\varphi_{jk}(\varphi_{ij}(a)+b)+c,k]\\ &= [\varphi_{ik}(a)+\varphi_{jk}(b)+c,k]\\ &= [a,i]+[\varphi_{jk}(b)+c,k]\\ &= [a,i]+([b,j]+[c,k]).\end{aligned}$$

We will now refer to the inductive limit of a directed system of abelian groups as the limit group of the directed system.

The limit group of a directed system of ordered groups can be shown to be an ordered group if we assume the maps φ_{ij} preserve the positivity of the elements in their domain. Homomorphisms with this property are called **positive**. In symbols, the homomorphism $\varphi : G_1 \to G_2$ is positive if $\varphi(G_1^+) \subset G_2^+$. We say that the ordered groups G_1 and G_2 are **order isomorphic** if φ is a group isomorphism and $\varphi(G_1^+) = G_2^+$.

Proposition 6.1.

Let (G_i, φ_{ij}) be a directed system of ordered groups such that the homomorphisms $(\varphi_{ij})_{1\le i<j}$ are positive and let $H = \overrightarrow{\lim}(G_i, \varphi_{ij})$ be the limit group of the directed system.

The group H is an ordered group with positive cone

$$H^+ := \{(a,i) \in H \mid \varphi_{ij}(a) \in G_j^+ \ \textit{for some}\ j > i\}.$$

Proof. We have already established that H is an abelian group. We need only check the following three conditions to show that H is an ordered group.

(i) $H^+ + H^+ \subset H^+$

Let $[a,i],[b,j] \in H^+$. By definition there exist k_1 and k_2 such that $\varphi_{ik_1}(a) \in G^+_{k_1}$ and $\varphi_{jk_2}(a) \in G^+_{k_2}$. Let $k = \max\{k_1,k_2\}$. It follows that

$$[a,i]+[b,j] = [\varphi_{ik_1}(a),k_1]+[\varphi_{jk_2}(b),k_2] = [\varphi_{ik}(a),k]+[\varphi_{jk}(b),k] = [\varphi_{ik}(a)+\varphi_{jk}(b),k].$$

Since $\varphi_{ik}(a), \varphi_{jk}(b) \in G^+_k$ and G_k is an ordered group we have that $\varphi_{ik}(a) + \varphi_{jk}(b) \in G^+_k$. We obtain $[a,i]+[b,j] \in H^+$ because the equivalence class has a positive representative. This is sufficient because the homomorphisms in the directed system preserve positivity.

(ii) $H^+ - H^+ = H$

The inclusion $H^+ - H^+ \subset H$ follows immediately because H is a group. We show that $H^+ - H^+ \supset H$. Let $[a,i] \in H$. Since G_i is an ordered group and $a \in G_i$, there exists $a_1, a_2 \in G^+_i$ such that $a = a_1 - a_2$. It follows that $[a,i] = [a_1,i] - [a_2,i]$ where $[a_1,i],[a_2,i] \in H^+$.

(iii) $H^+ \cap (-H^+) = \{[0_1,1]\}$

Let $[a,i] \in H^+ \cap (-H^+)$. This means that $[a,i],[-a,i] \in H^+$. We may choose k such that $\varphi_{ik}(a), \varphi_{ik}(-a) \in G^+_k$. However, $\varphi_{ik}(-a) = -\varphi_{ik}(a)$. This implies that $\varphi_{ik}(a) \in G^+_k \cap (-G^+_k)$. Therefore $\varphi_{ik}(a) = 0_k$ and thus $[a,i] = [0_k,k] = [0_1,1]$. $\square$

The ordered groups we consider in this chapter are all of the form $\mathbb{Z}^n$ for some $n \geq 1$ where $(\mathbb{Z}^n)^+ = \{(a_1,\ldots,a_n) \in \mathbb{Z}^n \mid a_i \geq 0, i = 1,\ldots,n\}$. This positive cone is called the **simplicial ordering** on $\mathbb{Z}^n$. We highlight that all homomorphisms $\varphi : \mathbb{Z}^n \to \mathbb{Z}^m$ can be expressed as m by n matrices with integer coefficients. We also comment that any element of the form $\sum_{i=1}^{n} c_i e_i$, where $e_1, \ldots, e_n$ are the generators of the group and c_i is a strictly positive integer for each i, is an order unit with respect to the simplicial ordering.

Definition 6.4.

The limit group of a directed system of simplicially ordered groups

$$\mathbb{Z}^{n_1} \xrightarrow{M_1} \mathbb{Z}^{n_2} \xrightarrow{M_2} \mathbb{Z}^{n_3} \xrightarrow{M_3} \cdots$$

where each matrix M_i has non-negative entries is called a **dimension group**.

If $n_i = n$ for all $i \geq 1$ in the definition above, the directed system is called an **Elliott system**([8]). An Elliott system is called **stationary** if $M_i = M$ for every $i \geq 1$. We provide the following elementary example of a dimension group.

Example 6.2. Consider the directed system

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{f} \cdots .$$

where $f(x) = 2x$. Using the isomorphism $\Phi([n,i]) = \frac{n}{2^{i-1}}$ we have that $\overrightarrow{\lim}(\mathbb{Z}, f)$ is order isomorphic to the set of dyadic rationals with positive cone equal to all non-negative dyadic rationals.

Given a Bratteli diagam $B = (V, E)$ we can form the directed system

$$\mathbb{Z} = \mathbb{Z}^{|V_0|} \xrightarrow{A_0} \mathbb{Z}^{|V_1|} \xrightarrow{A_1} \mathbb{Z}^{|V_2|} \xrightarrow{A_2} \cdots$$

where the matrices $A_n : \mathbb{Z}^{|V_n|} \to \mathbb{Z}^{|V_{n+1}|}$ are the incidence matrices of B and each group is endowed with the simplicial ordering. Incidence matrices are positive homomorphisms with respect to the simplicial ordering because the entries are always non-negative. The dimension group formed by taking the inductive limit of this directed system is the algebraic invariant for AF-algebras discovered by Elliott. Following [17], we will denote this group by $K_0(V, E)$.

The combined results of Herman, Giordano, Putnam, and Skau, presented in [17] and [13], assert that two Cantor minimal systems are strong orbit equivalent if and only if the dimension groups are order isomorphic by a map that preserves what they call the distinguished order unit. This element of $K_0(V, E)$ is the equivalence class with representative $1 \in \mathbb{Z}$. Consider the nth level V_n. The free abelian group generated by the elements of V_n is isomorphic to $\mathbb{Z}^{|V_n|}$. From this perspective, the representative 1 is identified with the root of the Bratteli diagram v_0.

Although we do not provide a precise treatment of the invariance theorems anywhere in this document (or orbit equivalence), we will take the distinguished order unit into account. To see that the equivalence class $[1, 1]$ is an order unit, observe that the first image of 1 along the directed system is a vector with strictly positive entries. This is because for every $v \in V_1$ there is at least one edge with source v_0 and range v. Furthermore, since the row of every incidence matrix has at least one non-zero element, all subsequent images along the directed system have strictly positive entries. Therefore all the representatives of the equivalence class $[1, 1]$ are order units in their respective groups. If $[v, i] \in K_0(V, E)^+$ then, since $u = A_{i-2} \cdots A_0$ is an order unit in $\mathbb{Z}^{|V_{i-1}|}$, there exists n such that $n[u, i] - [v, i] \in K_0^+(V, E)$. Therefore $[v, i] \leq n[1, 1]$.

Let $B = (V, E)$ be a Bratteli diagram. Up to a change in order unit, we argue that we can work with the dimension group H formed by the shifted directed system

$$\mathbb{Z}^{|V_1|} \xrightarrow{A_1} \mathbb{Z}^{|V_2|} \xrightarrow{A_2} \cdots$$

in place of

$$\mathbb{Z} \xrightarrow{A_0} \mathbb{Z}^{|V_1|} \xrightarrow{A_1} \mathbb{Z}^{|V_2|} \xrightarrow{A_2} \rightarrow \cdots .$$

Let us define the map $g : K_0(V, E) \to H$ by

$$g([v,i]) = \begin{cases} [v, i-1] & \text{if } i > 1, \\ [A_0 v, 1] & \text{if } i = 1. \end{cases}$$

It is clear that this map is surjective. We will argue that it is well-defined, injective, and preserves the binary operation for pairs of elements $[v, 1], [w, j] \in K_0(V, E)$ where $j > 1$. The other pairs follow similarly.

If $[v, 1] = [w, j]$ then there exists k such that $A_k \cdots A_1(A_0 v) = A_k \cdots A_{j-1} w$. This is precisely what is means for $[A_0 v, 1] = [w, j-1]$ in H. Therefore g is well defined. On the other hand if $g([v, 1]) = [A_0 v, 1] = [w, j-1] = g([w, j])$ in H then $[A_0 v, 2] = [w, j]$ in $K_0(V, E)$. Therefore $[v, 1] = [w, j]$ and we conclude that g is injective.

The image of the $[v, 1] + [w, j]$ under g is

$$\begin{aligned} g([A_{j-2} \cdots A_0 v + w, j]) &= [A_{j-2} \cdots A_0 v + w, j-1] \\ &= [A_{j-2} \cdots A_1(A_0 v), j-1] + [w, j-1] \\ &= [A_0 v, 1] + [w, j-1] \\ &= g([v, 1]) + g([w, j]) \end{aligned}$$

Lastly we establish $g(K_0(V,E)^+) = H^+$. If $[v,i] \in H^+$ then by definition there exists $k > i$ such that $A_{k-1} \cdots A_i v \in (\mathbb{Z}^{|V_k|})^+$. This is equivalent to $[v, i+1] \in K_0(V,E)^+$. Therefore $g([v, i+1]) \in H^+$ if and only if $[v, i+1] \in K_0(V,E)^+$. If $[v,1] \in K_0(V,E)^+$ then $A_{k-1} \cdots A_0 v \in (\mathbb{Z}^{|V_k|})^+$ for some k. This means $[A_0 v, 1] \in H^+$.

We choose the distinguished order unit of H to be the equivalence class represented by the column vector $A_0 \in \mathbb{Z}^{|v_1|}$. This ensures that any Bratteli-Vershik system on the diagram with the dimension group H is strong orbit equivalent to a system on the diagram with the dimension group $K_0(V,E)$. Although $\mathbb{Z}^{|V_0|}$ is removed, it is not forgotten. The information is transferred via A_0. If we shifted the directed system over m spots we would need to choose the representative $A_{m-1} \cdots A_0$. When this product is interpreted as the first (zeroth) incidence matrix of a Bratteli diagram followed by the remaining incidence matrices of the original diagram B, we obtain the telescoping $(0, m, m+1, m+2, \ldots)$ of B. In other words, when we factor in the distinguished order unit, the associated Bratteli diagrams are equivalent up to telescoping. We will see in Section 6.4 how different choices for the distinguished order unit of a dimension group affect the underlying Bratteli diagram. We proceed as if $K_0(V,E) = H$.

The rest of this chapter is devoted to computing dimension groups associated with $B_{\alpha,\beta}$. For every vertex level V_n we have $|V_n| = 3$. Therefore every directed system of ordered groups associated with $B_{\alpha,\beta}$ is an Elliott system. The following proposition partially characterizes the dimension groups of $B_{\alpha,\beta}$.

Proposition 6.3.

Let $B_{\alpha,\beta} = (V,E)$. There exists a group isomorphism $\Phi : K_0(V,E) \to \mathbb{Z}^3$ such that $\Phi\big(K_0^+(V,E)\big) = P := \{v \in \mathbb{Z}^3 \mid A_N \cdots A_1 v \in (\mathbb{Z}^3)^+ \text{ for some } N \geq 1\}$.

Proof. We first claim that for every $n \geq 1, |\det A_n| = 1$. We can prove this by computing only three determinants because for each possible scenario (see Definition 5.2 and Proposition 5.8) we know the general form of the incidence matrix. The following Laplace expansions are computed along the bottom row.

Scenario G: $\det \begin{pmatrix} b_n - 1 & a_n - b_n + 1 & 1 \\ b_n & a_n - b_n & 1 \\ 0 & 1 & 0 \end{pmatrix} = -1((b_n - 1)(1) - (1)(b_n)) = 1$

Scenario B_1: $\det \begin{pmatrix} a_n & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1(0 - (1)(1)) = -1$

Scenario B_2: $\det \begin{pmatrix} a_n & 0 & 1 \\ a_n & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1(0 - (1)(1)) = -1$

Therefore the incidence matrices are a sequence of automorphisms.

Let $[v, i] \in K_0(V, E)$. We define the isomorphism Φ by $\Phi([v, i]) = A_1^{-1} \cdots A_{i-1}^{-1} v$ where A_0 is the identity matrix.

We first verify that this map is well defined. Suppose that $[v, i] = [w, j]$ and without loss of generality that $i \leq j$. By definition there exists $k \geq j$ such that $A_{k-1} \cdots A_{i-1} v = A_{k-1} \cdots A_{j-1} w$. Using the fact that each incidence matrix is invertible we can multiply by $A_1^{-1} \cdots A_{k-1}^{-1}$ on both sides to obtain $A_1^{-1} \cdots A_{i-1}^{-1} v = A_1^{-1} \cdots A_{j-1}^{-1} w$. Therefore $\Phi([v, i]) = \Phi([w, j])$. Injectivity follows in a similar way.

If $\Phi([v, i]) = \Phi([w, j])$, then $A_1^{-1} \cdots A_{i-1}^{-1} v = A_1^{-1} \cdots A_{j-1}^{-1} w$ and $A_{j-1} \cdots A_i v = w$. This implies that $[v, i] = [w, j]$. Lastly, we demonstrate surjectivity. Given $v \in \mathbb{Z}^3$ we can choose $[v, 1]$ to obtain $\Phi([v, 1]) = v$. Therefore Φ is bijective.

Now we verify that Φ preserves the group operation. Let $[v, i], [w, j] \in K_0(V, E)$

and observe that for $i < j$

$$\begin{aligned}\Phi([v,i]+[w,j]) &= \Phi([A_{j-1}\cdots A_i v + w, j]) \\ &= A_1^{-1}\cdots A_{j-1}^{-1}(A_{j-1}\cdots A_i v + w) \\ &= A_1^{-1}\cdots A_{i-1}^{-1}v + A_1^{-1}\cdots A_{j-1}^{-1}w = \Phi([v,i]) + \Phi([w,j]).\end{aligned}$$

The case when $i = j$ follows similarly. All that remains is the assertion that the image of the positive elements in the limit group are the elements in P. If $[v,i] \in K_0^+(V,E)$ then there exists k such that $A_k \cdots A_i v \in (\mathbb{Z}^3)^+$. The image $\Phi([v,i]) = A_1^{-1}\cdots A_{i-1}^{-1}v \in P$ because $A_k \cdots A_1(A_1^{-1}\cdots A_{i-1}^{-1}v) = A_k \cdots A_i v \in (\mathbb{Z}^3)^+$.

Conversely if $v \in P$ then there exists N such that $A_N \cdots A_1 v \in (\mathbb{Z}^3)^+$. It follows by definition of the positive cone of the limit group and the map Φ that $[v,1] \in K_0^+(V,E)$ and $\Phi([v,1]) = v$. This completes the proof. $\square$

Remark 6.4. A nearly identical proof to the one above shows that given an Elliott system $(\mathbb{Z}^n, A_n)$, if for every n, $|\det A_n| = 1$ and A_n is positive, then the limit group is isomorphic to $\mathbb{Z}^n$ where the positive cone is the set of elements in $\mathbb{Z}^n$ that are positive with respect to the simplicial ordering after a finite number of steps along the directed system.

The proposition that we just proved reduces the problem of finding dimension groups of $B_{\alpha,\beta}$ to finding positive cones. Furthermore, it tells us that if e_1, e_2, and e_3 are the standard generators of $\mathbb{Z}^3$, then $K_0^+(V,E) \cong \bigcup_{n\geq 0}\{\sum_{i=1}^3 c_i(A_1^{-1}\cdots A_{n-1}^{-1}e_i) \mid c_i \in (\mathbb{Z})^+\}$. This is an increasing sequence of unions.

From this it can be seen that if $B'_{\alpha,\beta} = (V', E')$ is obtained by telescoping $B_{\alpha,\beta}$, then $K_0(V',E')$ and $K_0(V,E)$ are order isomorphic. The limit groups are the same because the composition of incidence matrices are still automorphisms on $\mathbb{Z}^3$. The positive cones do not change because a single step along the telescoped directed system corresponds to one or more steps along the original directed system. Any element (in this context) that is positive after N steps along a directed system will be positive after $K > N$ steps along the same directed system.

We present two examples associated with particular instances of $B_{\alpha,\beta}$. For these examples, when we speak of the dimension group of $B_{\alpha,\beta}$ we mean $\mathbb{Z}^3$ where the positive cone consists precisely of those elements which are eventually positive along the directed system of $(\mathbb{Z}^3, A_n)$ with respect to the simplicial ordering. The distinguished order unit is $e_1 + e_2 + e_3 = (1, 1, 1)$.

6.2 A Stationary Example

We let $\alpha_1 = [0; 2, 2, 2, \ldots]$ and β_1 be such that $s_n = G$ and $b_n = 1$ for all $n \geq 1$. This is the set up from Example 5.12 in the previous chapter, however we do not require any of the information we derived from that example. The Bratteli diagram $B_{\alpha,\beta}$ provides us with the stationary Elliott system $(\mathbb{Z}^3, A_G)$ where

$$A_G = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This matrix has the eigenbasis

$$\left\{ v_1 = \begin{pmatrix} 1+\sqrt{2} \\ 1+\sqrt{2} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1-\sqrt{2} \\ 1-\sqrt{2} \\ 1 \end{pmatrix} \right\}$$ with respective eigenvalues $\lambda_1 = 1+\sqrt{2}$, $\lambda_2 = -1$, and $\lambda_3 = 1-\sqrt{2}$. We draw attention to the limiting behaviour

of $|\lambda_i|^k$ as k gets arbitrarily large for each i:

(i) $|\lambda_1|^k = (1+\sqrt{2})^k \xrightarrow{k\to\infty} \infty$

(ii) $|\lambda_2|^k = |-1|^k = 1$

(iii) $|\lambda_3|^k = (\sqrt{2}-1)^k \xrightarrow{k\to\infty} 0$

Therefore, for any $w \in \mathbb{Z}^3$, we have $A_G^k w = \Sigma_{i=1}^3 \lambda_i^k c_i v_i$ where $c_i \in \mathbb{R}$. To achieve $A_G^k w \in (\mathbb{Z}^3)^+$ for some $k \geq 1$ we must have

$(A_G^k w)_1 = (1+\sqrt{2})^{k+1} c_1 + (-1)^k c_2 + (1-\sqrt{2})^{k+1} c_3 \geq 0,$

$(A_G^k w)_2 = (1+\sqrt{2})^{k+1} c_1 + (-1)^{k+1} c_2 + (1-\sqrt{2})^{k+1} c_3 \geq 0$, and

$(A_G^k w)_3 = (1+\sqrt{2})^{k} c_1 + (-1)^k c_2 + (1-\sqrt{2})^{k} c_3 \geq 0.$

Moreover, since A_G is a positive map, it must be that these inequalities hold for all $m \geq k$. Given the limit behaviour of (i) we see that this is achieved whenever $c_1 > 0$. Similarly, it will never be achieved if $c_1 < 0$. Let us examine the cases when $c_1 = 0$. The inequalities are then

$(A_G^k w)_1 = (-1)^k c_2 + (1-\sqrt{2})^{k+1} c_3 \geq 0,$

$(A_G^k w)_2 = (-1)^{k+1} c_2 + (1-\sqrt{2})^{k+1} c_3 \geq 0$, and

$(A_G^k w)_3 = (-1)^k c_2 + (1-\sqrt{2})^{k} c_3 \geq 0.$

If $c_2 = 0$ and $c_3 \neq 0$ then $(A_G^k w)_1 > 0$ if and only if $(A_G^k w)_3 < 0$. These coefficients will never yield positivity. If $c_2 \neq 0$, the it follows from the limiting behaviour of (ii) and (iii) above that there exists sufficiently large K such that whenever $k \geq K$ we have $(A_G^k w)_1 > 0$ if and only if $(A_G^k w)_2 < 0$. This leaves us with only two choices of coefficients: either $c_1 > 0$ or $c_1 = c_2 = c_3 = 0$.

If we want to express these conditions explicitly in terms of elements of $\mathbb{Z}^3$ we must solve the following system of equations for c_1:

$$x = (1+\sqrt{2})c_1 + c_2 + (1-\sqrt{2})c_3$$
$$y = (1+\sqrt{2})c_1 - c_2 + (1-\sqrt{2})c_3$$
$$z = c_1 + c_2 + c_3.$$

We find that $c_1(1+\frac{1+\sqrt{2}}{\sqrt{2}-1}) = \frac{x}{2}(\frac{1}{\sqrt{2}-1}-1)+\frac{y}{2}(\frac{1}{\sqrt{2}-1}+1)+z$. The left hand side is greater than zero exactly when $c_1 > 0$ and thus we have $c_1 > 0$ if and only if $(x, y, z) \cdot p > 0$ where $p = (\frac{1}{2}(\alpha_1+1), \frac{1}{2}(\alpha_1+3), 1)$.

To summarize, the positive cone is $\{v \in \mathbb{Z}^3 \mid v \cdot p > 0\} \cup \{(0,0,0)\}$. Since we did not telescope at all in this example, we maintain the distinguished order unit $(1, 1, 1)$.

6.3 A Non-Stationary Example

Let $\alpha_1 = \phi - 1$ where $\phi = (1+\sqrt{5})/2$, the golden ratio. This means $\alpha_1 = [0; 1, 1, \ldots]$. Suppose that we have chosen a β_1 which produces the itinerary

$$G(B_1B_2)G(B_1B_2)^2G(B_1B_2)^3G\cdots.$$

Our first step is a move to the telescoping $(0, 1, 4, 9, 16, 25, \ldots)$. This can be expressed by the following partition of the itinerary:

$$G(B_1B_2)\Big|G(B_1B_2)^2\Big|G(B_1B_2)^3\Big|G\cdots.$$

As an aside, we can capture β_1 by deriving its Ostrowski expansion from the itinerary. A simpler case was demonstrated in Example 5.12 in chapter 5. The parameter β_1 is approximately -0.78449 and β is approximately 0.13319.

Let $A_G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $A_B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

The matrix A_G is the incidence matrix corresponding to the G scenario where $a_n \equiv b_n \equiv 1$. The matrix A_B is the product of the incidence matrices corresponding to B_1 and B_2 where $a_n \equiv 1$. Since the itinerary reads $(B_1 B_2)$ the matrix A_B is computed by multiplying the B_1 matrix on the left by the B_2 matrix. Let $D_k = A_B^k A_G$. The Elliott system corresponding to the telescoping is

$$\mathbb{Z}^3 \xrightarrow{D_1} \mathbb{Z}^3 \xrightarrow{D_2} \mathbb{Z}^3 \xrightarrow{D_3} \cdots .$$

We present the dimension group of this system in the following theorem.

Theorem 6.5.

Let ϕ be the golden ratio and D_k be defined as it was above. The dimension group of the Elliott system $(\mathbb{Z}^3, D_k)$ is order isomorphic to $\mathbb{Z}^3$ with positive cone

$$P_\xi = \{u \in \mathbb{Z}^3 \mid u \cdot (\xi, \phi - \xi, 1) > 0\} \cup \{(0,0,0)\} \ \textit{where}\ \xi = (1+\phi) \sum_{i=1}^{\infty} \left(\frac{-1}{\phi}\right)^{(i+1)^2}$$

Observe that the infinite series defining ξ is an alternating series whose terms tend to zero. Therefore the series converges and ξ is a real number. In order to prove the theorem, we begin by establishing a formula for D_k.

Lemma 6.6.

Let $(F_i)_{i \geq 1}$ denote the Fibonacci sequence. That is $F_1 = F_2 = 1$ and $F_i = F_{i-1} +$

F_{i-2} for all $i \geq 3$. Then for all $k \geq 1$

$$D_k = \begin{pmatrix} 0 & F_{2k+2} & F_{2k+1} \\ 1 & F_{2k+2} - 1 & F_{2k+1} \\ 0 & F_{2k+1} & F_{2k} \end{pmatrix}$$

Proof. We proceed by induction. Observe that

$$D_1 = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & F_4 & F_3 \\ 1 & F_4 - 1 & F_3 \\ 0 & F_3 & F_2 \end{pmatrix}.$$

Let us assume that the lemma statement holds for some k. Then

$$\begin{aligned} D_{k+1} &= A_B^{k+1} A_G \\ &= A_B D_k \\ &= \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & F_{2k+2} & F_{2k+1} \\ 1 & F_{2k+2} - 1 & F_{2k+1} \\ 0 & F_{2k+1} & F_{2k} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2F_{2k+2} + F_{2k+1} & 2F_{2k+1} + F_{2k} \\ 1 & 2F_{2k+2} + F_{2k+1} - 1 & 2F_{2k+1} + F_{2k} \\ 0 & F_{2k+1} + F_{2k} & F_{2k+1} + F_{2k} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & F_{2k+4} & F_{2k+3} \\ 1 & F_{2k+4}-1 & F_{2k+3} \\ 0 & F_{2k+3} & F_{2k+2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & F_{2(k+1)+2} & F_{2(k+1)+1} \\ 1 & F_{2(k+1)+2}-1 & F_{2(k+1)-1} \\ 0 & F_{2(k+1)+1} & F_{2(k+1)} \end{pmatrix}.$$

$\square$

If $L_k := D_k \cdots D_1$ then the positive cone is captured by the set

$$\bigcup_{k=0}^{\infty} \{c_1 L_k^{-1} e_1 + c_2 L_k^{-1} e_2 + c_3 L_k^{-1} e_3 : c_i \in \mathbb{Z}^+\},$$

where $L_0 = I_3$, the identity matrix.

We would like to express this set using a half-space as was done for the positive cone of the previous example. This requires an explicit description of the linear transformation L_k^{-1}. For the remainder of this example the sequence $(F_i)_{i\geq 1}$ will denote the Fibonacci numbers as defined in Lemma 6.6.

Lemma 6.7.

The transformation $L_k^{-1} : \mathbb{Z}^3 \to \mathbb{Z}^3$ is defined by the matrix

$$L_k^{-1} = \begin{pmatrix} (-1)^k(1+\sum_{i=1}^{k} F_{(i+1)^2-2}) & (-1)^{k+1}(1+\sum_{i=1}^{k-1} F_{(i+1)^2-2}) & (-1)^{k+1}F_{(k+1)^2-1} \\ (-1)^k \sum_{i=1}^{k} F_{(i+1)^2-2} & (-1)^{k+1} \sum_{i=1}^{k-1} F_{(i+1)^2-2} & (-1)^{k+1}F_{(k+1)^2-1} \\ (-1)^{k+1} \sum_{i=1}^{k} F_{(i+1)^2-1} & (-1)^{k} \sum_{i=1}^{k-1} F_{(i+1)^2-1} & (-1)^{k}F_{(k+1)^2} \end{pmatrix}$$

We make use of the convention that if the upper index of a sum is less than the lower index, the sum is zero.

Proof. By definition $L_1^{-1} = D_1^{-1}$. Let us invert D_k.

$$D_k^{-1} = \begin{pmatrix} -(F_{2k}+1) & 1 & F_{2k+1} \\ -F_{2k} & 0 & F_{2k+1} \\ F_{2k+1} & 0 & -F_{2k+2} \end{pmatrix}$$

Verifying this computation requires Cassini's identity: $F_{i-1}F_{i+1} - F_i^2 = (-1)^i$ ([25]). Therefore

$$L_1^{-1} = \begin{pmatrix} -2 & 1 & 2 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{pmatrix} = \begin{pmatrix} -(1+F_{(4-2)}) & (1+0) & F_{(4-1)} \\ -F_{(4-2)} & 0 & F_{(4-1)} \\ F_{(4-1)} & 0 & -F_4 \end{pmatrix}.$$

Let us assume the lemma holds for some k and show that it holds for $k+1$. By definition $L_{k+1}^{-1} = L_k^{-1} D_{k+1}^{-1}$.

Using the formula for D_k^{-1} above we can compute D_{k+1}^{-1}. We will compute the matrix L_{k+1}^{-1} column by column. Observe that

$$L_{k+1}^{-1} e_1 = -(F_{2k+2}+1) L_k^{-1} e_1 - F_{2k+2} L_k^{-1} e_2 + F_{2k+3} L_k^{-1} e_3.$$

We will only demonstrate that the first entry of this vector satisfies the formula in the statement of the lemma. This is because the other two entries follow from nearly identical steps. The first entry of the first column is

$$- (F_{2k+2}+1)(-1)^k\Big(1+\sum_{i=1}^{k} F_{(i+1)^2-2}\Big) - F_{2k+2}(-1)^{k+1}\Big(1+\sum_{i=1}^{k-1} F_{(i+1)^2-2}\Big)$$
$$+ F_{2k+3}(-1)^{k+1}F_{(k+1)^2-1}$$
$$= (-1)^{k+1}(F_{2k+2}F_{(k+1)^2-2} + F_{2k+3}F_{(k+1)^2-1}) + (-1)^{k+1}\Big(1+\sum_{i=1}^{k} F_{(i+1)^2-2}\Big)$$
$$= (-1)^{k+1}(F_{(k+1)^2+2k+2}) + (-1)^{k+1}\Big(1+\sum_{i=1}^{k} F_{(k+1)^2-2}\Big)$$
$$= (-1)^{k+1}\Big(1+\sum_{i=1}^{k+1} F_{(k+1)^2-2}\Big).$$

The third line follows after applying the identity $F_kF_j + F_{k+1}F_{j+1} = F_{k+j+1}$. The final line holds because $(k+1)^2 + 2k + 1 = k^2 + 4k + 2 = (k+2)^2 - 2$.

For the second column we have

$$L_{k+1}^{-1}e_2 = L_k^{-1}D_{k+1}e_2 = L_k^{-1}e_1.$$

This matches the lemma statement immediately. For the third column we have

$$L_{k+1}^{-1}e_3 = F_{2k+3}L_k^{-1}e_1 + F_{2k+3}L_k^{-1}(e_2) - F_{2k+4}L_k^{-1}e_3.$$

Again we only explicitly compute the first entry. The other two follow in a similar fashion. The first entry of the third column is

$$F_{2k+3}(-1)^k\Big(1+\sum_{i=1}^{k} F_{(i+1)^2-2}\Big)+F_{2k+3}(-1)^{k+1}\Big(1+\sum_{i=1}^{k-1} F_{(i+1)^2-2}\Big)-F_{2k+4}(-1)^{k+1}F_{(k+1)^2-1}$$

$$= (-1)^k (F_{2k+3} F_{(k+1)^2-2} + F_{2k+4} F_{(k+1)^2-1})$$
$$= (-1)^{k+2} F_{(k+2)^2-1}.$$

This completes the induction. $\square$

In order to prove that the integer lattice points of a half-space are in the positive cone we would have to show that they become positive after finitely many steps along the Elliott system. In other words, for $u \in \mathbb{Z}^3$ in the half-space, there exists k such that $L_k u \in (\mathbb{Z}^3)^+$. This requires an expression for L_k.

Lemma 6.8.

The transformation $L_k : \mathbb{Z}^3 \to \mathbb{Z}^3$ is defined by

$$L_k e_1 = \begin{pmatrix} \sum_{i=1}^{k-1} (-1)^{i+1} F_{(k+1)^2-(i+1)^2+1} \\ (-1)^{k+1} + \sum_{i=1}^{k-1} (-1)^{i+1} F_{(k+1)^2-(i+1)^2+1} \\ \sum_{i=1}^{k-1} (-1)^{i+1} F_{(k+1)^2-(i+1)^2} \end{pmatrix}$$

$$L_k e_2 = \begin{pmatrix} F_{(k+1)^2} \\ F_{(k+1)^2} \\ F_{(k+1)^2-1} \end{pmatrix} - L_k e_1$$

$$L_k e_3 = \begin{pmatrix} F_{(k+1)^2-1} \\ F_{(k+1)^2-1} \\ F_{(k+1)^2-2} \end{pmatrix}.$$

As before, if the upper index of a sum is less than the lower index, the sum is zero.

Proof. We again use induction to establish these formulas. Observe that $L_1 = D_1$ satisfies the statement immediately (see 6.6). We assume the formulas hold for some k and compute the first entry of $L_{k+1}e_1$. The other two follow from similar computations. By definition $L_{k+1} = D_{k+1}L_k$. This implies

$$\begin{aligned}
(L_{k+1}e_1)_1 &= F_{2(k+1)+2}\Big((-1)^{k+1} + \sum_{i=1}^{k-1}(-1)^{i+1}F_{(k+1)^2-(i+1)^2+1}\Big) \\
&\quad + F_{2(k+1)+1}\sum_{i=1}^{k-1}(-1)^{i+1}F_{(k+1)^2-(i+1)^2} \\
&= (-1)^{k+1}F_{2(k+1)+2} \\
&\quad + \sum_{i=1}^{k-1}(-1)^{i+1}(F_{2(k+1)+2}F_{(k+1)^2-(i+1)^2+1} + F_{(k+1)^2-(i+1)^2}F_{2(K+1)+1}) \\
&= (-1)^{k+1}F_{2(k+1)+2} + \sum_{i=1}^{k-1}(-1)^{i+1}F_{(k+2)^2-(i+1)^2} \\
&= \sum_{i=1}^{k}(-1)^{i+1}F_{(k+2)^2-(i+1)^2}.
\end{aligned}$$

We now compute the first entry of $L_{k+1}e_3$, the image of e_2 will be argued for last. We quickly obtain

$$(L_{k+1}e_3)_1 = F_{2(k+1)+2}F_{(k+1)^2-1} + F_{2(k+1)+1}F_{(k+1)^2-2} = F_{(k+2)^2-1}.$$

The other entries follow similarly. To see that the formula holds for $L_{k+1}e_2$ we observe that the sum along the entries of each row in the matrix L_{k+1} is the height of a tower represented by L, M, or S.

To be precise, $D_k = A_B^k A_G$ represents $2k+1$ levels of the Bratteli diagram $B_{\alpha,\beta}$. Therefore L_k represents $\sum_{i=1}^{k}(2i+1) = k^2+2k$ levels. In other words, L_k encodes the number of paths from the vertices at the first level to vertices at level k^2+2k+1. The sum along the entries of the first and second row are $h^{(L)}_{k^2+2k+1} = q_{k^2+2k} = F_{k^2+2k+2} = F_{(k+1)^2+1}$ and the sum along the entries of the third row is $h^{(S)}_{(k+1)^2} = F_{(k+1)^2}$. We have $h^{(L)}_{(k+1)^2} = h^{(M)}_{(k+1)^2}$ because the product defining L_k ends with a B_2 incidence matrix from the left.

We return now to the induction and conclude that

$$(L_{k+1}e_1)_1 + (L_{k+1}e_2)_1 + (L_{k+1}e_3)_1 = F_{(k+2)^2+1}.$$

Therefore

$$\begin{aligned}(L_{k+1}e_2)_1 &= F_{(k+2)^2+1} - F_{(k+2)^2-1} - \sum_{i=1}^{k}(-1)^{i+1}F_{(k+2)^2-(i+1)^2} \\ &= F_{(k+2)^2} - \sum_{i=1}^{k}(-1)^{i+1}F_{(k+2)^2-(i+1)^2}.\end{aligned}$$

The remaining entries can be found in the same way. □

We now have all the ingredients needed to prove Theorem 6.5.

Proof of Theorem 6.5. It has already been discussed why the elements of the limit group H can be identified with $\mathbb{Z}^3$. We establish the claim $H^+ = P_\xi$. Let us move to the telescoping $(0, 1, 3, 5, 7, \ldots)$ of the Bratteli diagram corresponding to the Elliott system given in the theorem statement. As mentioned earlier, this has no effect on the dimension group. In order to prove the theorem, we must establish that

$$\bigcup_{k=0}^{\infty}\{c_1L_{2k}^{-1}e_1 + c_2L_{2k}^{-1}e_2 + c_3L_{2k}^{-1}e_3 \mid c_i \in \mathbb{Z}^+\} = P_\xi.$$

Let $v \in H^+$. We will have $v \in P_\xi$ if we can show that $L_{2k}^{-1}(e_1), L_{2k}^{-1}(e_2)$, and $L_{2k}^{-1}(e_3)$ are in P_ξ. The easiest of the three is $L_{2k}^{-1}e_3 = (-F_{(2k+1)^2-1}, -F_{(2k+1)^2-1}, F_{(2k+1)^2})$. The dot product is

$$L_{2k}^{-1}(e_3) \cdot (\xi, \phi - \xi, 1) = -\phi F_{(2k+1)^2-1} + F_{(2k+1)^2}.$$

The Fibonacci sequence encodes the continued fraction entries of ϕ. It is a well known fact that $F_{2m+1}/F_{2m} > \phi$ for all m (In [9] they establish inequalities for an irrational α). In particular, when $m = 2k(k+1)$, it follows that $L_{2k}^{-1}e_3 \cdot (\xi, \phi - \xi, 1) > 0$.

From Lemma 6.7 we have

$$L_{2k}^{-1}e_1 = \begin{pmatrix} 1 + \sum_{i=1}^{2k} F_{(i+1)^2-2} \\ \sum_{i=1}^{2k} F_{(i+1)^2-2} \\ -\sum_{i=1}^{2k} F_{(i+1)^2-1}) \end{pmatrix} \text{ and } L_{2k}^{-1}e_2 = \begin{pmatrix} -1 - \sum_{i=1}^{2k-1} F_{(i+1)^2-2} \\ -\sum_{i=1}^{2k-1} F_{(i+1)^2-2} \\ \sum_{i=1}^{2k-1} F_{(i+1)^2-1}) \end{pmatrix}.$$

Computing the respective dot products with $(\xi, \phi - \xi, 1)$ yields

$$L_{2k}^{-1}e_1 \cdot (\xi, \phi - \xi, 1) = \xi + \phi \sum_{i=1}^{2k} F_{(i+1)^2-2} - \sum_{i=1}^{2k} F_{(i+1)^2-1},$$

and

$$L_{2k}^{-1}e_2 \cdot (\xi, \phi - \xi, 1) = -\xi - \phi \sum_{i=1}^{2k-1} F_{(i+1)^2-2} + \sum_{i=1}^{2k-1} F_{(i+1)^2-1}.$$

Let $l_k = \sum_{i=1}^{2k} F_{(i+1)^2-1} - \phi \sum_{i=1}^{2k} F_{(i+1)^2-2}$ and $u_k = \sum_{i=1}^{2k-1} F_{(i+1)^2-1} - \phi \sum_{i=1}^{2k-1} F_{(i+1)^2-2}$. If we can show that $l_k < \xi < u_k$ then both of the dot products above are positive.

Notice that l_k and u_k are the even and odd partial sums of the same infinite series. Therefore if we show that the limit of l_k as k gets arbitrarily large exists, then u_k

must share the same limit. We claim that $\lim_{k\to\infty} l_k = \xi$. It is a well known fact that Fibonacci numbers have a closed form in terms of the golden ratio ϕ. This is called the Euler-Binet formula ([15]). Namely, $F_m = \frac{\phi^m - \hat{\phi}^m}{\sqrt{5}}$ where $\hat{\phi} = \frac{-1}{\phi}$.

We simplify l_k using this substitution to obtain

$$\begin{aligned}
l_k &= \frac{1}{\sqrt{5}} \sum_{i=1}^{2k} (\phi^{(i+1)^2-1} - \hat{\phi}^{(i+1)^2-1}) - \frac{\phi}{\sqrt{5}} \sum_{i=1}^{2k} (\phi^{(i+1)^2-2} - \hat{\phi}^{(i+1)^2-2}) \\
&= \frac{\phi}{\sqrt{5}} \sum_{i=1}^{2k} \hat{\phi}^{(i+1)^2-2} - \frac{1}{\sqrt{5}} \sum_{i=1}^{2k} \hat{\phi}^{(i+1)^2-1} \\
&= \frac{\phi}{\sqrt{5}} \sum_{i=1}^{2k} \left(\frac{-1}{\phi}\right)^{(i+1)^2-2} - \frac{1}{\sqrt{5}} \sum_{i=1}^{2k} \left(\frac{-1}{\phi}\right)^{(i+1)^2-1} \\
&= \frac{\phi^3}{\sqrt{5}} \sum_{i=1}^{2k} \left(\frac{-1}{\phi}\right)^{(i+1)^2} + \frac{\phi}{\sqrt{5}} \sum_{i=1}^{2k} \left(\frac{-1}{\phi}\right)^{(i+1)^2} \\
&= \frac{\phi^3 + \phi}{\sqrt{5}} \sum_{i=1}^{2k} \left(\frac{-1}{\phi}\right)^{(i+1)^2} \\
&= (1+\phi) \sum_{i=1}^{2k} \left(\frac{-1}{\phi}\right)^{(i+1)^2}.
\end{aligned}$$

It is now clear that $\lim_{k\to\infty} l_k = \lim_{k\to\infty} l_k = \xi$. It is not difficult to see that from the last line that $l_k < l_{k+1}$. If we change the upper index to $2k-1$ and perform the same analysis we obtain $u_k > u_{k+1}$. We conclude that $l_k < \xi < u_k$ and the dot products above are positive. Thus $H^+ \subset P_\xi$.

Conversely, let us assume that $v \in P_\xi$. We want to show that there exists K such that $L_{2K}v \in \mathbb{Z}^3$. Let us examine the first entry of $L_{2k}v$ for some $k > 1$. If $v = (x, y, z)$, then by Lemma 6.8 we have

$$
\begin{aligned}
(L_{2k}v)_1 &= x\Big(\sum_{i=1}^{2k-1}(-1)^{i+1}F_{(2k+1)^2-(i^2+2i)}\Big) + y\Big(F_{(2k+1)^2} - \sum_{i=1}^{2k-1}(-1)^{i+1}F_{(2k+1)^2-(i^2+2i)}\Big) \\
&\quad + zF_{(2k+1)^2-1} \\
&= (x-y)\Big(\sum_{i=1}^{2k-1}(-1)^{i+1}F_{(2k+1)^2-(i^2+2i)}\Big) + yF_{(2k+1)^2} + zF_{(2k+1)^2-1} \\
&= F_{(2k+1)^2-1}\Big((x-y)\Big(\sum_{i=1}^{2k-1}\frac{(-1)^{i+1}F_{(2k+1)^2-(i^2+2i)}}{F_{(2k+1)^2-1}}\Big) + y\frac{F_{(2k+1)^2}}{F_{(2k+1)^2-1}} + z\Big).
\end{aligned}
$$

Firstly $\frac{F_{(2k+1)^2}}{F_{(2k+1)^2-1}} \xrightarrow{k\to\infty} \phi$. $(*)$ (Again, the general case can be found in [9])

Let us examine the limiting behaviour of the series with coefficient $(x-y)$. Substituting in the closed form expressions for the Fibonacci numbers yields

$$
\begin{aligned}
&\sum_{i=1}^{2k-1}\frac{(-1)^{i+1}F_{(2k+1)^2-(i^2+2i)}}{F_{(2k+1)^2-1}} \\
&= \underbrace{\sum_{i=1}^{2k-1}\frac{(-1)^{i+1}\phi^{(2k+1)^2-(i^2+2i)}}{\phi^{(2k+1)^2-1}-\phi^{1-(2k+1)^2}}}_{a_k} - \underbrace{\sum_{i=1}^{2k-1}\frac{(-1)^{(i+1)^2+i+1}}{\phi^{(2k+1)^2-(i^2+2i)}F_{(2k+1)^2-1}}}_{b_k}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
|b_k| &\le \sum_{i=1}^{2k-1}\frac{1}{\phi^{(2k+1)^2-(i^2+2i)}F_{(2k+1)^2-1}} \\
&\le \sum_{i=1}^{2k-1}\frac{1}{\phi^{(2k+1)^2-((2k-1)^2+2k-1)}F_{(2k+1)^2-1}} \\
&= \frac{2k-1}{\phi^{4k+2}F_{(2k+1)^2-1}} \xrightarrow{k\to\infty} 0.(**)
\end{aligned}
$$

We move to a_k. Simplifying the expression reveals

$$\begin{aligned}
a_k &= \sum_{i=1}^{2k-1} \frac{(-1)^{i+1}\phi^{(2k+1)^2-(i^2+2i)}}{\phi^{(2k+1)^2-1} - \phi^{1-(2k+1)^2}} \\
&= \sum_{i=1}^{2k-1} \frac{(-1)^{i+1}\phi^{1-(i^2+2i)}}{1-\phi^{2-2(2k+1)^2}} \\
&= \sum_{i=1}^{2k-1} \frac{(-1)^{i+1}\phi^2\phi^{-(i+1)^2}}{1-\phi^{2-2(2k+1)^2}} \\
&= \frac{1+\phi}{1-\phi^{2-2(2k+1)^2}} \sum_{i=1}^{2k-1} \left(\frac{-1}{\phi}\right)^{(i+1)^2} \xrightarrow{k\to\infty} \xi. (***)
\end{aligned}$$

If we put $(*), (**)$, and $(***)$ together we have that for k sufficiently large

$$(L_{2k}v)_1 \approx F_{(2k+1)^2-1}\Big((x-y)\xi + y\phi + z\Big) = F_{(2k+1)^2-1}\Big(x\xi + y(\phi - \xi) + z\Big) > 0.$$

The last inequality follows from the assumption that $v \in P_\xi$. The second entry $(L_{2k}v)_2 = (L_{2k}v)_1 - (x-y)$. Therefore for k sufficiently large

$$(L_{2k}v)_2 \approx F_{(2k+1)^2-1}\Big((x-y)\xi + y\phi + z\Big) - \frac{x-y}{F_{(2k+1)^2-1}} > 0.$$

Lastly, the third entry has the same form as the first, with the exception that the index of every Fibonacci number in the expression is dropped by one. Since their relative position from one another mimics that of the first entry the analysis leads to the same conclusion. Therefore $P_\xi \subset H^+$.

This completes the proof of $H^+ = P_\xi$. $\square$

The set E_0 was unaffected by the telescopings taken in this example and thus the distinguished order unit representative is still $(1,1,1)$.

6.4 The Collapse of The Middle Tower

We conclude the chapter by discussing the case when β is in the forward orbit of $1-\alpha$ under the action of R_α. The plan is to begin with an example that demonstrates the effect this case has on the cutting and stacking procedure from chapter 5 and thus on the Bratteli diagram.

Let $\beta = \alpha = [0; a_1, a_2, \ldots]$. The image under ψ is $\beta_1 = T_{\alpha_1}(-1) = -1 + \alpha_1$. We draw attention to the fact that this is the left endpoint of the second bin in the long segment of I_1.

We define $Z_1^{(L)} = \{[-1, -1+\alpha_1)\}$, $Z_1^{(M)} = \{[-1+\alpha_1, 0)\}$, and $Z_1^{(S)} = \{[0, \alpha_1)\}$. These represent three vertices in V_1. The cutting and stacking procedure dictates that $Y_2^{(L)}$ is built by stacking the intervals $[-1, -1+\alpha_1), [-1+\alpha_1, -1+2\alpha_1), \ldots, [-1+(a_1-2)\alpha_1, -1+(a_1-1)\alpha_1)$ on top of $[0, \alpha_1)$. When we project β_1 down to the base of the tower we see that $\beta_2 = 0$.

There are two choices that one can make at this point. The first is to define $Z_2^{(L)} = \{T_{\alpha_1}^i((0, \alpha_n)) \mid 0 \leq i < q_1\}$ and $Z_2^{(M)} = \{\{T_{\alpha_1}^i(0)\} \mid 0 \leq i < q_1\}$. In other words, the width of the middle tower is zero.

The second choice is to claim that there is no middle tower. If we place the Lebesgue measure on I_1 then we could say that $Z_2^{(M)}$ has measure zero. Therefore $Z_2^{(L)} = Y_2^{(L)}$ modulo a set of measure zero and what remains is $Z_2^S = Y_2^{(S)} = \{[-\alpha_2, 0)\}$. We then define the second level by only two towers. In fact $\beta_n = 0$ for all $n \geq 2$ and so all the remaining levels will only have two vertices. The Bratteli diagram is no different from B_α from the second level onward. We call this the collapse of the middle tower. In this case the collapse occurs between the first and second level.

We will shelve the discussion of the first choice and proceed with the second. We will continue to use $K_0(V,E)$ to denote the dimension group of a Bratteli diagram in which a collapse has occurred.

From the cutting and stacking carried out above we obtain the substitutions $\sigma_2(L) = SLM^{a_1-2}$ and $\sigma_2(S) = M$. The incidence matrices are then $A_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$ for all $n \geq 2$ and $A_1 = \begin{pmatrix} 1 & a_1 - 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. We draw the levels one, two, and three of the underlying Bratteli diagram in Figure 6.1.

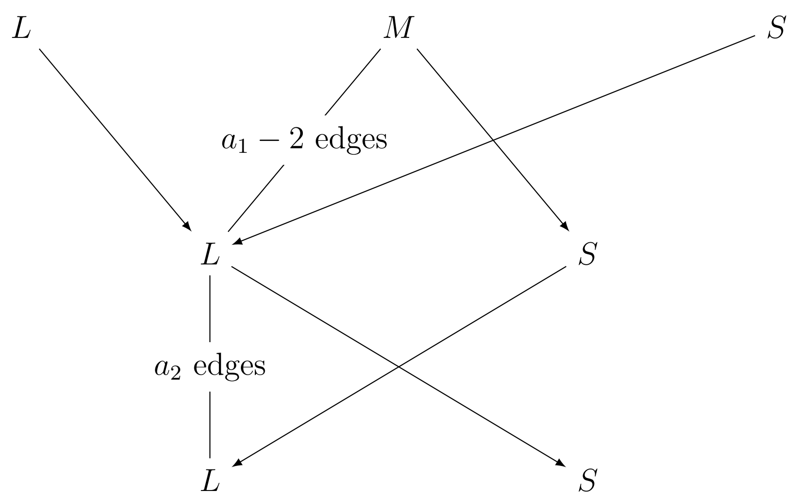


Figure 6.1: a collapse between the first and second level

The directed system associated with the collapsed diagram is

$$\mathbb{Z}^3 \xrightarrow{A_1} \mathbb{Z}^2 \xrightarrow{A_2} \mathbb{Z}^2 \xrightarrow{A_3} \cdots .$$

The matrix A_1 is surjective but not injective. In particular,

$$A_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A_1 \begin{pmatrix} 0 \\ y \\ x+z \end{pmatrix} \quad \text{for every} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{Z}^3.$$

The remaining incidence matrices are automorphisms of $\mathbb{Z}^2$. It follows that any equivalence class in $K_0(V,E)$ has infinitely many representatives in $G_1 = \mathbb{Z}^3$. Therefore the limit group is "smaller" than $\mathbb{Z}^3$. In fact we claim that it is isomorphic to $\mathbb{Z}^2$.

This is because we can shift the directed system of groups to the left (and "forget" $G_1 = \mathbb{Z}^3$) just as we did to "forget" $\mathbb{Z}$. In fact, we can essentially use the same map g. The distinguished order unit is now represented by

$$A_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ 1 \end{pmatrix}.$$

This is the distinguished order unit of $\mathbb{Z}^2$ with positive cone dictated by the Elliott system

$$\mathbb{Z}^2 \xrightarrow{A_2} \mathbb{Z}^2 \xrightarrow{A_3} \mathbb{Z}^2 \xrightarrow{A_4} \cdots .$$

The following theorem of Effros and Shen from [9] allows us to express $K_0(V,E)$ using a half-space.

Theorem 6.9 (Effros and Shen 1980)**.**

Suppose γ *is an irrational number and has continued fraction expansion* $[c_0; c_1, c_2, \ldots]$.

Let $C_n = \begin{pmatrix} c_{n-1} & 1 \\ 1 & 0 \end{pmatrix}$. *Then* $\overrightarrow{\lim}(\mathbb{Z}^2, C_n)$ *is order isomorphic to* $\mathbb{Z}^2$ *with positive cone* $P_\gamma = \{v \in \mathbb{Z}^2 \mid v \cdot (\gamma, 1) > 0\} \cup \{(0,0)\}$.

Remark 6.10. The product $C_m \cdots C_1 = \begin{pmatrix} p_{m-1} & q_{m-1} \\ p_{m-2} & q_{m-2} \end{pmatrix}$, where p_n and q_n are the nth numerator and denominator convergents of γ. Using this fact Effros and Shen gave a proof of Theorem 6.9 which follows the same approach as the proof of Theorem 6.5 which established the dimension group of our non-stationary example. It was also shown in [9] that every positive cone of $\mathbb{Z}^2$ can be identified with P_γ for some irrational parameter γ.

It follows from Theorem 6.9 that $K_0(V,E)^+ = P_{\hat{\alpha}}$ where $\hat{\alpha} = [a_2; a_3, a_4 \ldots]$.

The question is, how does this compare to the same ordered group but with distinguished order unit $(1,1)$ instead of $(a_1, 1)$?

The difference can be seen in the edge set E_0 of the underlying Bratteli diagram shown in Figure 6.2. On the left is the Bratteli diagram associated with the dimension group with distinguished order unit $(1,1)$. The diagram on the right is associated with the dimension group we computed with distinguished order unit $(a_1, 1)$.

Observe that the diagram on the right is the telescoping $(0, 2, 3, 4, \ldots)$ of the collapsed diagram we presented in Figure 6.1.

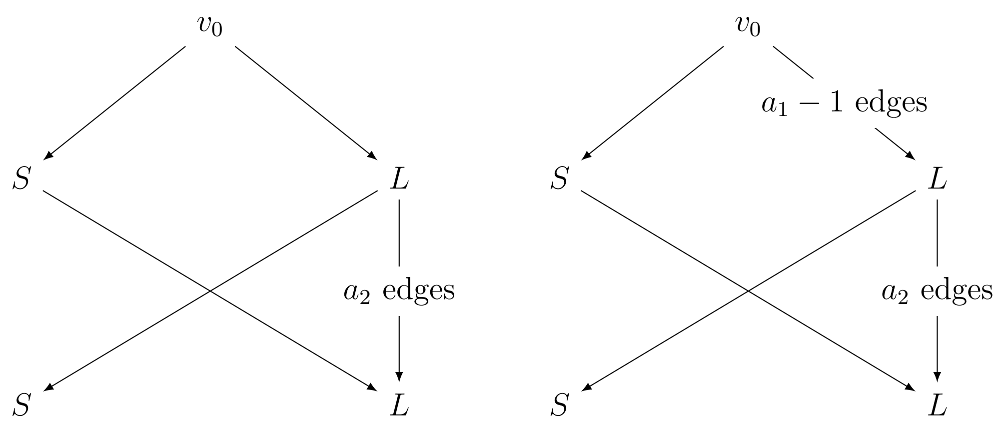


Figure 6.2: the order unit reflected in the Bratteli diagram

Proposition 6.11.

Let $\alpha = [0; a_1, a_2, \ldots]$ be an irrational number in $(0, 1/2)$ and suppose $\beta = R^t_\alpha(0)$ where $1 \leq t < a_1$. The dimension group of the collapsed Bratteli diagram is $\mathbb{Z}^2$ with positive cone $P_{\hat{\alpha}}$ where $\hat{\alpha} = [a_2; a_3, a_4, \ldots]$ and the distinguished order unit is $(a_1, 1) \in \mathbb{Z}^2$.

Proof. The collapse still occurs between levels one and two and we have

$$A_1 = \begin{pmatrix} t & a_1 - 1 - t & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The location of the collapse ensures that we can draw an order isomorphism from the dimension associated to the Bratteli diagram as given and the dimension group formed by the left shifted directed system $(\mathbb{Z}^2, M_n)$. The dimension group is order isomorphic with $(\mathbb{Z}^2, P_{\hat{\alpha}})$ because $M_n = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$ (application of Theorem 6.9). Let $u = (1, 1, 1)$. This is a representative of the order unit of the original system. Up to isomorphism, the element $(a_1, 1) = A_1 u$ is the distinguished order unit of the left shifted directed system (after applying Theorem 6.9).

We note that for $t = a_1 - 1$, we have $\beta_2 = -\alpha_2$, not zero. However, under the convention that we do not "see" towers of zero width (i.e. Lebesgue measure zero), then there are only two towers defining the second level and furthermore, $\beta_4 = \beta_5 = \cdots = 0$. $\square$

Remark 6.12. For the sake of completeness we mention the cases $\beta = 1 - \alpha$ or 0. In these cases $\beta_1 = 0$ or -1, so we would not acknowledge a middle tower from the outset and the Bratteli diagram would have two vertices at every level beyond the root. From Theorem 6.9 we gather that the dimension group is $\mathbb{Z}^2$ with positive cone $P_{\tilde{\alpha}}$ and distinguished order unit $(1, 1)$ where $\tilde{\alpha} = [a_1 - 1; a_2, a_3, \ldots]$.

Let us now extend to the case $t \geq a_1$. This means that the collapse occurs at a later stage of the cutting and stacking procedure and thus, in the diagram, vertex sets of size three persist beyond the first level. The timing of the collapse depends on both t and the continued fraction entries of α. If $t = \sum_{i=1}^{m-1} a_i + s - 1$ where s ranges from 1 to a_m, then the collapse will occur between vertex levels m and $m + 1$. The matrix responsible for the collapse is then $A_m = \begin{pmatrix} s & a_m - s & 1 \\ 0 & 1 & 0 \end{pmatrix}$ for all $t \geq a_1$. The case of the first level differs because of the discrepancy between α and α_1.

This corresponds to the directed system

$$\mathbb{Z}^3 \xrightarrow{A_1} \mathbb{Z}^3 \xrightarrow{A_2} \cdots \xrightarrow{A_{m-1}} \mathbb{Z}^3 \xrightarrow{A_m} \mathbb{Z}^2 \xrightarrow{A_{m+1}} \mathbb{Z}^2 \xrightarrow{A_{m+2}} \cdots .$$

where for $n < m$, the matrix A_n is a 3 by 3 matrix no different from the incidence matrices defining $B_{\alpha,\beta}$ when $\beta \neq R_\alpha^t(1 - \alpha)$ with $t \geq 0$. The incidence matrices for

$n > k$ are of the form $\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$. We move to the left shifted directed system

$$\mathbb{Z}^2 \xrightarrow{A_{m+1}} \mathbb{Z}^2 \xrightarrow{A_{m+2}} \cdots .$$

using the order isomorphism

$$f([v,i]) = \begin{cases} [v, i-m] & \text{if } i \geq m+1, \\ [A_m \cdots A_1 v, 1] & \text{if } i < m+1. \end{cases}$$

This is a generalization of the single left shift isomorphism g. It follows from Theorem 6.9 that the dimension group is $\mathbb{Z}^2$ with positive cone $P_{\hat{\alpha}}$ where $\hat{\alpha} = [a_{m+1}; a_{m+2}, a_{m+3} \ldots]$. The distinguished order unit is the element

$$A_m \cdots A_1 u \text{ where } u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This completes the discussion of the dimension groups of collapsed diagrams. Let us say a little about the problem when we allow the middle tower to represent a vertex even when it has no width. We return to the case when $\beta = \alpha$ and thus $\beta_n = 0$ for all $n \geq 2$. It follows that the substitutions which give the compositions of towers in the castle $\mathcal{C}_n$ for $n \geq 2$ are those corresponding to the sequence of scenarios $B_1, B_2, B_1, B_2, \ldots$. We use the telescoping $(0, 1, 2, 4, 6, \ldots)$ to produce the Elliott system

$$\mathbb{Z}^3 \xrightarrow{M} \mathbb{Z}^3 \xrightarrow{A_B^{(2)}} \mathbb{Z}^3 \xrightarrow{A_B^{(3)}} \cdots ,$$

where

$$M = \begin{pmatrix} 1 & a_1 - 2 & 1 \\ 1 & a_1 - 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_B^{(n)} = \begin{pmatrix} a_{n+1} & 0 & 1 \\ a_{n+1} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_n & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Notice that the matrix M maps $\mathbb{Z}^3$ into a subgroup isomorphic to $\mathbb{Z}^2$. In fact $M(\mathbb{Z}^3) = M(\{(0, y, z) \in \mathbb{Z}^3\})$. This suggests that the limit group is $\mathbb{Z}^2$. The positive cone is therefore P_γ for some γ, but finding this parameter is a more difficult problem than finding parameters for collapsed diagrams.

Chapter 7

Conclusion

Further study of the dimension groups of diagrams derived from the cutting and stacking procedure presented in chapter 5 would almost certainly lead to more general statements regarding the positive cones of their dimension groups. If it is not possible to describe the cone for parameters α and β assuming only that $\alpha \in (0, 1/2)$ is irrational and β is not in the forward orbit of $1 - \alpha$, then we envision a result that groups parameters into different classes. Such as periodic/non-periodic continued fraction expansions and/or periodic/non-periodic Ostrowski entries.

It was not explored how the quotient from $\Sigma_{\alpha,\beta}$ onto Σ_α interacts with the dimension groups (if at all). Further inquiry in this direction may be fruitful because Theorem 6.9 allows us to express the dimension group for B_α explicitly in terms of α.

With regards to dynamics, recent strides have been made in studying the dynamical invariants for $\mathbb{Z}^d$-odometers ([14]). It is natural to try to extend the Bratteli model for irrational rotations and ask what analogues exist for minimal $\mathbb{Z}^d$-actions on the circle. Perhaps even on more abstract compact groups.

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