Letters in Mathematical Physics
25(3), 189–201, 1992

Lagrangian submanifolds and an application to the reduced Schrödinger equation in central force problems

Andrew D. Lewis*

01/10/1991
Last updated: 11/10/1997

Abstract

In this paper a Lagrangian foliation of the zero energy level is constructed for a family of planar central force problems. The dynamics on the leaves are explicitly computed and these dynamics are given a simple interpretation in terms of the dynamics near the singularity of the potential. Lagrangian submanifolds also arise when seeking asymptotic solutions to certain partial differential equations with a large parameter. In determining such solutions, an operator between half densities on the Lagrangian submanifold and half densities on the configuration space is computed. This operator is derived for the given example, and the corresponding first order asymptotic solution to the reduced Schrödinger equation is given.

Keywords. classical mechanics, Lagrangian submanifolds, Schrödinger equation, Hamilton-Jacobi equation.

AMS Subject Classifications. 53B99, 70F99, 70H05, 70H20, 81Q20.

1. Introduction

In classical mechanical systems with a potential energy function that becomes unbounded from below at certain points in the configuration space, a problem of interest is the determination of the behaviour of trajectories which pass close to the singularities. A well-studied example of this type of potential is the gravitational potential used in celestial mechanics. Coordinate transformations like those introduced by McGehee [1974] and studied in more detail by Devaney [1980] give a good idea of how we may understand such a singular potential when the singular set consists of an isolated point. However, it is not clear how these coordinate transformations may be useful when the singular set is more complicated. As a first step towards trying to understand how to approach a more complicated problem, a more intrinsic meaning is given to the collision manifold equations of Devaney [1980], at least in the case when the configuration space is $\mathbb{R}^2 \setminus \{0\}$.

In Section 2 the simple problem that will be studied is presented and some notation is fixed. In Section 3 an explicit foliation is given of the zero energy level by Lagrangian submanifolds for the problem introduced in Section 2. It should be noted that since the problem is completely integrable, there is an obvious Lagrangian foliation of each energy

*Assistant Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew@mast.queensu.ca, URL: http://www.mast.queensu.ca/~andrew/
Work performed while a Graduate Research Assistant at the California Institute of Technology.

level by level sets of the other constants of motion. However, we find a foliation only of the zero energy level, and it is not the same as the natural one associated with two degree of freedom systems in which angular momentum is conserved. Since the leaves of the foliation are Lagrangian and are contained within a level set of the Hamiltonian, the associated Hamiltonian vector field is tangent to the leaves. This allows us to compute the restriction of the Hamiltonian vector field to the leaves and to determine the behaviour of orbits on the leaves. This is done in Section 4. In Section 5 we present an application of the computations carried out in Sections 3 and 4 to an asymptotic solution of the reduced Schrödinger equation. The relationship between the Lagrangian submanifolds discussed in Section 3, and the collision manifold equations of Devaney [1980], is presented in an appendix.

2. Preliminaries

We will study the one-parameter family of Hamiltonian systems on T^*Q with $Q = \mathbb{R}^2 \setminus \{0\}$ and Hamiltonian function

$$H(r,\theta,p_r,p_\theta) = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - r^{-k} \tag{2.1}$$

for $k \in \mathbb{R}^+$. Here (r,θ) are regarded as polar coordinates on Q and (p_r,p_θ) as their respective conjugate momenta. The case of $k = 1$ corresponds to the Kepler problem of gravitational interaction between two celestial bodies. For any k, the problem is completely integrable with two independent constants of motion being the Hamiltonian and the angular momentum p_θ. As is well known, this completely integrable structure gives rise to a stratification of the phase space by level sets of the constants of motion, and the strata are Lagrangian submanifolds for nondegenerate values of the constants of motion. In any case, the dynamics of the Hamiltonian system given by (2.1) are easily understood in the reduced phase space $T^*\mathbb{R}^+$ with the reduced Hamiltonian

$$H_\mu(r,p_r) = \frac{1}{2}(p_r^2 + \frac{\mu^2}{r^2}) - r^{-k} \tag{2.2}$$

Here the dynamics are simply those of a particle in the amended potential

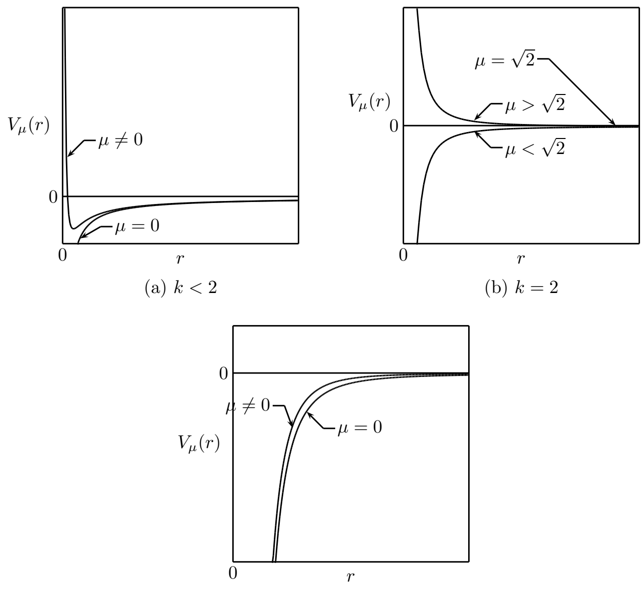
$$V_\mu(r) = \frac{\mu^2}{2r^2} - r^{-k} \tag{2.3}$$

with μ the value of the conserved angular momentum. In Figure 1 the graph of V_μ is shown for various k and μ.

For the purposes of the first three sections of this paper, it is more convenient to make use of Jacobi's metric for mechanical systems than to study the equations of motion in their original Hamiltonian form. For a general Riemannian manifold (M,g), and potential function $V : M \to \mathbb{R}$, a Hamiltonian on T^*M can be defined by

$$E(\alpha) = \frac{1}{2}g^\sharp(\alpha,\alpha) + V \circ \tau_M^*(\alpha) \tag{2.4}$$

where $\tau_M^* : T^*M \to M$ is the cotangent bundle projection, and $g^\sharp$ is the vector bundle metric on T^*M induced by the metric g on TM. Jacobi showed that the projected integral curves of



(a) $k < 2$ (b) $k = 2$ (c) $k > 2$

Figure 1. The amended potential

the Hamiltonian vector field on the surface $E^{-1}(e)$ are the same, up to reparameterisation, as geodesics of the Jacobi metric

$$g_e = (e - V)g \tag{2.5}$$

We must, of course, restrict ourselves to the subset (submanifold, if e is a regular value of V) of M where $e - V > 0$. For more on the use of the Jacobi metric see [Ong 1975].

If a Hamiltonian on T^*M is defined by

$$E_e(\alpha) = \frac{1}{2} g_e^\sharp(\alpha, \alpha) \tag{2.6}$$

we have the following

2.1 Lemma: $E_e^{-1}(1) = E^{-1}(e)$.

Proof: Let $\alpha \in E^{-1}(e)$. Then

$$\begin{aligned} & \frac{1}{2} g^\sharp(\alpha, \alpha) + V \circ \tau_M^*(\alpha) = e \\ \implies & \frac{\frac{1}{2} g^\sharp(\alpha, \alpha)}{(e - V \circ \tau_M^*(\alpha))} = 1 \\ \implies & \frac{1}{2} g_e^\sharp(\alpha, \alpha) = 1 \end{aligned}$$

Thus $\alpha \in E_e^{-1}(1)$. The other inclusion is similarly proved. ■

Note that (2.2) is of the same form as (2.4) with $M = Q$, g the standard metric on Q, and $V(r, \theta) = -r^{-k}$. Thus the Jacobi metric on Q for our problem is

$$g_e = (e + r^{-k})(\mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta) \tag{2.7}$$

Since we are claiming to be interested in the behaviour of the orbits near the origin, it seems plausible to let $e = 0$. This may be reasoned by observing that, near the origin, the kinetic energy is large and positive, and the potential energy is large and negative. Thus their difference, e, is of little consequence. This issue notwithstanding, we will let $e = 0$ in (2.7) and study geodesics of the Riemannian manifold $(Q, g_k) \triangleq (Q, g_0)$, or, equivalently, the orbits of the Hamiltonian system on T^*Q with Hamiltonian

$$H_k(r, \theta, p_r, p_\theta) = \frac{1}{2} r^k (p_r^2 + \frac{p_\theta^2}{r^2}) \tag{2.8}$$

To study this system it is sufficient to study integral curves on the energy manifold $H_k^{-1}(h)$ for any $h > 0$. For, if $\tilde{h} > 0$ is another value of the energy, then the integral curves projected from $H_k^{-1}(\tilde{h})$ onto Q via τ_Q^* will be the same, up to reparameterisation, as the integral curves projected from $H_k^{-1}(h)$. We will choose $h = 1$ so that $H_k^{-1}(h) = H^{-1}(0)$ by Lemma 2.1. For brevity we will denote $P_k = H_k^{-1}(1)$.

Coordinates will be needed on P_k. First note that $P_k \simeq Q \times \mathbb{S}^1$. Motivated by this, define the coordinate $\psi \in \mathbb{S}^1$ by

$$\psi = \arctan \frac{r p_r}{p_\theta} \tag{2.9}$$

for $(r, \theta, p_r, p_\theta) \in P_k$. Note that (2.8) gives

$$\begin{aligned} p_r &= \sqrt{2} r^{-k/2} \sin \psi \\ p_\theta &= \sqrt{2} r^{1-k/2} \cos \psi \end{aligned} \tag{2.10}$$

if $(r, \theta, p_r, p_\theta) \in P_k$. Thus (r, θ, ψ) are coordinates for P_k.

3. Definition of Lagrangian immersions

In this section we explicitly define a family of immersed Lagrangian submanifolds of P_k. Specifically, for $k \neq 2$ and rational, a foliation of P_k by embedded submanifolds, each diffeomorphic to $\mathbb{R}^+ \times \mathbb{S}^1$, is defined; for $k \neq 2$ and irrational, we define a foliation of P_k by weakly embedded submanifolds, each diffeomorphic to $\mathbb{R}^+ \times \mathbb{R}^1$ (We say that an immersed submanifold, N, of a manifold, M, is a weakly embedded submanifold if, for every manifold K and smooth mapping $f: K \to M$ with image in N, the mapping $f: K \to N$ is smooth. For example, the leaves of a foliation are weakly embedded submanifolds.); finally, for $k = 2$, a single embedding of $\mathbb{R}^1 \times \mathbb{S}^1$ in P_k is defined.

To define these immersions, the coordinates (r, θ, ψ) introduced in Section 2 will be used.

Case a) $k \neq 2$ *and rational:* If k is rational, then certainly $1 - k/2$ is rational. So suppose $1 - k/2 = m/n$ for $m, n \in \mathbb{Z}$ with $(m, n) = 1$, and $n > 0$. We let (R, ϕ) be standard coordinates for $\mathbb{R}^+ \times \mathbb{S}^1$, and let $\psi_0 \in \mathbb{S}^1$ be a parameter. Now we define the map

$$\begin{aligned} i_k \colon \mathbb{R}^+ \times \mathbb{S}^1 &\to P_k \\ (R, \phi) &\mapsto ((\mid 1 - k/2 \mid R)^{2/(2-k)}, n\phi (\mathrm{mod} 2\pi), \\ &\qquad ((1 - k/2)n\phi + \psi_0)(\mathrm{mod} 2\pi)) \end{aligned} \tag{3.1a}$$

Case b) $k \neq 2$ *and irrational:* If k is irrational, then so is $1 - k/2$. Let (R, s) be standard coordinates for $\mathbb{R}^+ \times \mathbb{R}^1$, and, as in a), let $\psi_0 \in \mathbb{S}^1$ be a parameter. We define the map

$$\begin{aligned} i_k \colon \mathbb{R}^+ \times \mathbb{R}^1 &\to P_k \\ (R, s) &\mapsto ((\mid 1 - k/2 \mid R)^{2/(2-k)}, s(\mathrm{mod} 2\pi), \\ &\qquad ((1 - k/2)s + \psi_0)(\mathrm{mod} 2\pi)) \end{aligned} \tag{3.1b}$$

Case c) $k = 2$*:* We let (R, ϕ) be standard coordinates for $\mathbb{R}^1 \times \mathbb{S}^1$, and define the map

$$\begin{aligned} i_k \colon \mathbb{R}^1 \times \mathbb{S}^1 &\to P_k \\ (R, \phi) &\mapsto (e^R, \phi, 0). \end{aligned} \tag{3.1c}$$

For brevity in the sequel, we will let $M_k = \mathbb{R}^+ \times \mathbb{S}^1$ for Case a), $M_k = \mathbb{R}^+ \times \mathbb{R}^1$ for Case b), and $M_k = \mathbb{R}^1 \times \mathbb{S}^1$ for Case c). Also define $\Lambda_k = i_k(M_k)$ and $\pi_k = \tau_Q^* \mid \Lambda_k$. The rôle of ψ_0 in Cases a) and b) is to parameterise the leaves of the foliation of P_k. However, as will be seen below, due to the rotational symmetry of the problem, the choice of ψ_0 is not important. When we speak of Λ_k, some value of ψ_0 will be assumed chosen and fixed.

We now proceed to verify that the subsets Λ_k of P_k are immersed Lagrangian submanifolds. That $\Lambda_k \subset P_k$ is clear from the definition of i_k. That the maps i_k are immersions is easily verified by computing that $rank(T_m i_k) = 2$ for all $m \in M_k$.

In fact, it can easily be seen that, for $k \neq 2$, Λ_k is locally the graph of the differential of the local function

$$S_j = \frac{\sqrt{2} r^{1-k/2}}{1 - k/2} \sin((1 - k/2)(\theta + 2\pi j) + \psi_0) \tag{3.2}$$

for some $j \in \mathbb{Z}$. More precisely, if $U \subset Q$ is a connected, simply connected open submanifold, then every connected component of $\pi_k^{-1}(U)$ is the graph of $\boldsymbol{d}S_j \mid U$ where $S_j \mid U$ is as given by (3.2) for some $j \in \mathbb{Z}$. Thus every such connected component of $\pi_k^{-1}(U)$ is diffeomorphic to U via the diffeomorphism $\boldsymbol{d}S_j \mid U$. This also verifies that Λ_k is Lagrangian since a submanifold of the cotangent bundle of a manifold which is diffeomorphic to the zero section

is Lagrangian if and only if it is the graph of a closed one-form. It is illustrative to write Λ_k in the following way

$$\begin{aligned} \Lambda_k = \{(r, \theta, \sqrt{2} r^{-k/2} &\sin((1 - k/2)(\theta + 2\pi j) + \psi_0), \\ &\sqrt{2} r^{1-k/2} \cos((1 - k/2)(\theta + 2\pi j) + \psi_0) \in T^*Q \mid j \in \mathbb{Z}\} \end{aligned} \tag{3.3}$$

Note that S_j is, formally, a solution of the Hamilton-Jacobi equation $H_k(\boldsymbol{d}S_j) = 1$, and hence by Lemma 2.1, also a solution of $H(\boldsymbol{d}S_j) = 0$.

Now we fix $(r, \theta) \in Q$ and look at the set $\pi_k^{-1}(r, \theta)$. The following proposition tells us the essential features of the map $\pi_k \colon \Lambda_k \to Q$, and, in particular, describes $\pi_k^{-1}(r, \theta)$ for all $(r, \theta) \in Q$.

3.1 Proposition: *$\pi_k \colon \Lambda_k \to Q$ is a covering map. Furthermore, if $k \neq 2$, then*

(i) if k is rational with $1 - k/2 = m/n$, $(m, n) = 1$, and $n > 0$, then π_k is an n-sheeted covering, and

(ii) if k is irrational, then π_k is an infinite-sheeted covering.

Proof: By our remarks following (3.2), π_k is a local diffeomorphism, and it is clearly surjective. Thus π_k is a covering map. For fixed $(r, \theta) \in Q$, (3.3) shows that there are the same number of points in $\pi_k^{-1}(r, \theta)$ as there are distinct elements in the sequence $\{2\pi(1 - k/2)l \mid l \in \mathbb{Z}\}(\mathrm{mod} 2\pi)$. This is clearly n if k satisfies the hypotheses of *i)*. If k is irrational then $\mid \pi_k^{-1}(r, \theta) \mid = \aleph_0$ and $\pi_k^{-1}(r, \theta)$ is dense in $(\tau_Q^* \mid P_k)^{-1}(r, \theta)$. ■

Observe that Proposition 3.1 confirms that i_k is an embedding for Case a), and a weak embedding for Case b).

This provides a clear description of Λ_k for $k \neq 2$. For $k = 2$ the situation is simpler since we have a single embedded copy of $\mathbb{R}^1 \times \mathbb{S}^1$ in P_k as a Lagrangian submanifold. In fact, it is easy to see from (3.1c) that Λ_2 is simply the Lagrangian submanifold which is the graph of the differential of the local function

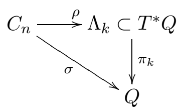
$$\tilde{S} = \sqrt{2}\theta \tag{3.4}$$

Observe that, unlike for the "function" S_j defined by (3.2), the differential of $\tilde{S}$ is well defined on all of Q, and so $\pi_k \colon \Lambda_k \to Q$ is a diffeomorphism. It is clear that the case $k = 2$ is something of a degenerate one. This will become more apparent when the behaviour of the Hamiltonian vector fields on Λ_k is discussed in the next section.

3.2 Remark: The fundamental group of Q is isomorphic to the group of integers since $\mathbb{S}^1$ is a deformation retract of Q. Thus, for every $n \in \mathbb{Z}^+$ there exists a unique, up to diffeomorphism, n-sheeted covering of Q, and an *infinite*-sheeted covering will be universal. The n-sheeted covering of Q will be denoted by C_n, and the universal covering by C_∞. Then Proposition 3.1 implies the following:

1. $C_n \simeq \mathbb{R}^+ \times \mathbb{S}^1 \simeq \mathbb{R}^1 \times \mathbb{S}^1$ for all $n \in \mathbb{Z}^+$, and $C_\infty \simeq \mathbb{R}^+ \times \mathbb{R}^1 \simeq \mathbb{R}^2$.

2. For any $n \in \mathbb{Z}^+$ (including $n = \infty$) there exists a diffeomorphism $\rho\colon C_n \to \Lambda_k$ for some, not necessarily unique, $k \in \mathbb{R}^+$ such that the following diagram commutes.



where $\sigma\colon C_n \to Q$ is the canonical projection. □

It will be desirable to put coordinates on the submanifolds Λ_k in terms of the coordinates (r, θ, ψ) on P_k. To do this we will be motivated by the definition, (3.1), of the maps i_k. Consider the curve

$$\begin{aligned} c\colon \mathbb{R} &\to Q \\ t &\mapsto (1, t(\mathrm{mod} 2\pi)) \end{aligned} \tag{3.5}$$

and let c_l be the unique ***lift***, via the covering map, of c to Λ_k such that $c_l(0) = (1, 0, \psi_0)$ (see (3.1) for the definition of ψ_0). A diffeomorphism of Λ_k with M_k for all $k \in \mathbb{R}^+$ can be defined as follows

Case a) $k \neq 2$ *and rational:* Assume $m, n \in \mathbb{Z}$ satisfy the hypotheses of Proposition 3.1i).

$$\begin{aligned} \rho_k\colon \Lambda_k &\to \mathbb{R}^+ \times \mathbb{S}^1 \\ (r, \theta, \psi) &\mapsto \Big(\frac{r^{1-k/2}}{\mid 1 - k/2 \mid}, \int_\gamma \mathrm{d}\psi (\mathrm{mod} 2\pi m)\Big) \end{aligned} \tag{3.6a}$$

where γ is the curve from ψ_0 to ψ along c_l.

Case b) $k \neq 2$ *and irrational:* Let γ be the same path as defined in a).

$$\begin{aligned} \rho_k\colon \Lambda_k &\to \mathbb{R}^+ \times \mathbb{R}^1 \\ (r, \theta, \psi) &\mapsto \Big(\frac{r^{1-k/2}}{\mid 1 - k/2 \mid}, \int_\gamma \mathrm{d}\psi\Big) \end{aligned} \tag{3.6b}$$

Case c) $k = 2$*:*

$$\begin{aligned} \rho_k\colon \Lambda_k &\to \mathbb{R}^1 \times \mathbb{S}^1 \\ (r, \theta, \psi) &\mapsto (\ln r, \theta) \end{aligned} \tag{3.6c}$$

Observe that these maps are essentially the "inverses" of the immersions defined by (3.1). Thus we will declare that these maps define coordinates $(R, \phi) \in \mathbb{R}^+ \times \mathbb{S}^1$ in Case a) (note that ϕ is defined mod$2\pi m$ in this case), $(R, s) \in \mathbb{R}^+ \times \mathbb{R}^1$ in Case b), and $(R, \phi) \in \mathbb{R}^1 \times \mathbb{S}^1$ in Case c). In Sections 4 and 5 we will see that the important quantities on Λ_k are essentially independent of k when written in these coordinates.

3.3 Remark: The meaning of the coordinate $R(r)$ defined in (3.6) is worthy of comment. Consider the Riemannian manifold (Q, g_k) where $g_k \triangleq g_0$ is given by (2.7). The Riemannian distance between two points $(r_1, \theta), (r_2, \theta) \in Q$ is easily computed to be

$$d_{g_k}((r_1, \theta), (r_2, \theta)) = \mid R(r_1) - R(r_2) \mid \tag{3.7}$$

So R can be thought of as measuring the distance from the origin when $k < 2$, measuring the distance from infinity when $k > 2$, and measuring the distance from $r = 1$ when $k = 2$. Thus the Riemannian manifold (Q, g_k) is complete if and only if $k = 2$. For $k < 2$ the point at the origin is a finite distance away from any point in Q, and for $k > 2$ infinity is a finite distance away from any point in Q. This has some relationship to the completeness condition for Riemannian manifolds given by Gordon [1973]. □

4. The vector field on Λ_k

As is well-known (see, for example [Abraham and Marsden 1978]), if a function, f, is constant on a co-isotropic submanifold, M, of a symplectic manifold, P, then the Hamiltonian vector field $\mathfrak{X}_f$ will be tangent to M. Thus, since Lagrangian submanifolds are minimal co-isotropic submanifolds, if a Lagrangian submanifold, L, is contained in $f^{-1}(a) \neq \emptyset$ for some $a \in \mathbb{R}$, then $\mathfrak{X}_f$ will be tangent to L. The Lagrangian submanifolds constructed in Section 3 are contained in the energy level P_k of the Hamiltonian H_k defined by (2.8). This implies that the Hamiltonian vector field $\mathfrak{X}_{H_k}$ will be tangent to Λ_k. Thus the vector field

$$\mathfrak{X}_k \triangleq \mathfrak{X}_{H_k} \mid \Lambda_k \tag{4.1}$$

is well-defined on Λ_k. The coordinates (R, ϕ) or (R, s) defined in Section 3 for Λ_k will be used to explicitly determine $\mathfrak{X}_k$.

First we look at $k \neq 2$. Define a metric G_k on Λ_k in these cases by

$$G_k = \begin{cases} \mathrm{d}R \otimes \mathrm{d}R + R^2 \mathrm{d}\phi \otimes \mathrm{d}\phi & \text{for Case a)} \\ \mathrm{d}R \otimes \mathrm{d}R + R^2 \mathrm{d}s \otimes \mathrm{d}s & \text{for Case b)} \end{cases} \tag{4.2a,b}$$

The following observation can be made about G_k.

4.1 Lemma: *G_k is the unique Riemannian metric on Λ_k such that $\pi_k\colon (\Lambda_k, G_k) \to (Q, g_k)$ is a Riemannian covering map.*

Proof: The existence and uniqueness of such a metric, G_k, is a consequence of the map π_k being a local diffeomorphism. To show that G_k is indeed as given by (4.2) it suffices to do the following: Let $U \subset \Lambda_k$ be an open submanifold such that $\pi_k \mid U$ is a diffeomorphism. Then, for Case a), we have

$$\begin{aligned} \pi_k \mid U\colon U &\to \pi_k(U) \\ (R, \phi) &\mapsto \Big((\mid 1 - k/2 \mid R)^{2/(2-k)}, \frac{\phi - \psi_o}{1 - k/2} - 2\pi j\Big) \end{aligned}$$

for some $j \in \mathbb{Z}$. A similar expression holds for Case b). A simple calculation then shows that $G_k \mid U = (\pi_k \mid U)^*(g_k \mid \pi_k(U))$, thus proving the lemma. ■

Now define a function on Λ_k by

$$\begin{aligned} \mathcal{S}_k : \Lambda_k &\to \mathbb{R} \\ &\begin{cases} (R, \phi) \mapsto R \sin\phi & \text{for Case a) and } k < 2 \\ (R, \phi) \mapsto -R \sin\phi & \text{for Case a) and } k > 2 \\ (R, s) \mapsto R \sin s & \text{for Case b) and } k < 2 \\ (R, s) \mapsto -R \sin s & \text{for Case b) and } k > 2 \end{cases} \end{aligned} \tag{4.3a,b}$$

The following proposition gives $\mathcal{X}_k$ for $k \neq 2$.

4.2 PROPOSITION: *$\mathcal{X}_k = G_k^\sharp(\boldsymbol{d}\mathcal{S}_k)$ where $G_k^\sharp \colon T^*\Lambda_k \to T\Lambda_k$ is the musical isomorphism associated to the metric G_k.*

Proof: As in the proof of Lemma 4.1, let $U \subset \Lambda_k$ be such that $\pi_k \mid U$ is a diffeomorphism. Recall from Section 3 that such a U is the graph of the differential of a function $S_j \mid \pi_k(U)$ of the form (3.2) for some $j \in \mathbb{Z}$. Now $(\pi_k \mid U)_*(\mathcal{X}_k \mid U)$ is a vector field on $\pi_k(U)$. Indeed, Hamilton-Jacobi theory (see [Abraham and Marsden 1978]) states that

$$(\pi_k \mid U)_*(\mathcal{X}_k \mid U) = g_k^\sharp(\boldsymbol{d}S_j \mid \pi_k(U))$$

Thus

$$\mathcal{X}_k \mid U = (\pi_k \mid U)^*(g_k^\sharp(\boldsymbol{d}S_j \mid \pi_k(U)))$$

A simple calculation gives $(\pi_k \mid U)^*(S_j \mid \pi_k(U)) = \mathcal{S}_k \mid U$ and Lemma 4.1 gives $(\pi_k \mid U)^*(g_k \mid \pi_k(U)) = G_k \mid U$. Using commutativity of exterior derivative with pullback and the musical isomorphism we get

$$\mathcal{X}_k \mid U = (G_k \mid U)^\sharp(\boldsymbol{d}\mathcal{S}_k \mid U)$$

as claimed. ■

For Case a) and $k < 2$, if $t \mapsto (R(t), \phi(t))$ is an integral curve for $\mathcal{X}_k$, then $(R(t), \phi(t))$ must satisfy the ordinary differential equation

$$\begin{aligned} \dot{R} &= \sin\phi \\ \dot{\phi} &= R^{-1}\cos\phi \end{aligned} \tag{4.4a}$$

Similarly, for Case b) and $k < 2$, an integral curve $t \mapsto (R(t), s(t))$ must satisfy

$$\begin{aligned} \dot{R} &= \sin s \\ \dot{s} &= R^{-1}\cos s \end{aligned} \tag{4.4b}$$

If $k > 2$ the differential equations for $(R(t), \phi(t))$ and $(R(t), s(t))$ are the same as (4.4a) and (4.4b), respectively, up to a multiplication by -1 on the right hand side.

This provides a pretty clear picture of the vector field on Λ_k for $k \neq 2$. For $k = 2$, the development of the vector field $\mathcal{X}_k$ is analogous to that for $k \neq 2$, but with different results. Define a metric on Λ_2 by

$$G_2 = \mathrm{d}R \otimes \mathrm{d}R + \mathrm{d}\phi \otimes \mathrm{d}\phi \tag{3.2c}$$

where (R, ϕ) are the coordinates on Λ_2 defined by (3.6c). As in Lemma 4.1, G_2 is the unique Riemannian metric on Λ_2 such that $\pi_2(\Lambda_2, G_2) \colon (Q, g_2) \to$ is a Riemannian covering map. Also define the local function

$$\mathcal{S}_2 = \sqrt{2}\phi \tag{3.3c}$$

on Λ_2, and, as in Proposition 4.2, find $\mathcal{X}_2 = G_2^\sharp(\boldsymbol{d}\mathcal{S}_2)$. So, if $t \mapsto (R(t), \phi(t))$ is an integral curve for $\mathcal{X}_2$, then $(R(t), \phi(t))$ satisfies

$$\begin{aligned} \dot{R} &= 0 \\ \dot{\phi} &= \sqrt{2} \end{aligned} \tag{3.4c}$$

The integral curves on Λ_k are shown in Figure 2. From these integral curves we can get a good idea of what happens to a typical orbit on P_k when $k \neq 2$.

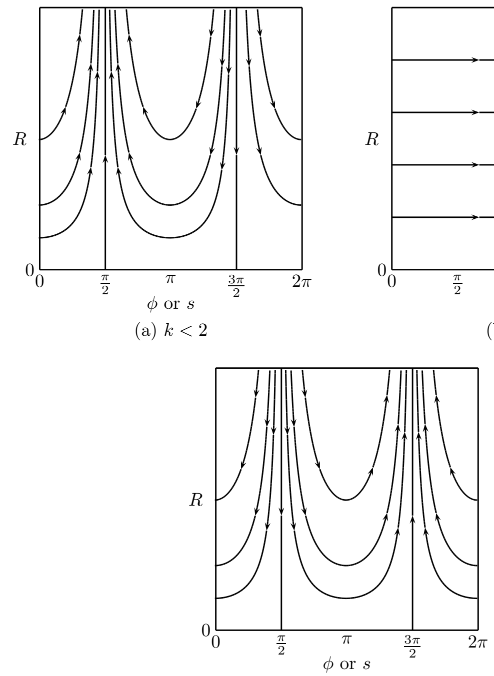


FIGURE 2. Integral curves on Λ_k.

1. *$k < 2$:* In this case a typical orbit comes from $R = \infty$ and ϕ (or s)$= (n + \frac{1}{2})\pi$ for some $n \in \mathbb{Z}$ and goes to $R = \infty$ and ϕ (or s) $= (m + \frac{1}{2})\pi$ where $m \in \{n-1, n+1\}$. From (3.6) we see that this implies that the orbit comes from $r = \infty$, undergoes a rotation in θ of $\pm\pi/(1 - k/2)$, then goes back out to$r = \infty$.

2. *$k > 2$:* Via the same argument as in I, we see that a typical orbit comes from $r = 0$, undergoes a rotation of $\pm\pi/(1 - k/2)$ in θ, then returns to $r = 0$.

In both cases, the orbits on ϕ (or s) $= (n + \frac{1}{2})\pi$ correspond to the zero angular momentum ***collision*** or ***ejection*** orbits.

For $k = 2$ it can be seen that the orbits on Λ_2 are all periodic orbits. In fact, Λ_2 contains all the periodic orbits for the case of $k = 2$ since $p_\theta = \sqrt{2}$ on Λ_2 and hence, from Figure 1, the amended potential, (2.3), vanishes. So, unlike the cases where $k \neq 2$, the orbits on Λ_2 are not 'typical'.

5. Application to the reduced Schrödinger equation

Throughout this section $k \neq 2$ unless otherwise stated.

It is well-known that Lagrangian submanifolds arise in the theory of asymptotic differential operators when one constructs a formally asymptotic solution to some linear partial differential equation with a large parameter. In this section we show how the Lagrangian submanifolds Λ_k constructed in Section 3 can be used to construct an asymptotic solution to the reduced Schrödinger equation

$$-\frac{\hbar^2}{2}\Delta\psi - r^{-k}\psi = 0 \tag{5.1}$$

as Planck's constant, $\hbar$, becomes small ($\hbar^{-1}$ is regarded as our large parameter). Here Δ is the Laplace-Beltrami operator on Q with respect to the standard metric.

Denote by L_k the differential operator in (5.1). Proceeding in the usual manner for problems of this type (see [Guillemin and Sternberg 1977]), a solution is sought to (5.1) of the form

$$\psi = e^{iS/\hbar}\sum_{j=0}^{\infty}\frac{\psi_j}{(i/\hbar)^j} \tag{5.2}$$

where $i = \sqrt{-1}$, and S and ψ_j are unknown functions on Q. Substituting (5.2) into (5.1) and setting the coefficient of $(i/\hbar)^0$ to zero we find

$$\sigma_k(\mathbf{d}S)\psi_0 = 0 \tag{5.3}$$

where $\sigma_k\colon T^*Q \to \mathbb{R}$ is the ***symbol*** of the asymptotic differential operator L_k, and is given by

$$\sigma_k(r,\theta,p_r,p_\theta) = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - r^{-k} \tag{5.4}$$

Similarly, setting the coefficient of $(i/\hbar)^{-1}$ to zero gives

$$\frac{\partial S}{\partial r}\frac{\partial\psi_0}{\partial r} + \frac{1}{r^2}\frac{\partial S}{\partial\theta}\frac{\partial\psi_0}{\partial\theta} + \sigma_k(\mathbf{d}S)\psi_1 = 0 \tag{5.5}$$

Note that if S is a solution to (5.3), then (5.5) can be written as

$$\frac{\partial\sigma_k}{\partial p_r}\frac{\partial\psi_0}{\partial r} + \frac{\partial\sigma_k}{\partial p_\theta}\frac{\partial\psi_0}{\partial\theta} = 0 \tag{5.6}$$

which is the ***transport equation***. If the Hamiltonian vector field corresponding to the symbol σ_k is denoted by $\mathfrak{X}_{\sigma_k}$, and if ψ_0 is regarded as a function on T^*Q, then (4.6) can be further simplified to get

$$\mathcal{L}_{\mathfrak{X}_{\sigma_k}}\psi_0 = 0 \tag{5.7}$$

where $\mathcal{L}_{\mathfrak{X}_{\sigma_k}}$ means Lie differentiation along $\mathfrak{X}_{\sigma_k}$. So, if S satisfies (5.3), and if ψ_0 satisfies the transport equation, then $L_k(\exp(iS/\hbar)\psi_0) = \mathcal{O}(\hbar^2)$.

It is well-known that a solution of the form (5.2) cannot be, in general, globally valid, and that there are techniques for constructing globally valid solutions which locally are of the form (5.2). A global solution to (5.1), valid to $\mathcal{O}(\hbar^2)$, will be constructed following Guillemin and Sternberg [1977]. Roughly, the procedure is as follows.

Step 1. Find a Lagrangian submanifold, L, on which the symbol σ_k vanishes. That is $L \subset \sigma_k^{-1}(0)$.

Step 2. Interpret the transport equation as an equation for half densities on L.

Step 3. Assign a half density on Q to every half density on L.

In Section 3 a family of Lagrangian submanifolds, Λ_k, was found which fulfill the requirements of Step 1 for all $k \in \mathbb{R}^+$, so this part of the problem is complete.

Step 2 asks that we think of the transport equation as an equation for half densities on Λ_k. This interpretation follows from the argument given by Guillemin and Sternberg [1977, pg. 53-57]. Note that in writing the transport equation in the form (5.7), ψ_0 was regarded as a function on T^*Q, so this may lead one to think that the transport equation should not naturally be thought of as an equation on Q.

The transport equation, (5.7), may be related to the vector fields $\mathfrak{X}_k$ computed in Section 4 as follows

5.1 Lemma: *Let u be a half density on Λ_k. Then $\mathcal{L}_{\mathfrak{X}_{\sigma_k}}u = 0$ if and only if $\mathcal{L}_{\mathfrak{X}_k}u = 0$.*

Proof: The integral curves of $\mathfrak{X}_{\sigma_k}$ are the same, up to reparameterisation, as those of $\mathfrak{X}_k$. Thus $\mathfrak{X}_{\sigma_k} = f\mathfrak{X}_k$ for some function $f > 0$ on Λ_k. Thus $\mathcal{L}_{\mathfrak{X}_{\sigma_k}}u = f\mathcal{L}_{\mathfrak{X}_k}u$ and this proves the lemma since $f > 0$. ■

Aided by Lemma 5.1, the form of solutions to the transport equation can be given. We will go through the calculations for Case a), and $k < 2$. The calculations for the other cases follow along the same lines, with similar results. It is convenient to introduce the coordinates $(\xi = R\cos\phi, \eta = R\sin\phi)$ for Λ_k. In these coordinates $\mathfrak{X}_k = \partial/\partial\eta$. It is now straightforward to verify that $\mathcal{L}_{\mathfrak{X}_k}u = 0$ implies that $u = f(\xi) \mid \mathrm{d}\xi\mathrm{d}\eta \mid^{1/2}$ where f is a strictly positive function of ξ. Going back to the coordinates (R,ϕ), the form of a solution to the transport equation is

$$u = f(R\cos\phi)\sqrt{R} \mid \mathrm{d}R\mathrm{d}\phi \mid^{1/2} \tag{5.8}$$

Now an operator between half densities on Λ_k and half densities on Q can be computed. (The bundle of half densities on a manifold M will be denoted by $\mid \bigwedge \mid^{1/2} M$, and the smooth sections of $\mid \bigwedge \mid^{1/2} M$ will be denoted by $C^\infty(\mid \bigwedge \mid^{1/2} M)$.) In order to define such an operator, it is necessary and sufficient to satisfy certain ***quantisation conditions*** for the Lagrangian submanifolds Λ_k. These conditions amount to the requirement that a certain differential one-form on Λ_k have an integral de Rham cohomology class. Specifically, if $\delta^*\colon H^1(\Lambda_k,\mathbb{Z}) \to H^1(\Lambda_k,\mathbb{R})$ is the map in cohomology induced by the natural homomorphism $\delta\colon \mathbb{Z} \to \mathbb{R}$, then we must have

$$\frac{\hbar}{2\pi}\beta + \frac{1}{4}\mathcal{M} \in \delta^*(H^1(\Lambda_k,\mathbb{Z})) \tag{5.9}$$

where $\mathcal{M}$ is the Maslov class of Λ_k and β is the canonical one-form $\alpha = p_r\mathrm{d}r + p_\theta\mathrm{d}\theta$ on T^*Q restricted to Λ_k (see [Guillemin and Sternberg 1977]). Since $\pi_k\colon \Lambda_k \to Q$ has no critical points (it is a covering), we have $\mathcal{M} = 0$. This is a consequence of the fact that the Lagrangian submanifold Λ_k does not "bend over" itself and so there are no ***caustics*** associated with it. The following lemma tells us when (5.9) is satisfied.

5.2 Lemma: *The cohomology class of β is zero.*

Proof: It suffices to check that

$$\int_\gamma \beta = 0 \tag{5.10}$$

where γ is any generator of $H_1(\Lambda_k, \mathbb{R})$. There are two cases.

Case a $\Lambda_k \simeq \mathbb{R}^+ \times \mathbb{S}^1$*:* In this case, $H_1(\Lambda_k, \mathbb{R}) \simeq \mathbb{R}$ and so $H_1(\Lambda_k, \mathbb{R})$ has a single generator, γ. In the coordinates (R, ϕ) introduced for Λ_k in Section 3 we compute

$$\beta = \sqrt{2} \sin\phi \mathrm{d}R + \sqrt{2} R \cos\phi \mathrm{d}\phi \tag{5.11}$$

Thus, in order to satisfy (5.10), we must have

$$\int_0^{2\pi m} \sqrt{2} R \cos\phi \mathrm{d}\phi = 0$$

which is indeed true (recall that ϕ is defined $\mathrm{mod} 2\pi m$).

Case b $\Lambda_k \simeq \mathbb{R}^+ \times \mathbb{R}^1$*:* In this case $H_1(\Lambda_k, \mathbb{R}) \simeq 0$, so (5.10) is trivially true.

This proves the lemma. ■

Thus (5.9) holds for all values of $\hbar$ and an operator

$$\Psi_k : C^\infty(|\bigwedge|^{1/2}\,\Lambda_k) \to C^\infty(|\bigwedge|^{1/2}\,Q) \tag{5.12}$$

can be defined. Since $\mathcal{M} = 0$, our task is particularly simple. Assume that $u \in C^\infty(|\bigwedge|^{1/2}\,\Lambda_k)$ has compact support, and define $v = \Psi_k(u) \in C^\infty(|\bigwedge|^{1/2}\,Q)$ at $q = (r, \theta) \in Q$ by

$$v(q) = \sum_{x_j \in \pi_k^{-1}(q)} u(x_j) \exp(i\hbar^{-1} \int_{\gamma_j} \beta) \tag{5.13}$$

where γ_j is a path from an arbitrary point on Λ_k to x_j. Since Λ_k satisfies the quantisation conditions (5.9), v is independent of the choice of paths γ_j, up to the arbitrary constant introduced by a choice of starting point for the paths. The condition of compact support for u can be weakened by requiring that the support of u be such that the sum in (5.13) be finite for all $q \in Q$.

Using the expression for β given by (5.11), (5.13) can be made more explicit. The computations will be carried out in Case a), when Λ_k is diffeomorphic to $\mathbb{R}^+ \times \mathbb{S}^1$. The computations are similar for Case b). Note that for Case a), the sum in (5.13) will always be finite since there are a finite number of elements in $\pi_k^{-1}(q)$ for all $q \in Q$. Fix $(R_0, \phi_0) \in \Lambda_k$, and let γ be a path from (R_0, ϕ_0) to $(R, \phi) \in \Lambda_k$ which lies in Λ_k. We compute

$$\int_\gamma \beta = \sqrt{2}(R \sin\phi - R_0 \sin\phi_0) \tag{5.14}$$

Note that if (R_1, ϕ_1) and (R_2, ϕ_2) are points in Λ_k such that both points are in $\pi_k^{-1}(q)$ for some $q \in Q$, then $R_1 = R_2$. With this and (5.14), (5.13) can be written as

$$v(q) = \sum_{(R_j,\phi_j) \in \pi_k^{-1}(q)} u(R, \phi_j) \exp(\sqrt{2} i\hbar^{-1}(R \sin\phi_j - R_0 \sin\phi_0)) \tag{5.15}$$

for $q \in Q$. Here R is such that $R_j = R$ for all j, and (R_0, ϕ_0) is an arbitrary point in Λ_k.

We can think of L_k as being a differential operator on $C^\infty(|\bigwedge|^{1/2}\ Q)$ which agrees with (5.1) when a trivialisation of $|\bigwedge|^{1/2}\,Q$ via some coordinate system is chosen. Then, if $u \in C^\infty(|\bigwedge|^{1/2}\,\Lambda_k)$ satisfies the transport equation, and if v satisfies (4.13), we have

$$L_k(v) = \mathcal{O}(\hbar^2) \tag{5.16}$$

and so v is a formally asymptotic solution to (5.1) (with ψ regarded as a half density) valid to $\mathcal{O}(\hbar^2)$.

Typically, one would specify some initial data with (5.1), and would proceed to construct an asymptotic solution which matches the initial data. However, in our construction of the solution (5.13), no initial data has been specified. At the moment, it is not clear to the author how to specify the appropriate Cauchy data in a meaningful manner. It would certainly be desirable to be able to formulate such a problem as it would give some clues about the significance of the Lagrangian foliation of P_k constructed in Section 3. It would also be interesting to see if the above methodology could be applied to a nonintegrable classical system given the observations made in the appendix of this paper.

A. Relationship with the collision manifold

Throughout this appendix $k \neq 2$ unless otherwise stated.

The problem of understanding dynamical behaviour near isolated singularities of potentials has been discussed quite extensively by Devaney [1980] using a transformation of coordinates motivated by one used by McGehee [1974]. The coordinate transformations serve to "slow down" the trajectories near the singularity and turn the singularity itself into an invariant boundary. If the configuration space is $\mathbb{R}^n \setminus \{0\}$, and if certain nondegeneracy conditions are met, then this boundary or ***collision manifold*** is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. One copy of $\mathbb{S}^{n-1}$ arises from the use of polar coordinates for $\mathbb{R}^n \setminus \{0\}$, and the other copy arises from restriction to an energy level which determines an $(n-1)$-sphere in momentum space at each point of the configuration space. This is in complete analogy with the choice of coordinates θ and ψ on P_k in Section 2.

In [Devaney 1980] the vector field on the collision manifold is computed and some statements are proven regarding its generic character (e.g., the vector field is generically Morse-Smale). In this section the relationship between trajectories on the collision manifold and the Lagrangian submanifolds Λ_k determined in section 2 will be discussed.

We begin by quickly reviewing the development of the vector field on the collision manifold for the Hamiltonian system given by the Hamiltonian

$$H(r, \theta, p_r, p_\theta) = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) - r^{-k} f(\theta) \tag{A.1}$$

on T^*Q. Here f is a periodic function of period 2π. Note that when $f(\theta) = 1$, (A.1) simplifies to (2.1).

The development proceeds most naturally beginning in Cartesian coordinates (x, y) on

Q and their conjugate momenta (p_x, p_y). The equations of motion are

$$\begin{aligned} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -\frac{\partial V}{\partial x} \\ \dot{p}_y &= -\frac{\partial V}{\partial y} \end{aligned} \tag{A.2}$$

where $V = -r^{-k} f(\theta)$. Now make the coordinate change

$$\begin{aligned} (x, y, p_x, p_y) \mapsto &(\sqrt{x^2+y^2}, \arctan\frac{y}{x}, \frac{x}{\sqrt{x^2+y^2}} p_x + \frac{y}{\sqrt{x^2+y^2}} p_y, \\ &\frac{-y}{\sqrt{x^2+y^2}} p_x + \frac{x}{\sqrt{x^2+y^2}} p_y) \\ &\triangleq (r, \theta, v_r, v_\theta) \end{aligned} \tag{A.3}$$

Under this coordinate change (A.2) becomes

$$\begin{aligned} \dot{r} &= v_r \\ \dot{\theta} &= r^{-1} v_\theta \\ \dot{v}_r &= r^{-1}(v_\theta^2 - k r^{-k} f(\theta)) \\ \dot{v}_\theta &= r^{-1}(f'(\theta) - v_r v_\theta) \end{aligned} \tag{A.4}$$

Note that $v_\theta \neq \dot{\theta}$, so the coordinate change (A.3) is not just the lift to TQ of the coordinate change $(x, y) \mapsto (r, \theta)$ on Q. The coordinate v_θ is a tangential velocity rather than an angular velocity. Now scale the momenta with the change of coordinates

$$\begin{aligned} (r, \theta, v_r, v_\theta) \mapsto &(r, \theta, r^{k/2} v_r, r^{k/2} v_\theta) \\ &\triangleq (r, \theta, V_r, V_\theta) \end{aligned} \tag{A.5}$$

Equations (A.4) transform to

$$\begin{aligned} \dot{r} &= r^{-k/2} V_r \\ \dot{\theta} &= r^{-1-k/2} V_\theta \\ \dot{V}_r &= r^{-1-k/2}(V_\theta^2 + \frac{k}{2} V_r^2 - k f(\theta)) \\ \dot{V}_\theta &= r^{-1-k/2}((k/2 - 1) V_r V_\theta + f'(\theta)) \end{aligned} \tag{A.6}$$

Next scale the independent variable by $r^{1+k/2}$ to get

$$\begin{aligned} \dot{r} &= r V_r \\ \dot{\theta} &= V_\theta \\ \dot{V}_r &= V_\theta^2 + \frac{k}{2} V_r^2 - k f(\theta) \\ \dot{V}_\theta &= (k/2 - 1) V_r V_\theta + f'(\theta) \end{aligned} \tag{A.7}$$

Note that the boundary $r = 0$ is now invariant. The vector field on this invariant manifold will be computed. For this purpose, it is convenient to introduce the variable $\psi = \arctan(V_r/V_\theta)$ analogous to (2.9) so that

$$\begin{aligned} V_r &= \sqrt{2f(\theta)} \sin\psi \\ V_\theta &= \sqrt{2f(\theta)} \cos\psi \end{aligned} \tag{A.8}$$

We compute the vector field on $\mathbb{S}^1 \times \mathbb{S}^1$ in the coordinates (θ, ψ) as

$$\begin{aligned} \dot{\theta} &= \sqrt{2f(\theta)} \cos\psi \\ \dot{\psi} &= (1 - k/2)\sqrt{2f(\theta)} \cos\psi - \frac{f'(\theta)}{\sqrt{2f(\theta)}} \sin\psi \end{aligned} \tag{A.9}$$

This finally leads to an ordinary differential equation for ψ in terms of θ of the form

$$\frac{\mathrm{d}\psi}{\mathrm{d}\theta} = (1 - k/2) - \frac{f'(\theta)}{2f(\theta)} \tan\psi \tag{A.10}$$

Now we investigate how (A.10) relates to the formal solution (3.2) of the Hamilton-Jacobi equation for the Hamiltonian (A.1). A solution is sought to

$$\frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - r^{-k} f(\theta) = 0 \tag{A.11}$$

of the form

$$S(r, \theta) = \frac{\sqrt{2} r^{1-k/2}}{1 - k/2} \sin\psi(\theta) \tag{A.12}$$

and we find that $\psi(\theta)$ must satisfy (A.10). Observe that if $f(\theta) = 1$ then (A.12) is essentially the formal solution, (3.2), of the Hamilton-Jacobi equation used to generate the Lagrangian submanifolds Λ_k in Section 3. This hints at some connection between the collision manifold analysis and the Lagrangian submanifolds discussed in this paper.

This relationship can be made more clear with a few observations about the Lagrangian submanifolds Λ_k and the flows on them as shown in Figure 2. We see that Λ_k is actually the union of $\mathfrak{X}_k$-invariant submanifolds, each lying in a strip φ (or s)$\in [(n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi)]$ for $n \in \mathbb{Z}$. Each of these submanifolds is Lagrangian since they are submanifolds of Λ_k. If we denote by $\Lambda_{k,n}$ one of these $\mathfrak{X}_k$-invariant submanifolds of Λ_k, we can say the following: *When $f(\theta) = 1$ there is a 1-1 correspondence between solutions $(\theta(t), \psi(t))$ of (A.9), and the Lagrangian submanifolds $\Lambda_{k,n}$.* There is no reason to believe that a similar correspondence does not exist when $f(\theta) \neq 1$. However, in such cases a description of the Lagrangian submanifolds could be expected to be somewhat more complicated. A well-studied example of such a problem is the anisotropic Kepler problem (see [Devaney 1978], and [Gutzwiller 1973]).

References

Abraham, R. and Marsden, J. E. [1978] *Foundations of Mechanics*, second edition, Addison Wesley, Reading, MA, ISBN 0-8053-0102-X.

Devaney, R. L. [1978] *Collision orbits in the anisotropic Kepler problem*, Inventiones Mathematicae, **45**(3), 221–251.

— [1980] *Morse-Smale singularities in simple mechanical systems*, Journal of Differential Geometry, **15**, 285–305.

Gordon, W. B. [1973] *An analytic criterion for the completeness of Riemannian manifolds*, Proceedings of the American Mathematical Society, **37**, 221–225.

Guillemin, V. and Sternberg, S. [1977] *Geometric Asymptotics*, number 14 in American Mathematical Society Mathematical Survey, American Mathematical Society, Providence, Rhode Island.

Gutzwiller, M. C. [1973] *The anisotropic Kepler problem in two dimensions*, Journal of Mathematical Physics, **14**, 139–152.

McGehee, R. [1974] *Triple collision in the collinear three-body problem*, Inventiones Mathematicae, **27**, 191–227.

Ong, C. P. [1975] *Curvature and mechanics*, Advances in Mathematics, **15**, 269–311.