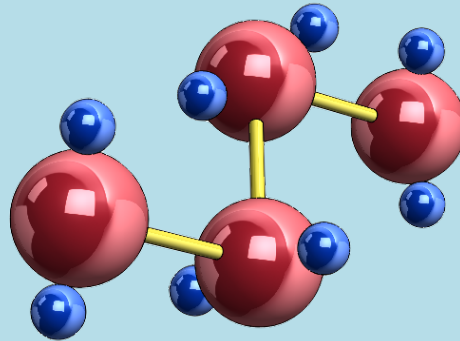


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**NORMAL STRESS DIFFERENCES FROM OLDROYD 8-CONSTANT
FRAMEWORK: EXACT ANALYTICAL SOLUTION FOR
LARGE-AMPLITUDE OSCILLATORY SHEAR FLOW**

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This report is circulated to persons believed to have an active interest in the subject matter; it is intended to furnish rapid communication and to stimulate comment, including corrections of possible errors.

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ABSTRACT

The Oldroyd 8-constant framework for continuum constitutive theory contains a rich diversity of popular special cases for polymeric liquids. In this paper, we use part of our exact solution for shear stress to arrive at unique exact analytical solutions for the normal stress difference responses to large-amplitude oscillatory shear flow (LAOS). The nonlinearity of the polymeric liquids, triggered by LAOS, causes these responses at even multiples of the test frequency. We call responses at frequency higher than twice the test frequency *higher harmonics*. We find the new exact analytical solutions to be compact and intrinsically beautiful. These solutions reduce to those of our previous work on the special case of the corotational Maxwell fluid. Our solutions also agree with our new truncated Goddard integral expansion for the special case of the corotational Jeffreys fluid. The limiting behaviors of our exact solutions for the Oldroyd 8-constant framework yield new explicit expressions for the normal stress difference responses in small-amplitude amplitude oscillatory shear flow (SAOS). Finally, we use our exact solutions to see how η_{∞} affects the normal stress differences in LAOS.

Keywords: Large-amplitude oscillatory shear; LAOS; Oldroyd 8-constant model; normal stress differences; exact analytical solution.

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I. INTRODUCTION

Large-amplitude oscillatory shear (LAOS) flow experiments have been used to investigate the physics of complex liquids (see [1]; Ch. 11 of [2]). Since its conception in 1935 [3,4,5] oscillatory shear flow has become by far the most popular laboratory method for exploring the physics of polymeric liquids. Table I summarizes the literature on experimental measurements of normal stress differences in LAOS, and from Table I we learn that few have undertaken these measurements. Of these, twelve undertook measurements of just the first normal stress difference, only three, both the first and second, and just two, the sum of the first and second. In this paper, we arrive at unique exact analytical solutions for the normal stress difference responses to LAOS. The limiting behaviors of these exact solutions for the Oldroyd 8-constant framework yield new explicit expressions for the normal stress difference responses in small-amplitude oscillatory shear flow (SAOS).

We generate oscillatory shear flow by confining the fluid to a simple shear apparatus (see Figure 1 of [6]), and we then subject one solid-liquid boundary to a coplanar sinusoidal displacement (see Figure 3 of [6]). The corresponding cosinusoidal shear rate is given by:

$$\dot{\gamma}(t) = \dot{\gamma}^0 \cos \omega t \quad (1)$$

Using the characteristic relaxation time of the viscoelastic fluid, λ_1 , we can nondimensionalize Eq. (1):

$$\lambda_1 \dot{\gamma}(t) = Wi \cos De(t / \lambda_1) \quad (2)$$

where:

$$De \equiv \lambda_1 \omega \quad (3)$$

and:

$$Wi \equiv \lambda_1 \dot{\gamma}^0 \quad (4)$$

are the Deborah number and the Weissenberg number. In this paper, we define dimensional symbols in Table III, and dimensionless ones in Table IV (which follow Tables 2 and 3 of [124] with adaptations for [7]).

When higher harmonics are observed in the normal stress responses, we call the oscillatory experiment *large-amplitude*. By *higher harmonics*, we mean contributions to the normal stress at even multiples of the test frequency that are higher than twice the test frequency:

$$\frac{N_1(\omega, \dot{\gamma}^0, \tau)}{(\dot{\gamma}^0)^2} \equiv \frac{\tau_{xx} - \tau_{yy}}{(\dot{\gamma}^0)^2} = - \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \Psi'_{1,n}(\omega, \dot{\gamma}^0) \cos n\tau + \Psi''_{1,n}(\omega, \dot{\gamma}^0) \sin n\tau \quad (5)$$

and:

$$\frac{N_2(\omega, \dot{\gamma}^0, \tau)}{(\dot{\gamma}^0)^2} \equiv \frac{\tau_{yy} - \tau_{zz}}{(\dot{\gamma}^0)^2} = - \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \Psi'_{2,n}(\omega, \dot{\gamma}^0) \cos n\tau + \Psi''_{2,n}(\omega, \dot{\gamma}^0) \sin n\tau \quad (6)$$

where $\tau \equiv \omega t$. For polymeric liquids, these higher harmonics are commonly observed when:

$$Wi > 1 \quad (7)$$

Eq. (7) with $De > 0$ is thus our working definition of LAOS and improves upon previous definitions ([1,8]; Eq. (8) of [124]). With recent advances, rheometers for experiments satisfying Eq. (7) are now available [9]. Many notations alternative to Eq. (5) have been introduced for analyzing the higher harmonics in the normal stress differences (see Section 10 of [124]).

In this paper, we follow Dealy *et al.* (1973) in plotting loops of normal stress differences *versus* $\dot{\gamma}(2\tau) = \dot{\gamma}^0 \cos 2\tau$, since these best bring out material nonlinearities as distortions from ellipticity [10,11,12]. We call these loops frequency-matched. By *frequency-matched*, we mean that the abscissa frequency matches twice the test frequency. To deepen our understanding of frequency-matched loops, we consider the loop area $\oint N_1 d\dot{\gamma}(2\tau)$ and $\oint N_2 d\dot{\gamma}(2\tau)$ for the general normal stress difference responses given by Eqs. (5) and (6):

$$\oint N_1 d\dot{\gamma}(2\tau) = \pi (\dot{\gamma}^0)^3 \Psi''_{1,1}(\omega, \dot{\gamma}^0) \quad (8)$$

and:

$$\oint N_2 d\dot{\gamma}(2\tau) = \pi (\dot{\gamma}^0)^3 \Psi''_{2,1}(\omega, \dot{\gamma}^0) \quad (9)$$

from which we learn that all of the loop area $\oint N_i d\dot{\gamma}(2\tau)$ is in the first harmonic, and that this area is proportional to the elastic part of the first normal stress difference harmonic, $\Psi''_{i,1}(\omega, \dot{\gamma}^0)$. By *elastic*, we mean the part that is out-of-phase with $\dot{\gamma}(2\tau)$. Frequency-matched loops also clearly display the phase difference between the normal stress differences and $\dot{\gamma}(2\tau)$.

In some of the entries in Table I, only part of a higher harmonic of the normal stress differences is measured. We use Columns 9 and 10 to detail these entries. For instance, in Ref. [87], only the parts of the normal stress differences that are in-phase with $\cos 2\tau$, that is, $\Psi'_{i,2}$, are measured. By contrast, in Ref. [90], only the magnitudes of the oscillating parts of the normal stress differences, that is, $|\Psi^*_{i,2}|$, are measured. Column 6 entries indicate measurements of loops of the normal stress differences *versus* either the shear strain ($\frac{Wi}{De} \sin \tau$) or the shear rate ($Wi \cos \tau$). To obtain Fourier series from loops, including loops of normal stress differences, the graphical method of [13] is suggested. Columns 9 and 10 of Table I correct columns 6 and 7 of Table 1 in [126].

Table II chronicles the constitutive equations that yielded to analytical solution to the normal stress difference responses in LAOS. From column 8 of Table II, we glean that the literature contains one and only one exact solution

for LAOS, and this, for the corotational Maxwell model [132]. With some difficulty, this solution was arrived at using the method of Kovacic [14] for the homogeneous part, and of variation of variables [15], for the particular. A method has also been proposed for arriving at approximate analytical solutions to K-BKZ integral models in LAOS including the normal stress differences (see Appendix B of [16] or Subsection 2.2.5 of Chapter 11 of [2]).

In this paper, we continue this work with the more ambitious project of arriving at the exact analytical solution for both normal stress differences for LAOS for an entire framework of constitutive equations. Specifically, we choose the Oldroyd 8-constant framework for its rich diversity of important special cases, 14 of which are tabulated in Table IV of [6].

a. Oscillatory Shear Flow

In the absence of fluid inertia [17], for isothermal [17,18,19] oscillatory simple shear flow, the rate of deformation tensor is given by:

$$\dot{\gamma} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & \dot{\gamma}^0 \cos \omega t & 0 \\ \dot{\gamma}^0 \cos \omega t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (10)$$

and for *any* simple shear flow (see Eq. (19) of [20]):

$$\frac{1}{2} \{ \boldsymbol{\omega} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega} \} = \frac{1}{2} \begin{bmatrix} -2\tau_{yx} & \tau_{xx} - \tau_{yy} & 0 \\ \tau_{xx} - \tau_{yy} & 2\tau_{yx} & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma} \quad (11)$$

and:

$$\frac{1}{2} \{ \boldsymbol{\omega} \cdot \dot{\gamma} - \dot{\gamma} \cdot \boldsymbol{\omega} \} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}^2 \quad (12)$$

where the vorticity tensor, $\boldsymbol{\omega}$, is defined in Eq. (16) below.

b. Oldroyd 8-Constant Framework

The Oldroyd 8-constant framework is given by (see Eq. (8.1-2) in [21];[22]):

$$\begin{aligned} \boldsymbol{\tau} + \lambda_1 \frac{\mathcal{D}\boldsymbol{\tau}}{\mathcal{D}t} + \frac{1}{2}\mu_0 (\text{tr } \boldsymbol{\tau}) \dot{\gamma} - \frac{1}{2}\mu_1 \{ \boldsymbol{\tau} \cdot \dot{\gamma} + \dot{\gamma} \cdot \boldsymbol{\tau} \} + \frac{1}{2}\nu_1 (\boldsymbol{\tau} : \dot{\gamma}) \boldsymbol{\delta} \\ = -\eta_0 \left(\dot{\gamma} + \lambda_2 \frac{\mathcal{D}\dot{\gamma}}{\mathcal{D}t} - \mu_2 \{ \dot{\gamma} \cdot \dot{\gamma} \} + \frac{1}{2}\nu_2 (\dot{\gamma} : \dot{\gamma}) \boldsymbol{\delta} \right) \end{aligned} \quad (13)$$

Eq. (13) reduces exactly to a rich diversity of special cases, including (1) Oldroyd 6-constant [22], (2) Oldroyd 4-constant [22], (3) Gordon-Schowalter [23], (4) Johnson-Segalman [24,25], (5) Oldroyd Fluid A (lower convected Jeffreys) (see Eq. (A) 9.1-9 of [21]), (6) Oldroyd B (upper convected Jeffreys) (see

Eq. (B) 9.1-9 of [21]), (7) second-order (see Eq. 8.4-3 of [21]), (8) arbitrary normal stress ratio (see Eq. (10) of [126]), (9) corotational Jeffreys [26 or 27], (10) Williams 3-constant Oldroyd [110,28], (11) Denn modified convected Maxwell [29], (12) lower convected Maxwell (Eq. (3.23) of [30]), (13) upper convected Maxwell [31], and (14) corotational Maxwell [124] (see Table IV of [6]).

The Oldroyd 8-constant framework has also been closely connected, albeit approximately, with macromolecular theory ([117,32]; see Table 1 of [33]; see Eqs. (32) of [34],[35]; see Tables 6.2-1 and 6.2-2 of [36]; Problems 11B.9 and 11B.10 of [37]; §IV and §V. of [38]; §9.5 of [21]). With Eqs. (13), the Oldroyd 8-constant framework thus provides a useful approximation to the polymer contribution to the stresses for a suspension of rigid dumbbells, and for FENE dumbbells [33].

The total stress tensor is defined by:

$$\boldsymbol{\pi} \equiv \boldsymbol{\tau} + p\boldsymbol{\delta} \quad (14)$$

the rate-of-deformation tensor by:

$$\dot{\boldsymbol{\gamma}} \equiv \nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger \quad (15)$$

the vorticity tensor by:

$$\boldsymbol{\omega} \equiv \nabla \mathbf{v} - (\nabla \mathbf{v})^\dagger \quad (16)$$

the corotational derivative by:

$$\frac{\mathcal{D}\mathbf{b}}{\mathcal{D}t} \equiv \frac{D\mathbf{b}}{Dt} + \frac{1}{2}\{\boldsymbol{\omega} \cdot \mathbf{b} - \mathbf{b} \cdot \boldsymbol{\omega}\} \quad (17)$$

In Eq. (14), $\boldsymbol{\delta}$ is the *kroncker delta* and p is the hydrodynamic pressure. When it leads to analytical solutions, the Oldroyd 8-constant model can be remarkably useful for polymer processing, as it is for wire coating (Case III in [39]), and for corrugated wire coated through a corrugated die (see Section 3. of [40]; Section 2. of [41]), and for plastic pipe extrusion for elliptical pipe ([42,43]), and also for extrusion from an eccentric annular pipe die (Case II in [39];[44];[45,46];[47]).

i. Steady Shear

For steady shear flow, the viscometric functions for the Oldroyd 8-constant framework are given by (see Eqs. (12)–(14) of [22]; Eqs. (14), (22) and (23) of [48]):

$$\frac{\eta(\dot{\gamma})}{\eta_0} = -\frac{\Psi_1(\dot{\gamma})}{\Psi_{10}} \left(\frac{\lambda_2}{\lambda_1} - 1 \right) + \frac{\lambda_2}{\lambda_1} = \frac{\Psi_2(\dot{\gamma})}{\Psi_{20}} \left(1 - \frac{\lambda_2 - \mu_2}{\lambda_1 - \mu_1} \right) + \frac{\lambda_2 - \mu_2}{\lambda_1 - \mu_1} = \frac{1 + \sigma_2 \dot{\gamma}^2}{1 + \sigma_1 \dot{\gamma}^2} \quad (18)$$

where:

$$\Psi_{10} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_1 = 2\eta_0(\lambda_1 - \lambda_2) \quad (19)$$

$$\Psi_{20} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_2 = -\eta_0(\lambda_1 - \lambda_2 - \mu_1 + \mu_2) \quad (20)$$

and where:

$$\sigma_1 \equiv \lambda_1^2 + \mu_0 \left(\mu_1 - \frac{3}{2} v_1 \right) - \mu_1 (\mu_1 - v_1) \quad (21)$$

$$\sigma_2 \equiv \lambda_1 \lambda_2 + \mu_0 \left(\mu_2 - \frac{3}{2} v_2 \right) - \mu_1 (\mu_2 - v_2) \quad (22)$$

As $Wi \rightarrow \infty$, Eq. (18) becomes

$$\frac{\eta_\infty}{\eta_0} = -\frac{\Psi_{1\infty}}{\Psi_{10}} \left(\frac{\lambda_2}{\lambda_1} - 1 \right) + \frac{\lambda_2}{\lambda_1} = \frac{\Psi_{2\infty}}{\Psi_{20}} \left(\frac{\lambda_1 - \lambda_2 - \mu_1 + \mu_2}{\lambda_1 - \mu_1} \right) + \frac{\lambda_2 - \mu_2}{\lambda_1 - \mu_1} = \frac{\sigma_2}{\sigma_1} \quad (23)$$

which for the special case of the corotational Jeffreys fluid, where $\mu_0 = \mu_1 = \mu_2 = v_1 = v_2 = 0$:

$$\frac{\Psi_{1\infty}}{\Psi_{10}} = \frac{\Psi_{2\infty}}{\Psi_{20}} = 0 \quad (24)$$

From Eqs. (23) and (24), we learn that only λ_2 affects η_∞ , and we will explore the role of η_∞ later in the Worked Example.

Rewriting Eq. (18) for the dimensionless first and second normal stress difference responses:

$$\frac{\sigma_1 N_1}{\Psi_{10}} = -\frac{1 + \sigma \left(\sqrt{\sigma_1} \dot{\gamma} \right)^2}{1 + \left(\sqrt{\sigma_1} \dot{\gamma} \right)^2} - \frac{\lambda_2}{\lambda_1} \frac{1}{1 - \frac{\lambda_2}{\lambda_1}} \left(\sqrt{\sigma_1} \dot{\gamma} \right)^2 \quad (25)$$

and:

$$\frac{\sigma_1 N_2}{\Psi_{20}} = -\frac{1 + \sigma \left(\sqrt{\sigma_1} \dot{\gamma} \right)^2}{1 + \left(\sqrt{\sigma_1} \dot{\gamma} \right)^2} - \frac{\lambda_2 - \mu_2}{\lambda_1 - \mu_1} \frac{1}{1 - \frac{\lambda_2 - \mu_2}{\lambda_1 - \mu_1}} \left(\sqrt{\sigma_1} \dot{\gamma} \right)^2 \quad (26)$$

where:

$$\sigma \equiv \frac{\sigma_2}{\sigma_1} = \frac{\lambda_1 \lambda_2 + \mu_0 \left(\mu_2 - \frac{3}{2} v_2 \right) - \mu_1 (\mu_2 - v_2)}{\lambda_1^2 + \mu_0 \left(\mu_1 - \frac{3}{2} v_1 \right) - \mu_1 (\mu_1 - v_1)} \quad (27)$$

In Subsection IV.b, we will use Eqs. (25) and (26) to check our main results for consistency.

ii. Small-Amplitude Oscillatory Shear (SAOS)

For the special case of the corotational Jeffreys fluid, where $\mu_0 = \mu_1 = \mu_2 = v_1 = v_2 = 0$, Eq. (13) reduces to:

$$\tau + \lambda_1 \frac{\mathcal{D}\tau}{\mathcal{D}t} = -\eta_0 \left(\dot{\gamma} + \lambda_2 \frac{\mathcal{D}\dot{\gamma}}{\mathcal{D}t} \right) \quad (28)$$

For SAOS, the expressions for the dimensionless normal stress differences are:

$$\mathbb{N}_1 \equiv \frac{\tau_{xx} - \tau_{yy}}{\eta_0 \dot{\gamma}^0} = -\lambda_1 \dot{\gamma}^0 \left[\frac{\Psi_1^d}{\eta_0 \lambda_1} + \frac{\Psi_1'}{\eta_0 \lambda_1} \cos 2\omega t + \frac{\Psi_1''}{\eta_0 \lambda_1} \sin 2\omega t \right] \quad (29)$$

and:

$$\mathbb{N}_2 \equiv \frac{\tau_{yy} - \tau_{zz}}{\eta_0 \dot{\gamma}^0} = -\lambda_1 \dot{\gamma}^0 \left[\frac{\Psi_2^d}{\eta_0 \lambda_1} + \frac{\Psi_2'}{\eta_0 \lambda_1} \cos 2\omega t + \frac{\Psi_2''}{\eta_0 \lambda_1} \sin 2\omega t \right] \quad (30)$$

where, comparing Eqs. (29) and (30) with Eqs. (5) and (6), gives $\Psi'_{i,0} \equiv \Psi_i^d$, $\Psi'_{i,2} \equiv \Psi_i'$ and $\Psi''_{i,2} \equiv \Psi_i''$.

We next use the well-known Goddard integral expansion employing the method of **Section 8.** of [124] to the leading order term in Eq. (66) of [124], to get (see also detailed work for \mathbb{N}_i in Subsection IV.c.i):

$$\mathbb{N}_1 = -2\mathbb{N}_2 = -\text{Wi} \left[\left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{1}{1 + \text{De}^2} + \left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{1 - 2\text{De}^2}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \cos 2\tau \right. \\ \left. + \left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{3\text{De}}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \sin 2\tau \right] \quad (31)$$

For SAOS, the coefficient of the displacement terms of the normal stress differences is given by:

$$\Psi_1^d = -2\Psi_2^d = \left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{\eta_0 \lambda_1}{1 + \text{De}^2} \quad (32)$$

and of the parts that are in-phase with $\cos 2\tau$:

$$\Psi_1' = -2\Psi_2' = \left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{\eta_0 \lambda_1 (1 - 2\text{De}^2)}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \quad (33)$$

and the parts that are out-of-phase with $\cos 2\tau$:

$$\Psi_1'' = -2\Psi_2'' = \left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{3\eta_0 \lambda_1 \text{De}}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \quad (34)$$

so that:

$$\mathbb{N}_1 = -2\mathbb{N}_2 = -\text{Wi} \left[\left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{1}{1 + \text{De}^2} + \left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{1 - 2\text{De}^2}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \cos 2\tau \right. \\ \left. + \left(1 - \frac{\lambda_2}{\lambda_1} \right) \frac{3\text{De}}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \sin 2\tau \right] \quad (35)$$

which we will use in Subsection IV.a to check the consistency of our new exact solutions to the Oldroyd 8-constant framework for LAOS.

II. METHOD

For oscillatory shear flow, Eq. (13) gives (see **APPENDIX A** of [6]):

$$\begin{aligned} \frac{d}{dt} \tau_{yx} = & -\frac{1}{2} \left(1 - \frac{\mu_1}{\lambda_1} + \frac{\mu_0}{\lambda_1} \right) N_1 \dot{\gamma}^0 \cos \omega t + \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_0}{\lambda_1} \right) N_2 \dot{\gamma}^0 \cos \omega t \\ & - \left(\frac{3\mu_0}{2\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \tau_{zz} \dot{\gamma}^0 \cos \omega t - \frac{1}{\lambda_1} \tau_{yx} - \eta_0 \left(\frac{1}{\lambda_1} \dot{\gamma}^0 \cos \omega t - \frac{\lambda_2}{\lambda_1} \dot{\gamma}^0 \omega \sin \omega t \right) \end{aligned} \quad (36)$$

$$\frac{d}{dt} N_1 = -\frac{1}{\lambda_1} N_1 + 2\tau_{yx} \dot{\gamma}^0 \cos \omega t + 2\frac{\lambda_2}{\lambda_1} \eta_0 (\dot{\gamma}^0)^2 \cos^2 \omega t \quad (37)$$

$$\frac{d}{dt} N_2 = -\frac{1}{\lambda_1} N_2 - \left(1 - \frac{\mu_1}{\lambda_1} \right) \tau_{yx} \dot{\gamma}^0 \cos \omega t - \eta_0 \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) (\dot{\gamma}^0)^2 \cos^2 \omega t \quad (38)$$

$$\frac{d}{dt} \tau_{zz} = -\frac{1}{\lambda_1} \tau_{zz} - \frac{\nu_1}{\lambda_1} \tau_{yx} \dot{\gamma}^0 \cos \omega t - \eta_0 \frac{\nu_2}{\lambda_1} (\dot{\gamma}^0)^2 \cos^2 \omega t \quad (39)$$

which can be non-dimensionalized to:

$$\begin{aligned} \frac{d\mathbb{S}}{d\tau} = & -\frac{1}{2} \left(1 - \frac{\mu_1}{\lambda_1} + \frac{\mu_0}{\lambda_1} \right) \frac{\text{Wi}}{\text{De}} \mathbb{N}_1 \cos \tau + \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_0}{\lambda_1} \right) \frac{\text{Wi}}{\text{De}} \mathbb{N}_2 \cos \tau \\ & - \left(\frac{3\mu_0}{2\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \frac{\text{Wi}}{\text{De}} \tilde{\tau}_{zz} \cos \tau - \frac{1}{\text{De}} \mathbb{S} - \left(\frac{1}{\text{De}} \cos \tau - \frac{\lambda_2}{\lambda_1} \sin \tau \right) \end{aligned} \quad (40)$$

$$\frac{d\mathbb{N}_1}{d\tau} = -\frac{1}{\text{De}} \mathbb{N}_1 + 2\frac{\text{Wi}}{\text{De}} \mathbb{S} \cos \tau + 2\frac{\lambda_2}{\lambda_1} \frac{\text{Wi}}{\text{De}} \cos^2 \tau \quad (41)$$

$$\frac{d\mathbb{N}_2}{d\tau} = -\frac{1}{\text{De}} \mathbb{N}_2 - \frac{\text{Wi}}{\text{De}} \left(1 - \frac{\mu_1}{\lambda_1} \right) \mathbb{S} \cos \tau - \frac{\text{Wi}}{\text{De}} \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) \cos^2 \tau \quad (42)$$

$$\frac{d\tilde{\tau}_{zz}}{d\tau} = -\frac{1}{\text{De}} \tilde{\tau}_{zz} - \frac{\nu_1}{\lambda_1} \frac{\text{Wi}}{\text{De}} \mathbb{S} \cos \tau - \frac{\nu_2}{\lambda_1} \frac{\text{Wi}}{\text{De}} \cos^2 \tau \quad (43)$$

Eqs. (40)–(43) are thus our coupled system of differential equations that governs the fluid responses in LAOS. By *coupled*, we mean that the independent variables, (\mathbb{S} , \mathbb{N}_1 , \mathbb{N}_2 and $\tilde{\tau}_{zz}$), are not separated.

The first and second normal stress responses have the form of:

$$\mathbb{N}_1 = \mathbb{N}_{1,h} + \mathbb{N}_{1,p} \quad (44)$$

and:

$$\mathbb{N}_2 = \mathbb{N}_{2,h} + \mathbb{N}_{2,p} \quad (45)$$

where $\mathbb{N}_{i,h}$ are the transient part of the normal stress differences, and $\mathbb{N}_{i,p}$ are the particular parts. In Subsection II.a, we will solve Eqs. (40)–(43) for \mathbb{S}_h , $\mathbb{N}_{1,h}$, $\mathbb{N}_{2,h}$ and $\tilde{\tau}_{zz,h}$ simultaneously, uniquely and exactly, and then in Subsection II.b, particularly with $\mathbb{N}_{i,p}$.

a. Transient Parts of the Stress Responses

As intermediate results for the particular parts of the normal stress response difference responses in Subsection II.b, we need the transient responses of S_h , $N_{1,h}$, $N_{2,h}$ and $\tilde{\tau}_{zz,h}$. Specifically, we will use these to craft the fundamental matrix, Φ , for calculating $N_{i,p}$. We begin with Eq. (61) of [6]:

$$\begin{bmatrix} S_h(\tau) \\ N_{1,h}(\tau) \\ N_{2,h}(\tau) \\ \tilde{\tau}_{zz,h}(\tau) \end{bmatrix} = e^{\frac{-\tau}{De}} \begin{bmatrix} S & C & 0 & 0 \\ -\frac{2}{\alpha} \frac{Wi}{De} C & \frac{2}{\alpha} \frac{Wi}{De} S & 0 & 1 \\ \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{Wi}{De} C & -\frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{Wi}{De} S & 1 & 0 \\ \frac{1}{\alpha} \frac{v_1}{\lambda_1} \frac{Wi}{De} C & -\frac{1}{\alpha} \frac{v_1}{\lambda_1} \frac{Wi}{De} S & -2 \frac{\mu_0 - \mu_1}{3\mu_0 - 2\mu_1} & -\frac{\lambda_1 + \mu_0 - \mu_1}{3\mu_0 - 2\mu_1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \quad (46)$$

from which the fundamental matrix:

$$\Phi = e^{\frac{-\tau}{De}} \begin{bmatrix} S & C & 0 & 0 \\ -\frac{2}{\alpha} \frac{Wi}{De} C & \frac{2}{\alpha} \frac{Wi}{De} S & 0 & 1 \\ \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{Wi}{De} C & -\frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{Wi}{De} S & 1 & 0 \\ \frac{1}{\alpha} \frac{v_1}{\lambda_1} \frac{Wi}{De} C & -\frac{1}{\alpha} \frac{v_1}{\lambda_1} \frac{Wi}{De} S & -2 \frac{\mu_0 - \mu_1}{3\mu_0 - 2\mu_1} & -\frac{\lambda_1 + \mu_0 - \mu_1}{3\mu_0 - 2\mu_1} \end{bmatrix} \quad (47)$$

can be extracted, where the following trig functions of trig functions arose:

$$S \equiv \sin(\alpha \sin \tau) \quad (48)$$

$$C \equiv \cos(\alpha \sin \tau) \quad (49)$$

and where:

$$\alpha \equiv \frac{Wi}{De} \tilde{\lambda} \equiv \frac{Wi}{De} \sqrt{1 + \frac{\mu_0 \mu_1}{\lambda_1^2} - \frac{\mu_1^2}{\lambda_1^2} - \frac{3}{2} \frac{\mu_0 v_1}{\lambda_1^2} + \frac{\mu_1 v_1}{\lambda_1^2}} \quad (50)$$

For the special case of the corotational Maxwell fluid, $\lambda_2 = \mu_0 = \mu_1 = \mu_2 = v_1 = v_2 = 0$, Eq. (47) reduces to Eq. (41) of [132] as it should. In Subsection II.b, we will use this fundamental matrix, Φ , to calculate the alternant normal stress difference responses to Eqs. (40)–(43).

b. Particular Part of the Stress Responses

The particular part of our stress response is given by (see Eq. (10) on p.711 of [15]):

$$\begin{bmatrix} \mathbb{S}_p(\tau) \\ \mathbb{N}_{1,p}(\tau) \\ \mathbb{N}_{2,p}(\tau) \\ \tilde{\tau}_{zz,p}(\tau) \end{bmatrix} = \Phi(\tau) \int_0^\tau \Phi^{-1}(\tau') \begin{bmatrix} 2 \frac{\lambda_2}{\lambda_1} \frac{\text{Wi}}{\text{De}} \cos^2 \tau' \\ -\frac{\text{Wi}}{\text{De}} \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) \cos^2 \tau' \\ -\frac{\nu_2}{\lambda_1} \frac{\text{Wi}}{\text{De}} \cos^2 \tau' \\ -\left(\frac{1}{\text{De}} \cos \tau' - \frac{\lambda_2}{\lambda_1} \sin \tau' \right) \end{bmatrix} d\tau' \quad (51)$$

which matched Eq. (63) of [6], and where Φ is defined in Eq. (47). Previously, we solved exactly and uniquely for the shear stress response, $\mathbb{S}_p(\tau)$, in Eq. (51). In this paper, we attack both normal stress difference responses, $\mathbb{N}_{i,p}(\tau)$, in Eq. (51).

III. EXACT SOLUTION: NORMAL STRESS DIFFERENCES

We next focus on the alternant normal stress difference responses. By *alternant*, we mean the response after the startup transient has vanished. To get the alternant normal stress differences, where $\mathbb{N}_{i,h} = 0$ and $\mathbb{N}_i = \mathbb{N}_{i,p}(\tau \gg 0)$, we first evaluate the $\mathbb{N}_{1,p}(\tau)$ component of Eq. (51) to get:

$$\begin{aligned} \mathbb{N}_1 = & -\frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} e^{-\frac{\tau}{\text{De}}} \text{C} \left[\begin{aligned} & \frac{-1}{\text{De}} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos \tau' \sin(\alpha \sin \tau') d\tau' + \frac{\lambda_2}{\lambda_1} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \sin \tau' \sin(\alpha \sin \tau') d\tau' \\ & + \frac{\text{Wi}}{\text{De}} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos^2 \tau' \cos(\alpha \sin \tau') d\tau' \end{aligned} \right] \\ & + \frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} e^{-\frac{\tau}{\text{De}}} \text{S} \left[\begin{aligned} & \frac{-1}{\text{De}} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos \tau' \cos(\alpha \sin \tau') d\tau' + \frac{\lambda_2}{\lambda_1} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \sin \tau' \cos(\alpha \sin \tau') d\tau' \\ & - \frac{\text{Wi}}{\text{De}} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos^2 \tau' \sin(\alpha \sin \tau') d\tau' \end{aligned} \right] \quad (52) \\ & + \frac{2}{\tilde{\lambda}^2} \left[\begin{aligned} & -\left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} - \tilde{\lambda}^2 \right) \frac{\lambda_2}{\lambda_1} \\ & + \left(\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) + \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \frac{\nu_2}{\lambda_1} \end{aligned} \right] \frac{\text{Wi}}{\text{De}} e^{-\frac{\tau}{\text{De}}} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos^2 \tau' d\tau' \end{aligned}$$

and then the $\mathbb{N}_{2,p}(\tau)$ component:

$$\begin{aligned}
\mathbb{N}_2 = & \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \text{C} \left[\begin{aligned} & \frac{-1}{\text{De}} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos \tau' \sin(\alpha \sin \tau') d\tau' + \frac{\lambda_2}{\lambda_1} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \sin \tau' \sin(\alpha \sin \tau') d\tau' \\ & + \frac{\text{Wi}}{\text{De}} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos^2 \tau' \cos(\alpha \sin \tau') d\tau' \end{aligned} \right] \\
& - \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \text{S} \left[\begin{aligned} & \frac{-1}{\text{De}} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos \tau' \cos(\alpha \sin \tau') d\tau' + \frac{\lambda_2}{\lambda_1} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \sin \tau' \cos(\alpha \sin \tau') d\tau' \\ & - \frac{\text{Wi}}{\text{De}} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos^2 \tau' \sin(\alpha \sin \tau') d\tau' \end{aligned} \right] \\
& + \frac{1}{\tilde{\lambda}^2} \left[\begin{aligned} & \left(1 - \frac{\mu_1}{\lambda_1}\right) \left[\left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{\lambda_2}{\lambda_1} - \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{\nu_2}{\lambda_1} \right] \\ & - \left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} - \frac{3}{2} \frac{\mu_0 \nu_1}{\lambda_1^2} + \frac{\mu_1 \nu_1}{\lambda_1^2}\right) \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1}\right) \right] \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \int_0^\tau e^{\frac{\tau'}{\text{De}}} \cos^2 \tau' d\tau' \end{aligned} \right] \quad (53)
\end{aligned}$$

We define $I_1 - I_7$ as:

$$\begin{aligned}
\mathbb{N}_1 = & -\frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \text{C} \left[\begin{aligned} & \frac{-1}{\text{De}} I_1 + \frac{\lambda_2}{\lambda_1} I_3 + \frac{\text{Wi}}{\text{De}} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) I_5 \end{aligned} \right] \\
& + \frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \text{S} \left[\begin{aligned} & \frac{-1}{\text{De}} I_2 + \frac{\lambda_2}{\lambda_1} I_4 - \frac{\text{Wi}}{\text{De}} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) I_6 \end{aligned} \right] \\
& + \frac{2}{\tilde{\lambda}^2} \left[-\left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} - \tilde{\lambda}^2\right) \frac{\lambda_2}{\lambda_1} + \left(\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1}\right) + \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{\nu_2}{\lambda_1} \right] \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} I_7 \quad (54)
\end{aligned}$$

and:

$$\begin{aligned}
\mathbb{N}_2 = & \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \text{C} \left[\begin{aligned} & \frac{-1}{\text{De}} I_1 + \frac{\lambda_2}{\lambda_1} I_3 + \frac{\text{Wi}}{\text{De}} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) I_5 \end{aligned} \right] \\
& - \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \text{S} \left[\begin{aligned} & \frac{-1}{\text{De}} I_2 + \frac{\lambda_2}{\lambda_1} I_4 - \frac{\text{Wi}}{\text{De}} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) I_6 \end{aligned} \right] \\
& + \frac{1}{\tilde{\lambda}^2} \left[\begin{aligned} & \left(1 - \frac{\mu_1}{\lambda_1}\right) \left[\left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{\lambda_2}{\lambda_1} - \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{\nu_2}{\lambda_1} \right] \\ & - \left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} - \frac{3}{2} \frac{\mu_0 \nu_1}{\lambda_1^2} + \frac{\mu_1 \nu_1}{\lambda_1^2}\right) \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1}\right) \right] \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} I_7 \end{aligned} \right] \quad (55)
\end{aligned}$$

Evaluating the seven integrals in Eq. (54) [or Eq. (55)], and then omitting the lower limit gives:

$$I_1 = \frac{4\text{De}}{\alpha} e^{\tau/\text{De}} \sum_{k=1}^{\infty} \left[-\frac{2\text{De}k^2 J_{2k}}{1+4\text{De}^2 k^2} \cos 2k\tau + \frac{k J_{2k}}{1+4\text{De}^2 k^2} \sin 2k\tau \right] \quad (56)$$

$$I_2 = \frac{2\text{De}}{\alpha} e^{\tau/\text{De}} \sum_{k=1}^{\infty} \left[\frac{\text{De}(2k-1)^2 J_{2k-1}}{\text{De}^2(2k-1)^2 + 1} \sin(2k-1)\tau + \frac{(2k-1) J_{2k-1}}{\text{De}^2(2k-1)^2 + 1} \cos(2k-1)\tau \right] \quad (57)$$

$$I_3 = \text{De} e^{\tau/\text{De}} J_1 + \text{De} e^{\tau/\text{De}} \sum_{k=1}^{\infty} \left[\frac{-J_{2k-1} + J_{2k+1}}{1+4k^2 \text{De}^2} \cos 2k\tau + (2\text{De}k) \frac{-J_{2k-1} + J_{2k+1}}{1+4k^2 \text{De}^2} \sin 2k\tau \right] \quad (58)$$

$$I_4 = \text{De} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{-(2k-1)\text{De}(J_{2k-2} - J_{2k})}{1 + (2k-1)^2 \text{De}^2} \cos(2k-1)\tau + \frac{J_{2k-2} - J_{2k}}{1 + (2k-1)^2 \text{De}^2} \sin(2k-1)\tau \right] \quad (59)$$

$$I_5 = \frac{\text{De} e^{\frac{\tau}{\text{De}}} J_1}{\alpha} + \text{De} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{J_{2k-2} + 2J_{2k} + J_{2k+2}}{2(1 + 4\text{De}^2 k^2)} \cos 2k\tau + (\text{De} k) \frac{J_{2k-2} + 2J_{2k} + J_{2k+2}}{1 + 4\text{De}^2 k^2} \sin 2k\tau \right] \quad (60)$$

$$I_6 = \frac{\text{De}}{2} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{-\text{De}(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \cos(2k-1)\tau \right. \\ \left. + \frac{J_{2k-3} + 2J_{2k-1} + J_{2k+1}}{1 + (2k-1)^2 \text{De}^2} \sin(2k-1)\tau \right] \quad (61)$$

$$I_7 = \frac{\text{De} e^{\frac{\tau}{\text{De}}}}{2} + \frac{\text{De} e^{\frac{\tau}{\text{De}}}}{2(1 + 4\text{De}^2)} \cos 2\tau + \frac{\text{De}^2 e^{\frac{\tau}{\text{De}}}}{1 + 4\text{De}^2} \sin 2\tau \quad (62)$$

For brevity, we just write J_m to mean $J_m(\alpha)$ where α is given by Eq. (50) (see after Eq. (5) of [49] or before Eq. (123) of [132]).

Eqs. (54) and (55) [with Eqs. (56)–(62)] are the main results of this paper. For the special case of the corotational Maxwell fluid, where $\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, Eqs. (54) and (55) [with Eqs. (56)–(62)] reduce to Eq. (56) [with Eqs. (47) and (48)] of [132] as it must. We can report that our exact solutions are both integrable and differentiable, and they thus provide suitable starting points for exploring, analytically and exactly, many nonlinear problems in fluid physics.

Our exact solution takes neither of the usual forms of expansion supposed for approximate solutions, in odd powers of $\lambda\dot{\gamma}^0$, or of $\dot{\gamma}^0/\omega$ (see column 11 in Table 1 of [132]). Instead, the exact solutions take the form of same two trig functions of trig functions [Eqs. (48) and (49)] that arose in our exact solution for the shear stress (Eq. (65) of [6]). We find this form to be intrinsically beautiful. Furthermore, whereas for the shear stress in SAOS, the $\sin(\alpha \sin \tau)$ dropped out, for the normal stress difference responses in SAOS, neither $\cos(\alpha \sin \tau)$ nor $\sin(\alpha \sin \tau)$ drop.

For the special case of the corotational Maxwell model, Eqs. (54) and (55) [with Eqs. (56)–(62)] has been rewritten as a Fourier series (see Eq. (66) with **Appendix: Fourier Analysis of Compact Forms** of [132]). We have yet to rewrite the main results of this paper as a Fourier series.

IV. CONSISTENCY CHECKS

In Section III, we found that our main results Eqs. (54) and (55) [with Eqs. (56)–(62)] reduce to the exact solution for the corotational Maxwell fluid, Eq.

(56) [with Eqs. (47) and (48)] of [132]. In this section, we check our main results for the limiting behavior both when $Wi \rightarrow 0$ (Subsection IV.a) and $De \rightarrow 0$ (Subsection IV.b). We then check our main results with our new approximate solutions for the corotational Jeffreys fluid (Subsection IV.c.i), and finally, with our finite difference solution for the corotational Jeffreys fluid (Subsection IV.c.ii).

a. SAOS

In the Introduction, we learnt that, for any case of the Oldroyd 8-constant framework, the fluid response in SAOS is given by Eq. (35). The following consistency check for SAOS thus applies for any case of the Oldroyd 8-constant framework, including for instance the 14 constitutive equations listed in the subhead of Subsection I.b.

In oscillatory shear flow, for small strain rate amplitude, where $Wi \rightarrow 0$, the normal stress difference responses for the Oldroyd 8-constant framework are given by Eq. (31). With this limiting behaviour, we will next validate our main finding for any special case of the Oldroyd 8-constant framework. Dividing Eqs. (54) and (55) by Wi , then taking the limit of the result as $Wi \rightarrow 0$ gives (see detailed work in Appendix VIII):

$$\begin{aligned} N_1^{\text{SAOS}} \equiv \lim_{Wi \rightarrow 0} \frac{N_1}{Wi} = & - \left[1 + \frac{\phi}{\tilde{\lambda}^2} - \frac{\psi}{\tilde{\lambda}^2} + De^2 \frac{\lambda_2}{\lambda_1} + De^2 \frac{\phi}{\tilde{\lambda}^2} - De^2 \frac{\psi}{\tilde{\lambda}^2} \right] \frac{1}{1 + De^2} \\ & + \left[-1 - \frac{\phi}{\tilde{\lambda}^2} + \frac{\psi}{\tilde{\lambda}^2} + 2De^2 - 3De^2 \frac{\lambda_2}{\lambda_1} - De^2 \frac{\phi}{\tilde{\lambda}^2} + De^2 \frac{\psi}{\tilde{\lambda}^2} \right] \frac{\cos 2\tau}{(1 + De^2)(1 + 4De^2)} \\ & - \left[3 - \frac{\lambda_2}{\lambda_1} + 2\frac{\phi}{\tilde{\lambda}^2} - 2\frac{\psi}{\tilde{\lambda}^2} + 2De^2 \frac{\lambda_2}{\lambda_1} + 2De^2 \frac{\phi}{\tilde{\lambda}^2} - 2De^2 \frac{\psi}{\tilde{\lambda}^2} \right] \frac{De \sin 2\tau}{(1 + De^2)(1 + 4De^2)} \end{aligned} \quad (63)$$

and:

$$\begin{aligned} N_2^{\text{SAOS}} \equiv \lim_{Wi \rightarrow 0} \frac{N_2}{Wi} = & \frac{1}{2} \left(1 - \frac{\mu_1}{\lambda_1} \right) \left[\frac{\lambda_2}{\lambda_1} + \frac{\phi}{\tilde{\lambda}^2} + \frac{1}{1 + De^2} - \frac{\lambda_2}{\lambda_1} \frac{1}{1 + De^2} \right] + \frac{\Omega}{2\tilde{\lambda}^2} \\ & + \frac{1}{2} \left[\left(1 - \frac{\mu_1}{\lambda_1} \right) \left(1 + \frac{\phi}{\tilde{\lambda}^2} - 2De^2 + De^2 \frac{\phi}{\tilde{\lambda}^2} + 3De^2 \frac{\lambda_2}{\lambda_1} \right) + \frac{\Omega}{\tilde{\lambda}^2} + De^2 \frac{\Omega}{\tilde{\lambda}^2} \right] \frac{\cos 2\tau}{(1 + De^2)(1 + 4De^2)} \\ & - \frac{De}{2} \left[\left(1 - \frac{\mu_1}{\lambda_1} \right) \left(-3 + \frac{\lambda_2}{\lambda_1} - 2\frac{\phi}{\tilde{\lambda}^2} - 2\frac{\lambda_2}{\lambda_1} De^2 - 2\frac{\phi}{\tilde{\lambda}^2} De^2 \right) - 2\frac{\Omega}{\tilde{\lambda}^2} - 2De^2 \frac{\Omega}{\tilde{\lambda}^2} \right] \frac{\sin 2\tau}{(1 + De^2)(1 + 4De^2)} \end{aligned} \quad (64)$$

where $\tilde{\lambda}$ is defined in (50) and:

$$\phi \equiv -\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \quad (65)$$

$$\psi \equiv -\left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} - \tilde{\lambda}^2 \right) \frac{\lambda_2}{\lambda_1} + \left(\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) + \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \frac{\nu_2}{\lambda_1} \quad (66)$$

$$\Omega \equiv \left(1 - \frac{\mu_1}{\lambda_1}\right) \left[\left(1 + \frac{\mu_0 - \mu_1}{\lambda_1}\right) \frac{\lambda_2}{\lambda_1} - \left(\frac{3\mu_0 - \mu_1}{2\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{v_2}{\lambda_1} \right] - \left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} - \frac{3\mu_0 v_1}{2\lambda_1^2} + \frac{\mu_1 v_1}{\lambda_1^2}\right) \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1}\right) \quad (67)$$

By contrast with the well-known results for the shear stress response (Eq. (77) of [6]), whose SAOS behavior depends on λ_1 and λ_2 only, Eqs. (63) and (64) show that every constant in the Oldroyd 8-constant framework [Eq. (13)] contributes to the normal stress difference responses in SAOS.

For the special case of the corotational Jeffreys fluid, where $\mu_0 = \mu_1 = \mu_2 = v_1 = v_2 = 0$, Eqs. (63) and (64) reduce to:

$$\mathbb{N}_1^{\text{SAOS}} = -2\mathbb{N}_2^{\text{SAOS}} = -\left(1 - \frac{\lambda_2}{\lambda_1}\right) \left[\frac{1}{1 + \text{De}^2} + \frac{(1 - 2\text{De}^2)\cos 2\tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)} + \frac{3\text{De}\sin 2\tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \right] \quad (68)$$

Further, for the special case of the corotational Maxwell fluid, $\lambda_2 = 0$, Eq. (68) further reduces to:

$$\mathbb{N}_1^{\text{SAOS}} = -2\mathbb{N}_2^{\text{SAOS}} = -\left[\frac{1}{1 + \text{De}^2} + \frac{(1 - 2\text{De}^2)\cos 2\tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)} + \frac{3\text{De}\sin 2\tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \right] \quad (69)$$

which agrees with the leading order term of Eq. (66) of [124] as it should.

b. Steady Shear Flow

In steady shear flow, for the special case of the corotational Maxwell model [Eq. (18) with $\lambda_2 = \mu_0 = \mu_1 = \mu_2 = v_1 = v_2 = 0$], the viscosity function is given by (Eq. (84) of [124]):

$$\frac{\Psi_1(\dot{\gamma})}{2\eta_0\lambda_1} = \frac{-\Psi_2(\dot{\gamma})}{\eta_0\lambda_1} = \frac{1}{1 + (\lambda_1\dot{\gamma})^2} \quad (70)$$

and thus, the normal stress differences, by:

$$\frac{\tau_{xx} - \tau_{yy}}{\eta_0\dot{\gamma}^0} = -2\frac{\tau_{yy} - \tau_{zz}}{\eta_0\dot{\gamma}^0} = \frac{-2}{1 + (\lambda_1\dot{\gamma})^2} \frac{(\lambda_1\dot{\gamma})^2}{\text{Wi}} \quad (71)$$

which we will next use to validate our main result, Eqs. (54) and (55) [with Eqs. (56)–(62)]. We compare the limiting behaviour of our main result for steady shear flow, where $\text{De} \rightarrow 0$, with Eq. (71) in Figure 1. The close agreement in Figure 1, all within a line width, shows that our main results, Eqs. (54) and (55) [with Eqs. (56)–(62)], are consistent with the well-known results [Eqs. (70) and (71)] for steady shear flow of a corotational Maxwell fluid.

c. LAOS

Our main results, Eqs. (54) and (55) [with Eqs. (56)–(62)], can also be checked for consistency in LAOS for the special case of the corotational Jeffreys fluid, where $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. For this, our main results reduce to:

$$\begin{aligned} \mathbb{N}_1 = -2\mathbb{N}_2 = & -2e^{\frac{-\tau}{\text{De}}} \cos\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[\frac{-1}{\text{De}} I_1 + \frac{\lambda_2}{\lambda_1} I_3 - \frac{\text{Wi}}{\text{De}} \frac{\lambda_2}{\lambda_1} I_5 \right] \\ & + 2e^{\frac{-\tau}{\text{De}}} \sin\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[\frac{-1}{\text{De}} I_2 + \frac{\lambda_2}{\lambda_1} I_4 + \frac{\text{Wi}}{\text{De}} \frac{\lambda_2}{\lambda_1} I_6 \right] \end{aligned} \quad (72)$$

where I_1 – I_6 are given by Eqs. (56)–(61).

i. Goddard Integral Approximation

To validate our main results, we will need approximate solutions for the corotational Jeffreys fluid. Specifically, to get these approximations, we employ the well-known Goddard integral expansion following the method of **Section 8.** of [124].

We begin with:

$$\mathbb{N}_1 = -2\mathbb{N}_2 = \sum_{m=1}^{\infty} \left(1 - \frac{\lambda_2}{\lambda_1}\right) \mathbb{N}_1[ma] \quad (73)$$

where $\mathbb{N}_1[ma]$ is the m th term of the *corotational Maxwell contribution* to \mathbb{N}_1 . In contrast with our previous work on the Goddard integral expansion for shear stress (Eq. (80) of [6]), the *Dirac delta function contribution* is zero. To get $\mathbb{N}_1[ma]$, we follow the method of **Section 3.** of [124] to extend Eq. (66) of [124] to the next order (fifth power of Wi):

$$\begin{aligned} \sum_{m=1}^4 \mathbb{N}_1[ma] = & -\text{Wi} \left[\frac{1}{1+\text{De}^2} + \frac{(1-2\text{De}^2)\cos 2\tau + 3\text{De}\sin 2\tau}{(1+\text{De}^2)(1+4\text{De}^2)} \right] \\ & + \frac{\text{Wi}^3}{4} \left[\frac{3}{(1+\text{De}^2)(1+4\text{De}^2)} + \frac{(4-24\text{De}^2)\cos 2\tau + 20\text{De}\sin 2\tau}{(1+\text{De}^2)(1+4\text{De}^2)(1+9\text{De}^2)} \right. \\ & \left. + \frac{(1-35\text{De}^2+24\text{De}^4)\cos 4\tau + (10-50\text{De}^2)\text{De}\sin 4\tau}{(1+\text{De}^2)(1+4\text{De}^2)(1+9\text{De}^2)(1+16\text{De}^2)} \right] \\ & - \frac{\text{Wi}^5}{16} \left[\frac{10}{(1+\text{De}^2)(1+4\text{De}^2)(1+9\text{De}^2)} \right. \\ & \left. + \frac{15(1-12\text{De}^2)\cos 2\tau + 105\text{De}\sin 2\tau}{(1+\text{De}^2)(1+4\text{De}^2)(1+9\text{De}^2)(1+16\text{De}^2)} \right. \\ & \left. + \frac{6(1-71\text{De}^2+120\text{De}^4)\cos 4\tau + 84(1-11\text{De}^2)\text{De}\sin 4\tau}{(1+\text{De}^2)(1+4\text{De}^2)(1+9\text{De}^2)(1+16\text{De}^2)(1+25\text{De}^2)} \right. \\ & \left. + \frac{(1-175\text{De}^2+1624\text{De}^4-720\text{De}^6)\cos 6\tau + 21(1-35\text{De}^2+84\text{De}^4)\text{De}\sin 6\tau}{(1+\text{De}^2)(1+4\text{De}^2)(1+9\text{De}^2)(1+16\text{De}^2)(1+25\text{De}^2)(1+36\text{De}^2)} \right] \end{aligned} \quad (74)$$

When the Wi^5 term is omitted, Eq. (74) reduces to Eq. (66) of [124] as it must. Figure 2 shows the improvement of Eq. (74) over Eq. (66) of [124] by Wi^5 term. Figure 2 also shows that Eqs. (74) and our main results (Eqs. (54) and (55) [with Eqs. (56)–(62)]) agree within a pen width. Eq. (74) thus suffices to check the consistency of our main results.

Substituting Eq. (74) into Eq. (73) gives the Goddard integral approximation to the corotational Jeffreys fluid (Eq. (13) with $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$):

$$\begin{aligned}
\mathbb{N}_1 = -2\mathbb{N}_2 = & -\left(1 - \frac{\lambda_2}{\lambda_1}\right) Wi \left[\frac{1}{1+De^2} + \frac{(1-2De^2)\cos 2\tau + 3De\sin 2\tau}{(1+De^2)(1+4De^2)} \right] \\
& + \left(1 - \frac{\lambda_2}{\lambda_1}\right) \frac{Wi^3}{4} \left[\frac{3}{(1+De^2)(1+4De^2)} + \frac{(4-24De^2)\cos 2\tau + 20De\sin 2\tau}{(1+De^2)(1+4De^2)(1+9De^2)} \right. \\
& \quad \left. + \frac{(1-35De^2+24De^4)\cos 4\tau + (10-50De^2)De\sin 4\tau}{(1+De^2)(1+4De^2)(1+9De^2)(1+16De^2)} \right] \\
& - \left(1 - \frac{\lambda_2}{\lambda_1}\right) \frac{Wi^5}{16} \left[\frac{10}{(1+De^2)(1+4De^2)(1+9De^2)} \right. \\
& \quad + \frac{15(1-12De^2)\cos 2\tau + 105De\sin 2\tau}{(1+De^2)(1+4De^2)(1+9De^2)(1+16De^2)} \\
& \quad + \frac{6(1-71De^2+120De^4)\cos 4\tau + 84(1-11De^2)De\sin 4\tau}{(1+De^2)(1+4De^2)(1+9De^2)(1+16De^2)(1+25De^2)} \\
& \quad \left. + \frac{(1-175De^2+1624De^4-720De^6)\cos 6\tau + 21(1-35De^2+84De^4)De\sin 6\tau}{(1+De^2)(1+4De^2)(1+9De^2)(1+16De^2)(1+25De^2)(1+36De^2)} \right] \\
& + \dots
\end{aligned} \tag{75}$$

which we will next use to validate our main result for the special case of the corotational Jeffreys fluid. The accuracies of Eqs. (74) and (75) can be further improve with the use of Padé approximants [131], though we find no need for this here (see Figure 4 through Figure 6).

When the Wi^3 and Wi^5 terms are omitted from Eq. (75), the remainder matches Eq. (131) of [124] as it should. To our knowledge, Eq. (75) is new. Whereas our exact solution does not take the form of a Fourier series, Eq. (75) does so. Eq. (75) is thus the first, albeit approximate, expression for each of the normal stress difference harmonics up to and including the sixth, for the corotational Jeffreys fluid. Figure 4 through Figure 6 show that Eq. (75) agrees closely, well within a pen width, with our new exact solution, Eqs. (54) and (55) [with Eqs. (56)–(62)], thus validating our main results.

ii. Finite Difference Solution

Viscoelastic models in LAOS can and have also been evaluated approximately numerically by the finite element method [50,51], or by numerical integration [52,53,54,55,56], or by finite difference [6,57,58,59,60,61,62, 63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,92]. Here, as a

consistency check on Eqs. (54) and (55) [with Eqs. (56)–(62)], we use the Runge-Kutta 5(4) finite difference scheme [83] to solve Eqs. (40)–(43) for the first and second normal stress differences subject to the initial conditions:

$$\mathbb{S}(0) = \mathbb{N}_1(0) = \mathbb{N}_2(0) = 0 \quad (76)$$

Our computation is, of course, independent of $\check{\tau}_{zz}(0)$, since for the corotational Jeffreys model, $v_1 = v_2 = 0$, or since, for the corotational Jeffreys model, $\mu_0 = \frac{2}{3}\mu_1 = 0$. The conditions $v_1 = v_2 = 0$ or $\mu_0 = \frac{2}{3}\mu_1 = 0$ satisfy Eqs. (75) and (76), our conditions for $\check{\tau}_{zz}$ invariance.

For our finite difference solution to Eqs. (40)–(43) subject to Eq. (76), we coded the *ode45* scheme into MATLAB (Version R2012b) on a MacBook Air (1.3 GHz Intel Core i5 processor with 4 GB 1600 MHz DDR3 memory) employing the OS X (Version 10.12.3) operating system. For each point in Figure 7 through Figure 9, we find such an evaluation to consume less than 1 second of CPU time. Figure 7 through Figure 9 show the consistency between our numerical and exact solutions, well within a pen width, for the Oldroyd 8-constant framework in LAOS. Indeed, we find that our finite difference and exact solutions agree well within a pen width.

V. WORKED EXAMPLE: η_∞ AND THE HIGHER HARMONICS

In this section, we illustrate the use of our exact solution for the Oldroyd 8-constant framework in LAOS (Eqs. (54) and (55) [with Eqs. (56)–(62)]). Specifically, we use this exact solution to explore the role of η_∞ on the higher harmonics of the normal stress differences. For this exploration, we choose the special case of the corotational Jeffreys model, the simplest relevant model containing η_∞ . By *relevant*, we still mean that the model at least predicts normal stress difference harmonics higher than the second. Recall that, for the corotational Jeffreys model, $\eta_\infty/\eta_0 = \lambda_2/\lambda_1$ (see Eq. (28) of [6]).

Figure 3 through Figure 6 employ Eq. (72) with [Eqs. (56)–(61)] to illustrate the role of η_∞ on the loop shapes. We find our new frequency-matched loops of minus N_1 *versus* $\dot{\gamma}(2\tau)$ to be counter-clockwise, and N_2 *versus* $\dot{\gamma}(2\tau)$, counter-clockwise too. Comparing Figure 3 successively with Figure 4 through Figure 6, we learn that increasing η_∞ lowers the normal stress difference *versus* shear rate loops. Both the magnitudes of the oscillating parts and the loop areas diminish as η_∞ increases. In other words, inasmuch as the normal stress differences reflect the fluid elasticity, increasing η_∞ decreases fluid elasticity. From Figure 3 through Figure 6, we see that increasing η_∞ does not produce self-intersection, at least not over the ranges $\frac{1}{10} < Wi < \frac{5}{4}$, $0.1 < De < 10$ and

$0 < \lambda_2/\lambda_1 < 1/3$. To our knowledge, whereas self-intersection of $\tau_{yx} - \dot{\gamma}$ loops is observed [11,12,84], self-intersection of the normal stress differences in LAOS have yet to be observed, and Figure 3 through Figure 6 show that the corotational Jeffreys model behaves accordingly.

VI. CONCLUSION

In this work, we arrive at exact and unique analytical solutions for both normal stress differences in LAOS, Eqs. (54) and (55) [with Eqs. (56)–(62)], for the Oldroyd 8-constant constitutive framework. We chose the Oldroyd 8-constant framework for its rich diversity of popular constitutive special cases. To our knowledge, our exact analytical solution Eqs. (54) and (55) [with Eqs. (56)–(62)] are the first exact solutions to a framework of constitutive equations in LAOS. Our new exact analytical solution reduces to our previous result for the corotational Maxwell fluid (Eq. (56) with Eqs. (127) and (134) of [132]), as it should.

To our knowledge, our Subsection IV.c.i provides the first integral expansion [Eq. (75)] for the normal stress difference responses of a corotational Jeffreys fluid in LAOS, and this up to and including the sixth harmonic. We use this Eq. (75) to validate our main result, Eqs. (54) and (55) [with Eqs. (56)–(62)], for the special case of the corotational Jeffreys fluid in LAOS. We find the Goddard integral expansion to be remarkably accurate.

Some special cases of the Oldroyd 8-constant framework (corotational Maxwell or corotational Jeffreys), are used with multiple relaxation times, λ_1 . Extending the results of this work on the normal stress differences in LAOS to multiple λ_1 might thus be a useful next step, and for this, we would begin with the Spriggs relations (see Eqs. (6.1-14) and (6.1-15) of [21]; see also Appendix of [85]; §8.5 of [135]). Other special cases of the Oldroyd 8-constant framework (Johnson-Segalman or Gordon-Schowalter) are often useful without extension to multiple λ_1 .

In this paper, we have limited the scope to just the alternant part of the normal stress differences in LAOS. Previously, for the exact solution to the corotational Maxwell model, we solved exactly for both the shear stress and the normal stress differences, and these for both start-up and alternance, using the Kovacic method [132]. We expect start-up for the Oldroyd 8-constant framework to yield to the Kovacic method. However, for the Oldroyd 8-constant framework, we leave the exact solution for the normal stress differences in start-up for another day.

In our previous work, we succeeded in rewriting our exact solutions for the corotational Maxwell fluid (Eq. (56) with Eqs. (127) and (134) of [132]) as a Fourier series (Eq. (57) of [132]). However, in this work, we have yet to rewrite

our main result Eqs. (54) and (55) [with Eqs. (56)–(62)] as a Fourier series. This too we shall leave for another day.

We have a strong preference for the approach that we have taken in this paper, which produces the exact solution for an entire framework, over the approach in Ref. [132], which produces an exact solution for just one constitutive equation. We prefer the framework approach because it at once generates exact solutions in LAOS to whole sets of constitutive equations, rather than generating solutions for one constitutive equation at a time. We can report that our exact solutions are both integrable and differentiable, and they thus provide suitable starting points for exploring, analytically and exactly, many nonlinear phenomena in fluid physics, including for instance, the normal thrust in oscillatory sliding plate flow.

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VIII. APPENDIX: EVALUATING $\lim_{Wi \rightarrow 0} \mathbb{N}_1/Wi$ AND $\lim_{Wi \rightarrow 0} \mathbb{N}_2/Wi$ FOR SAOS

In this appendix, we detail our evaluation of the limit. Dividing Eqs. (54) and (55) by Wi , and then taking the limit as $Wi \rightarrow 0$ [with Eqs. (56)–(61)] gives:

$$\mathbb{N}_1^{\text{SAOS}} \equiv \lim_{Wi \rightarrow 0} \frac{\mathbb{N}_1}{Wi} = \mathbb{N}_1^{(1)} + \mathbb{N}_1^{(2)} + \mathbb{N}_1^{(3)} + \mathbb{N}_1^{(4)} + \mathbb{N}_1^{(5)} + \mathbb{N}_1^{(6)} + \mathbb{N}_1^{(7)} \quad (77)$$

and:

$$\mathbb{N}_2^{\text{SAOS}} \equiv \lim_{Wi \rightarrow 0} \frac{\mathbb{N}_2}{Wi} = \mathbb{N}_2^{(1)} + \mathbb{N}_2^{(2)} + \mathbb{N}_2^{(3)} + \mathbb{N}_2^{(4)} + \mathbb{N}_2^{(5)} + \mathbb{N}_2^{(6)} + \mathbb{N}_2^{(7)} \quad (78)$$

where $\mathbb{N}_1^{(m)}$ are defined in Eqs. (79), (83), (87), (91), (95), (99) and (103), and $\mathbb{N}_2^{(m)}$ are defined in Eqs. (105), (109), (113), (117), (121), (125) and (129). We will next evaluate $\mathbb{N}_1^{(m)}$ in Appendix VIII.a, and $\mathbb{N}_2^{(m)}$ in Appendix VIII.b.

a. $\lim_{Wi \rightarrow 0} \mathbb{N}_1/Wi$

The first term in Eq. (77) is defined as:

$$\mathbb{N}_1^{(1)} = \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\alpha} \frac{Wi}{De} e^{\frac{-\tau}{De}} C \frac{1}{De} I_1 \quad (79)$$

Substituting Eq. (56) into Eq. (79) gives:

$$N_1^{(1)} \equiv \frac{8De}{\tilde{\lambda}^2} \sum_{k=1}^{\infty} \left[-\frac{2De k^2 \cos 2k\tau}{1+4De^2 k^2} \lim_{Wi \rightarrow 0} \frac{1}{Wi^2} \cos\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_{2k}\left(\frac{Wi}{De} \tilde{\lambda}\right) \right. \\ \left. + \frac{k \sin 2k\tau}{1+4De^2 k^2} \lim_{Wi \rightarrow 0} \frac{1}{Wi^2} \cos\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_{2k}\left(\frac{Wi}{De} \tilde{\lambda}\right) \right] \quad (80)$$

Applying:

$$\lim_{Wi \rightarrow 0} \frac{1}{Wi^2} \cos\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_{2k}\left(\frac{Wi}{De} \tilde{\lambda}\right) = \begin{cases} \frac{\tilde{\lambda}^2}{8De^2} & ; k=1 \\ 0 & ; k=2,3,4,\dots \end{cases} \quad (81)$$

to Eq. (80) gives:

$$N_1^{(1)} = \frac{1}{De} \left[-\frac{2De \cos 2\tau}{1+4De^2} + \frac{\sin 2\tau}{1+4De^2} \right] \quad (82)$$

The second term in Eq. (77) is defined as:

$$N_1^{(2)} \equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\alpha} \frac{Wi}{De} e^{-\frac{\tau}{De}} S \frac{-1}{De} I_2 \quad (83)$$

Substituting Eq. (57) into Eq. (83) gives:

$$N_1^{(2)} = -\frac{4De}{\tilde{\lambda}^2} \sum_{k=1}^{\infty} \left[\frac{De(2k-1)^2 \sin(2k-1)\tau}{De^2(2k-1)^2+1} \lim_{Wi \rightarrow 0} \frac{1}{Wi^2} \sin\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_{2k-1} \right. \\ \left. + \frac{(2k-1)\cos(2k-1)\tau}{De^2(2k-1)^2+1} \lim_{Wi \rightarrow 0} \frac{1}{Wi^2} \sin\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_{2k-1} \right] \quad (84)$$

Applying:

$$\lim_{Wi \rightarrow 0} \frac{1}{Wi^2} \sin\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_{2k-1}\left(\frac{Wi}{De} \tilde{\lambda}\right) = \begin{cases} \frac{\tilde{\lambda}^2 \sin \tau}{2De^2} & ; k=1 \\ 0 & ; k=2,3,4,\dots \end{cases} \quad (85)$$

to Eq. (84) gives:

$$N_1^{(2)} = -\frac{1}{De} \left[\frac{De - De \cos 2\tau + \sin 2\tau}{1+De^2} \right] \quad (86)$$

The third term in Eq. (77) is defined as:

$$N_1^{(3)} \equiv -\lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\alpha} \frac{Wi}{De} e^{-\frac{\tau}{De}} C \frac{\lambda_2}{\lambda_1} I_3 \quad (87)$$

Substituting Eq. (58) into Eq. (87) gives:

$$N_1^{(3)} = -\frac{2De}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \left[\lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_1 + \sum_{k=1}^{\infty} \left[\frac{-\lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_{2k-1} + \lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_{2k+1}}{1+4k^2 De^2} \cos 2k\tau \right. \right. \\ \left. \left. + (2De k) \frac{-\lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_{2k-1} + \lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_{2k+1}}{1+4k^2 De^2} \sin 2k\tau \right] \right] \quad (88)$$

Applying:

$$\left\{ \begin{array}{l} \lim_{Wi \rightarrow 0} \frac{1}{Wi} CJ_1 = \frac{\tilde{\lambda}}{2De} \\ \lim_{Wi \rightarrow 0} \frac{1}{Wi} CJ_{2k-1} = \begin{cases} \frac{\tilde{\lambda}}{2De} & ; n = 1 \\ 0 & ; n = 2, 3, 4, \dots \end{cases} \\ \lim_{Wi \rightarrow 0} \frac{1}{Wi} CJ_{2k+1} = 0 \end{array} \right. \quad (89)$$

to Eq. (88) gives:

$$N_1^{(3)} = -\frac{\lambda_2}{\lambda_1} \left[1 - \frac{1}{1+4De^2} \cos 2\tau - \frac{2De}{1+4De^2} \sin 2\tau \right] \quad (90)$$

The fourth term in Eq. (77) is defined as:

$$N_1^{(4)} \equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\alpha} \frac{Wi}{De} e^{\frac{-\tau}{De}} S \frac{\lambda_2}{\lambda_1} I_4 \quad (91)$$

Substituting Eq. (59) into Eq. (91) gives:

$$N_1^{(4)} = \frac{2De}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \sum_{k=1}^{\infty} \left[\begin{array}{l} \frac{-(2k-1)De \left(\lim_{Wi \rightarrow 0} \frac{1}{Wi} SJ_{2k-2} - \lim_{Wi \rightarrow 0} \frac{1}{Wi} SJ_{2k} \right)}{1+(2k-1)^2 De^2} \cos(2k-1)\tau \\ + \frac{\lim_{Wi \rightarrow 0} \frac{1}{Wi} SJ_{2k-2} - \lim_{Wi \rightarrow 0} \frac{1}{Wi} SJ_{2k}}{1+(2k-1)^2 De^2} \sin(2k-1)\tau \end{array} \right] \quad (92)$$

Applying:

$$\left\{ \begin{array}{l} \lim_{Wi \rightarrow 0} \frac{1}{Wi} SJ_{2k-2} = \begin{cases} \frac{\tilde{\lambda} \sin \tau}{De} & ; n = 1 \\ 0 & ; n = 2, 3, 4, \dots \end{cases} \\ \lim_{Wi \rightarrow 0} \frac{1}{Wi} SJ_{2k} = 0 \end{array} \right. \quad (93)$$

to Eq. (92) gives:

$$N_1^{(4)} = \frac{\lambda_2}{\lambda_1} \left[\frac{1 - \cos 2\tau - De \sin 2\tau}{1 + De^2} \right] \quad (94)$$

The fifth term in Eq. (77) is defined as:

$$\begin{aligned} N_1^{(5)} &\equiv - \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\alpha} \frac{Wi}{De} e^{\frac{-\tau}{De}} C \frac{Wi}{De} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 v_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 v_2}{\lambda_1^2} \right) I_5 \\ &\equiv - \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\alpha} \frac{Wi}{De} e^{\frac{-\tau}{De}} C \frac{Wi}{De} \frac{\Phi}{\tilde{\lambda}} I_5 \end{aligned} \quad (95)$$

Substituting Eq. (60) into Eq. (95) gives:

$$N_1^{(5)} = -\frac{2\Phi}{\tilde{\lambda}^2} \lim_{Wi \rightarrow 0} \left\{ \frac{De}{\tilde{\lambda}} \frac{1}{Wi} CJ_1 + \sum_{k=1}^{\infty} \left[\frac{CJ_{2k-2} + 2CJ_{2k} + CJ_{2k+2} \cos 2k\tau}{2(1+4De^2 k^2)} + (De k) \frac{CJ_{2k-2} + 2CJ_{2k} + CJ_{2k+2} \sin 2k\tau}{1+4De^2 k^2} \right] \right\} \quad (96)$$

Applying:

$$\left\{ \begin{array}{l} \lim_{Wi \rightarrow 0} \frac{1}{Wi} CJ_1 = \frac{\tilde{\lambda}}{2De} \\ \lim_{Wi \rightarrow 0} CJ_{2k-2} = \begin{cases} 1 & ; n=1 \\ 0 & ; n=2,3,4\dots \end{cases} \\ \lim_{Wi \rightarrow 0} CJ_{2k} = 0 \\ \lim_{Wi \rightarrow 0} CJ_{2k+2} = 0 \end{array} \right. \quad (97)$$

to Eq. (96) becomes:

$$N_1^{(5)} = -\frac{2\Phi}{\tilde{\lambda}^2} \left[\frac{1}{2} + \frac{1}{2(1+4De^2)} \cos 2\tau + \frac{De}{1+4De^2} \sin 2\tau \right] \quad (98)$$

The sixth term in Eq. (77) is defined as:

$$\begin{aligned} N_1^{(6)} &\equiv -\lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\alpha} \frac{Wi}{De} e^{\frac{-\tau}{De}} S \frac{Wi}{De} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 v_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 v_2}{\lambda_1^2} \right) I_6 \\ &\equiv -\lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\alpha} \frac{Wi}{De} e^{\frac{-\tau}{De}} S \frac{Wi}{De} \frac{\Phi}{\tilde{\lambda}} I_6 \end{aligned} \quad (99)$$

Substituting Eq. (61) into Eq. (99) gives:

$$N_1^{(6)} = -\frac{\Phi}{\tilde{\lambda}^2} \lim_{Wi \rightarrow 0} \sum_{k=1}^{\infty} \left[\frac{-De(2k-1)(SJ_{2k-3} + 2SJ_{2k-1} + SJ_{2k+1}) \cos(2k-1)\tau}{1+(2k-1)^2 De^2} + \frac{SJ_{2k-3} + 2SJ_{2k-1} + SJ_{2k+1} \sin(2k-1)\tau}{1+(2k-1)^2 De^2} \right] \quad (100)$$

Applying:

$$\left\{ \begin{array}{l} \lim_{Wi \rightarrow 0} SJ_{2k-3} = 0 \\ \lim_{Wi \rightarrow 0} SJ_{2k-1} = 0 \\ \lim_{Wi \rightarrow 0} Wi SJ_{2k-3} = 0 \end{array} \right. \quad (101)$$

to Eq. (100) gives:

$$N_1^{(6)} = 0 \quad (102)$$

The seventh term in Eq. (77) is defined as:

$$\begin{aligned} N_1^{(7)} &\equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2}{\tilde{\lambda}^2} \left[-\left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} - \tilde{\lambda}^2 \right) \frac{\lambda_2}{\lambda_1} + \left(\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) + \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \frac{v_2}{\lambda_1} \right] \frac{Wi}{De} e^{\frac{-\tau}{De}} I_7 \\ &\equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{2\Psi}{\tilde{\lambda}^2} \frac{Wi}{De} e^{\frac{-\tau}{De}} I_7 \end{aligned} \quad (103)$$

Substituting Eq. (62) into Eq. (103) gives:

$$\mathbb{N}_1^{(\tau)} = \frac{2\Psi}{\tilde{\lambda}^2} \left[\frac{1}{2} + \frac{1}{2(1+4\text{De}^2)} \cos 2\tau + \frac{\text{De}}{1+4\text{De}^2} \sin 2\tau \right] \quad (104)$$

Substituting Eqs. (82), (86), (90), (94), (98), (102) and (104) into Eq. (77) gives Eq. (63) above.

b. $\lim_{\text{Wi} \rightarrow 0} \mathbb{N}_2/\text{Wi}$

The first term in Eq. (78) is defined as:

$$\mathbb{N}_2^{(1)} \equiv \lim_{\text{Wi} \rightarrow 0} \frac{1}{\text{Wi}} \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1} \right) \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \text{C} \frac{-1}{\text{De}} I_1 \quad (105)$$

Substituting Eq. (56) into Eq. (105) gives:

$$\mathbb{N}_2^{(1)} = \frac{-4\text{De}}{\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1} \right) \sum_{k=1}^{\infty} \left[-\frac{2\text{De}k^2 \lim_{\text{Wi} \rightarrow 0} \frac{1}{\text{Wi}^2} \text{C} J_{2k}}{1+4\text{De}^2 k^2} \cos 2k\tau + \frac{k \lim_{\text{Wi} \rightarrow 0} \frac{1}{\text{Wi}^2} \text{C} J_{2k}}{1+4\text{De}^2 k^2} \sin 2k\tau \right] \quad (106)$$

Applying:

$$\lim_{\text{Wi} \rightarrow 0} \frac{1}{\text{Wi}^2} \text{C} J_{2k} = \begin{cases} \frac{\tilde{\lambda}^2}{8\text{De}^2} & ; k=1 \\ 0 & ; k=2,3,4,\dots \end{cases} \quad (107)$$

to Eq. (106) gives:

$$\mathbb{N}_2^{(1)} = \frac{-1}{2\text{De}} \left(1 - \frac{\mu_1}{\lambda_1} \right) \left[-\frac{2\text{De}}{1+4\text{De}^2} \cos 2\tau + \frac{1}{1+4\text{De}^2} \sin 2\tau \right] \quad (108)$$

The second term in Eq. (78) is defined as:

$$\mathbb{N}_2^{(2)} \equiv -\lim_{\text{Wi} \rightarrow 0} \frac{1}{\text{Wi}} \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1} \right) \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \text{S} \frac{-1}{\text{De}} I_2 \quad (109)$$

Substituting Eq. (57) into Eq. (109) gives:

$$\mathbb{N}_2^{(2)} = \left(1 - \frac{\mu_1}{\lambda_1} \right) \frac{2\text{De}}{\tilde{\lambda}^2} \sum_{k=1}^{\infty} \left[\frac{\text{De}(2k-1)^2 \lim_{\text{Wi} \rightarrow 0} \frac{1}{\text{Wi}^2} \text{S} J_{2k-1}}{\text{De}^2(2k-1)^2 + 1} \sin(2k-1)\tau \right. \\ \left. + \frac{(2k-1) \lim_{\text{Wi} \rightarrow 0} \frac{1}{\text{Wi}^2} \text{S} J_{2k-1}}{\text{De}^2(2k-1)^2 + 1} \cos(2k-1)\tau \right] \quad (110)$$

Applying:

$$\lim_{\text{Wi} \rightarrow 0} \frac{1}{\text{Wi}^2} \text{S} J_{2k-1} = \begin{cases} \frac{\tilde{\lambda}^2 \sin \tau}{2\text{De}^2} & ; k=1 \\ 0 & ; k=2,3,4,\dots \end{cases} \quad (111)$$

to Eq. (110) gives:

$$N_2^{(2)} = \frac{1}{2De} \left(1 - \frac{\mu_1}{\lambda_1} \right) \left[\frac{De}{1+De^2} - \frac{De}{1+De^2} \cos 2\tau + \frac{1}{1+De^2} \sin 2\tau \right] \quad (112)$$

The third term in Eq. (78) is defined by:

$$N_2^{(3)} \equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1} \right) \frac{Wi}{De} e^{\frac{-\tau}{De}} C \frac{\lambda_2}{\lambda_1} I_3 \quad (113)$$

Substituting Eq. (58) into Eq. (113) gives:

$$N_2^{(3)} = \frac{1}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \left(1 - \frac{\mu_1}{\lambda_1} \right) \lim_{Wi \rightarrow 0} \frac{1}{Wi} C \left\{ De J_1 + De \sum_{k=1}^{\infty} \left[\begin{array}{l} \frac{-J_{2k-1} + J_{2k+1}}{1+4k^2 De^2} \cos 2k\tau \\ + (2De k) \frac{-J_{2k-1} + J_{2k+1}}{1+4k^2 De^2} \sin 2k\tau \end{array} \right] \right\} \quad (114)$$

Applying:

$$\left\{ \begin{array}{l} \lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_1 = \frac{\tilde{\lambda}}{2De} \\ \lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_{2k-1} = \begin{cases} \frac{\tilde{\lambda}}{2De} & ; k=1 \\ 0 & ; k=2,3,4,\dots \end{cases} \\ \lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_{2k+1} = 0 \end{array} \right. \quad (115)$$

to Eq. (114) gives:

$$N_2^{(3)} = \frac{1}{2} \frac{\lambda_2}{\lambda_1} \left(1 - \frac{\mu_1}{\lambda_1} \right) \left[1 - \frac{1}{1+4De^2} \cos 2\tau - \frac{2De}{1+4De^2} \sin 2\tau \right] \quad (116)$$

The fourth term in Eq. (78) is given by:

$$N_2^{(4)} \equiv - \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1} \right) \frac{Wi}{De} e^{\frac{-\tau}{De}} S \frac{\lambda_2}{\lambda_1} I_4 \quad (117)$$

Substituting Eq. (59) into Eq. (117) gives:

$$N_2^{(4)} = - \frac{De}{\tilde{\lambda}} \left(1 - \frac{\mu_1}{\lambda_1} \right) \frac{\lambda_2}{\lambda_1} \lim_{Wi \rightarrow 0} \frac{1}{Wi} S \sum_{k=1}^{\infty} \left[\begin{array}{l} \frac{-(2k-1)De(J_{2k-2} - J_{2k})}{1+(2k-1)^2 De^2} \cos(2k-1)\tau \\ + \frac{J_{2k-2} - J_{2k}}{1+(2k-1)^2 De^2} \sin(2k-1)\tau \end{array} \right] \quad (118)$$

Applying:

$$\left\{ \begin{array}{l} \lim_{Wi \rightarrow 0} \frac{1}{Wi} S J_{2k-2} = \begin{cases} \frac{\tilde{\lambda} \sin \tau}{De} & ; k=1 \\ 0 & ; k=2,3,4,\dots \end{cases} \\ \lim_{Wi \rightarrow 0} \frac{1}{Wi} S J_{2k} = 0 \end{array} \right. \quad (119)$$

to Eq. (118) gives:

$$N_2^{(4)} = -\frac{1}{2} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\lambda_2}{\lambda_1} \left[\frac{-De}{1+De^2} \sin 2\tau + \frac{1}{1+De^2} - \frac{1}{1+De^2} \cos 2\tau \right] \quad (120)$$

The fifth term in Eq. (78) is defined by:

$$\begin{aligned} N_2^{(5)} &\equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{Wi}{De} e^{\frac{-\tau}{De} C} \frac{Wi}{De} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) I_5 \\ &\equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{Wi}{De} e^{\frac{-\tau}{De} C} \frac{Wi}{De} \frac{\Phi}{\tilde{\lambda}} I_5 \end{aligned} \quad (121)$$

Substituting Eq. (60) into Eq. (121) gives:

$$N_2^{(5)} = \frac{\Phi}{\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1}\right) \left\{ \frac{De \lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_1}{\tilde{\lambda}} + \lim_{Wi \rightarrow 0} C \sum_{k=1}^{\infty} \left[\frac{J_{2k-2} + 2J_{2k} + J_{2k+2}}{2(1+4De^2 k^2)} \cos 2k\tau \right. \right. \\ \left. \left. + (De k) \frac{J_{2k-2} + 2J_{2k} + J_{2k+2}}{1+4De^2 k^2} \sin 2k\tau \right] \right\} \quad (122)$$

Applying:

$$\begin{cases} \lim_{Wi \rightarrow 0} \frac{1}{Wi} C J_1 = \frac{\tilde{\lambda}}{2De} \\ \lim_{Wi \rightarrow 0} C J_{2k-2} = \begin{cases} 1 & ; k=1 \\ 0 & ; k=2,3,4,\dots \end{cases} \\ \lim_{Wi \rightarrow 0} C J_{2k} = 0 \\ \lim_{Wi \rightarrow 0} C J_{2k+2} = 0 \end{cases} \quad (123)$$

to Eq. (122) gives:

$$N_2^{(5)} = \frac{\Phi}{\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1}\right) \left[\frac{1}{2} + \frac{1}{2(1+4De^2)} \cos 2\tau + \frac{De}{1+4De^2} \sin 2\tau \right] \quad (124)$$

The sixth term in Eq. (78).

$$\begin{aligned} N_2^{(6)} &\equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{Wi}{De} e^{\frac{-\tau}{De} S} \frac{Wi}{De} \frac{1}{\tilde{\lambda}} \left(-\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 \nu_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 \nu_2}{\lambda_1^2} \right) I_6 \\ &\equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{Wi}{De} e^{\frac{-\tau}{De} S} \frac{Wi}{De} \frac{\Phi}{\tilde{\lambda}} I_6 \end{aligned} \quad (125)$$

Substituting Eq. (61) into Eq. (125) gives:

$$N_2^{(6)} = \frac{\Phi}{2\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1}\right) \lim_{Wi \rightarrow 0} S \sum_{k=1}^{\infty} \left[\frac{-De(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1+(2k-1)^2 De^2} \cos(2k-1)\tau \right. \\ \left. + \frac{J_{2k-3} + 2J_{2k-1} + J_{2k+1}}{1+(2k-1)^2 De^2} \sin(2k-1)\tau \right] \quad (126)$$

Applying:

$$\begin{cases} \lim_{Wi \rightarrow 0} SJ_{2k-3} = 0 \\ \lim_{Wi \rightarrow 0} SJ_{2k-1} = 0 \\ \lim_{Wi \rightarrow 0} SJ_{2k+1} = 0 \end{cases} \quad (127)$$

to Eq. (126) gives:

$$N_2^{(6)} = 0 \quad (128)$$

The seventh term in Eq. (78) is given by:

$$\begin{aligned} N_2^{(7)} &\equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{1}{\tilde{\lambda}^2} \left[\left(1 - \frac{\mu_1}{\lambda_1} \right) \left[\left(1 + \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \frac{\lambda_2}{\lambda_1} - \left(\frac{3\mu_0}{2\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \frac{v_2}{\lambda_1} \right] \right] \frac{Wi}{De} e^{\frac{-\tau}{De}} I_7 \\ &\equiv \lim_{Wi \rightarrow 0} \frac{1}{Wi} \frac{\Omega}{\tilde{\lambda}^2} \frac{Wi}{De} e^{\frac{-\tau}{De}} I_7 \end{aligned} \quad (129)$$

Substituting Eq. (62) into Eq. (129) gives:

$$N_2^{(7)} = \frac{\Omega}{\tilde{\lambda}^2} \left[\frac{1}{2} + \frac{1}{2(1+4De^2)} \cos 2\tau + \frac{De}{1+4De^2} \sin 2\tau \right] \quad (130)$$

Substituting Eq. (108), (112), (116), (120), (124), (128) and (130) into Eq. (78) gives Eq. (64) above.

Table I: Literature on Experimental Measurements of Normal Stress Differences in Large-Amplitude Oscillatory Shear Flow

Authors (year)	Materials	Normal Stress Difference Harmonic			Loops	Startup	Method	$\Psi'_{i,2}$	$ \Psi^*_{i,2} $	[Ref.] (Correction to)
		Zeroth	Second	Fourth						
Endo and Nagasawa (1970)	PS ^S	N_1	N_1				Me			[86]
Christiansen and Leppard (1974); Leppard (1975)	PEO ^S , PAA ^S	N_1 N_2	N_1 N_2				Me	X	X	[87,88]
Vinogradov <i>et al.</i> (1978)	PB; PI	N_1					Op			[89]
Gao (1979); Gao <i>et al.</i> (1981)	PS ^S	N_1 N_2	N_1 N_2				Me	X	X	[90,91]
Isayev and Hieber (1982)	PB	N_1					Me			[92]
Kornfield <i>et al.</i> (1991)	PI ^M	N_3	N_3				Op			[93,94]
Oakley (1992)	LDPE ^M , HDPE ^M	N_1	N_1	N_1	N_1	N_1	Me			[120]
Oakley and Giacomin (1992)	HDPE ^M	N_1	N_1	N_1	N_1	N_1	Me			[95]
Kannan and Kornfield (1992); Kannan (1995)	PI ^M , PEP ^M	N_3	N_3				Op			[96,97]
Oakley and Giacomin (1994)	HDPE ^M	N_1	N_1	N_1	N_1	N_1	Me			[121]
Venerus (1989); Vrentas <i>et al.</i> (1991)	PS ^S	N_1	N_1	N_1			Me			[98,99]
Reimers (1996); Reimers and Dealy (1998)	PS ^S	N_3	N_3	N_3			Op			[100, 101]
Labiausse <i>et al.</i> (2007)	PEO ^{SF}		N_1				Me			[102]
Férec <i>et al.</i> (2008)	PB ^M , PP ^M , PP ^{M_{ff}}	N_1	N_1	N_1			Me			[103]
Nam <i>et al.</i> (2008); Nam <i>et al.</i> (2010)	PEO ^S , PAA ^S , PB/PIB ^M	N_1	N_1	N_1			Me			[104, 105]
Nam <i>et al.</i> (2010)	PIB ^S	N_1 N_2	N_1 N_2	N_1 N_2			Me	I	I	[106]

Legend: *ff* ≡ fiber-filled; F ≡ foam; HDPE ≡ high-density polyethylene; I ≡ quantity is implicitly available; LDPE ≡ low-density polyethylene; M ≡ melt; Me ≡ Mechanical; N_1, N_2, N_3 ≡ first, second and third normal stress differences; Op ≡ Optical; PAA ≡ polyacrylamide; PB ≡ polybutadiene; PEO ≡ polyethylene oxide; PEP ≡ ethylene / propylene copolymer; PI ≡ polyisoprene; PIB ≡ polyisobutylene; PP ≡ polypropylene; PS ≡ polystyrene; S ≡ solution; X ≡

quantity is explicitly available;

Table II: Literature on Analytical Solutions for Normal Stress Differences in Large-Amplitude Oscillatory Shear Flow

Authors (year)	Model	Normal Stress Difference Harmonic			Orientation	Startup	Form	[Ref.] (Correction to)
		Zeroth	Second	Fourth				
Lodge (1961, 1964)	L^\dagger	N_1	N_1				=	[107,108]
Spriggs (1966)	NGJ	N_1	N_1				=	[109]
Spriggs (1966)	CJ	N_1 N_2	N_1 N_2				=	[109]
Williams and Bird (1962)	O_3		N_1				=	[110]
Williams and Bird (1964)	O_3	N_1	N_1				=	[111]
Spriggs (1965)	O_3^\dagger	N_1 N_2	N_1 N_2				=	[112]
Akers and Williams (1969)	RZ	N_1	N_1				=	[113]
Bird, Warner and Evans (1971)	RD	N_1 N_2	N_1 N_2		ψ		\cong	[114]
Leal and Hinch (1972)	RSD	N_1 N_2	N_1 N_2		ψ		\cong	Section 3.3 of [115]
Abdel-Khalik <i>et al.</i> (1974); Bird <i>et al.</i> (1974)	GE+SK	N_1 N_2	N_1 N_2				\cong	[116,117]
Mou and Mazo (1977)	RR	N_1 N_2	N_1 N_2		ψ		\cong	[118](119)119
Oakley (1992); Oakley and Giacomin (1994)	L^\dagger	N_1	N_1				=	APPENDIX B of [120]; [121](107)
Yu <i>et al.</i> (2002); Zhou (2004)	SE	N_1	N_1	N_1			\cong	[122,123]
Giacomin <i>et al.</i> (2011)	CM^\dagger , CJ	N_1 N_2	N_1 N_2	N_1 N_2		X	\cong	[124] (125)
Giacomin and Bird (2011)	ANSR	N_1 N_2	N_1 N_2	N_1 N_2		X	\cong	[126]
Gurnon and Wagner (2012)	G	N_1 N_2	N_1 N_2				\cong	[127]
Schmalzer <i>et al.</i> (2014)	RD	N_1 N_2	N_1 N_2	N_1 N_2	ψ		\cong	[128] (124,126); [129]

Thompson and de Souza Mendes (2015)	MSJ	0	0	0		0	\cong	See Section 5.2.2 of [130]
Giacomin <i>et al.</i> (2015)	CM ^P	N_1 N_2	N_1 N_2	N_1 N_2		X	\cong	[131]
Saengow <i>et al.</i> (2015)	CM [†]	N_1 N_2	N_1 N_2	N_1 N_2		X	=	[132]
This paper	O ₈						=	[133]

Legend: ANSR \equiv corotational arbitrary normal stress ratio; CJ \equiv corotational Jeffreys; CM \equiv corotational Maxwell; G \equiv Giesekus; GE \equiv Goddard integral expansion; L \equiv Lodge rubberlike; NGJ \equiv nonlinear Generalized Jeffreys; O₃ \equiv Oldroyd 3-constant; O₈ \equiv Oldroyd 8-constant; P \equiv Padé approximants; RD \equiv rigid dumbbell; RR \equiv planar rigid ring; RSD \equiv rods, spheres or disks; RZ \equiv Rouse-Zimm; SE \equiv simple emulsion; SK \equiv shish-kebab; N_1, N_2 \equiv first and second normal stress differences; = \equiv exact; \cong \equiv approximate; ψ \equiv orientation distribution.

Table III: Dimensional Variables

Name	Unit	Symbol
Angular frequency [Eq. (1)]	t^{-1}	ω
Cartesian coordinate	L	x, y, z
Distance between two oscillating plates	L	h
Extra stress tensor* [Eq. (14)]	M/Lt^2	τ
Extra stress, ij th component	M/Lt^2	τ_{ij}
First normal stress coefficient, in-phase with shear rate, n th harmonic [Eq. (5)]	M/L	$\Psi'_{1,n}$
First normal stress coefficient, out-of-phase with shear rate, n th harmonic [Eq. (5)]	M/L	$\Psi''_{1,n}$
First normal stress coefficient, small amplitude oscillatory shear response, distance term [Eq. (29)]	M/L	$\Psi_1^d \equiv \Psi'_{1,0}$
First normal stress coefficient, small amplitude oscillatory shear response, in-phase with shear rate [Eq. (29)]	M/L	$\Psi'_1 \equiv \Psi'_{1,1}$
First normal stress coefficient, small amplitude oscillatory shear response, out-of-phase with shear rate [Eq. (29)]	M/L	$\Psi''_1 \equiv \Psi''_{1,1}$
First normal stress coefficient, steady shear [Eq. (18)]	M/L	$\Psi_1 \equiv -N_1/\dot{\gamma}^2$
First normal stress coefficient, steady shear, infinite shear rate [Eq. (23)]	M/L	$\Psi_{1\infty} \equiv \lim_{\dot{\gamma} \rightarrow \infty} \Psi_1$
First normal stress coefficient, steady shear, zero shear rate [Eq. (19)]	M/L	$\Psi_{10} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_1$
First normal stress difference [Eq. (5)]	M/Lt^2	$N_1 \equiv \tau_{xx} - \tau_{yy}$
Hydrodynamic pressure [Eq. (14)]	M/Lt^2	p
Oldroyd constant [Eq. (13)]	t	μ_0
Oldroyd constant [Eq. (13)]	t	μ_1
Oldroyd constant [Eq. (13)]	t	μ_2
Oldroyd constant [Eq. (13)]	t	ν_1
Oldroyd constant [Eq. (13)]	t	ν_2
Oldroyd constant, relaxation time [Eq. (13)]	t	λ_1
Oldroyd constant, retardation time [Eq. (13)]	t	λ_2
Oldroyd steady shear constant [Eq. (22)]	t^2	σ_2
Oldroyd steady shear constant [Eq. (21)]	t^2	σ_1
Second normal stress coefficient [Eq. (18)]	M/L	$\Psi_2 \equiv -N_2/\dot{\gamma}^2$
Second normal stress coefficient, in-phase with shear rate, n th harmonic [Eq. (6)]	M/L	$\Psi'_{2,n}$

Second normal stress coefficient, out-of-phase with shear rate, n th harmonic [Eq. (6)]	M/L	$\Psi''_{2,n}$
Second normal stress coefficient, small amplitude oscillatory shear response, displacement term [Eq. (30)]	M/L	$\Psi_2^d \equiv \Psi'_{2,0}$
Second normal stress coefficient, small amplitude oscillatory shear response, out-of-phase with shear rate [Eq. (30)]	M/L	$\Psi'_2 \equiv \Psi'_{2,1}$
Second normal stress coefficient, small amplitude oscillatory shear response, out-of-phase with shear rate [Eq. (30)]	M/L	$\Psi''_2 \equiv \Psi''_{2,1}$
Second normal stress coefficient, zero shear rate [Eq. (20)]	M/L	$\Psi_{20} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_2$
Second normal stress coefficient, infinite shear rate [Eq. (23)]	M/L	$\Psi_{2\infty} \equiv \lim_{\dot{\gamma} \rightarrow \infty} \Psi_2$
Second normal stress difference [Eq. (6)]	M/Lt^2	$N_2 \equiv \tau_{yy} - \tau_{zz}$
Shear rate, amplitude [Eq. (1)]	t^{-1}	$\dot{\gamma}^0$
Strain rate tensor [Eq. (15)]	t^{-1}	$\dot{\gamma}$
Strain rate, yx -component [Eq. (1)]	t^{-1}	$\dot{\gamma}$
Time	t	t
Total stress tensor* [Eq. (14)]	M/Lt^2	π
Velocity vector	L/t	\mathbf{v}
Velocity, upper plate	L/t	V_0
Viscosity, steady shear [Eq. (18)]	M/Lt	η
Viscosity, steady shear, infinite shear rate [Eq. (23)]	M/Lt	$\eta_\infty \equiv \lim_{\dot{\gamma} \rightarrow \infty} \eta$
Viscosity, steady shear, zero shear rate	M/Lt	$\eta_0 \equiv \lim_{\dot{\gamma} \rightarrow 0} \eta$
Vorticity tensor [Eq. (16)]	t^{-1}	ω

Legend: $M \equiv$ mass; $L \equiv$ length; $t \equiv$ time; $T \equiv$ temperature

*Where τ_{ij} is the force exerted in the j th direction on a unit area of fluid surface of constant x_i by fluid in the region lesser x_i on fluid in the region greater x_i (see "Note on the Sign Convention for the Stress Tensor" on pp. 19–20 of [134], or pp. 24–25 of [135]).

Table IV: Dimensionless Variables and Groups

Bessel function of first kind, m th order, argument α	$J_m \equiv J_m(\alpha) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{(m+k)!k!} \left(\frac{\alpha}{2}\right)^{m+2k}$
Combined fluid parameter [Eq. (50)]	$\tilde{\lambda}$
Constant group [Eq. (65)]	ϕ
Constant group [Eq. (66)]	ψ
Constant group [Eq. (67)]	Ω
Constant in Eq. (50)	$\alpha \equiv (\text{Wi}/\text{De})\tilde{\lambda}$
Constant, i th, in Eq. (46)	C_i
Deborah number [Eq. (3)]	$\text{De} \equiv \lambda_1\omega$
Extra stress, zz -component [Eq. (43)]	$\tilde{\tau}_{zz} \equiv \tau_{zz}/\eta_0\dot{\gamma}^0$
Extra stress, zz -component, homogeneous part [Eq. (46)]	$\tilde{\tau}_{zz,h}$
Extra stress, zz -component, particular part [Eq. (51)]	$\tilde{\tau}_{zz,p}$
First normal stress difference [Eq. (44)]	$\mathbb{N}_1 \equiv N_1/\eta_0\dot{\gamma}^0$
First normal stress difference, corotational Maxwell contribution, n th harmonic [Eq. (73)]	$\mathbb{N}_1[ma]$
First normal stress difference, homogeneous part [Eq. (44)]	$\mathbb{N}_{1,h}$
First normal stress difference, particular part [Eq. (44)]	$\mathbb{N}_{1,p}$
First normal stress difference, small-amplitude oscillatory shear response [Eq. (63)]	$\mathbb{N}_1^{\text{SAOS}} \equiv \lim_{\text{Wi} \rightarrow 0} [\mathbb{N}_1/\text{Wi}]$
First normal stress difference, steady shear response	$\mathbb{N}_{1,ss}$
First normal stress difference, steady shear response, m th term [Eqs. (79), (83), (87), (91), (95), (99) and (103)]	$\mathbb{N}_1^{(m)}$
Fundamental matrix [Eq. (47)]	Φ
Integral-defined functions [Eqs. (54) or (55)]	I_1, I_2, \dots, I_7
Kronecker delta	δ
Oldroyd constant ratio [Eq. (27)]	$\sigma \equiv \sigma_2/\sigma_1$
Second normal stress difference [Eq. (45)]	$\mathbb{N}_2 \equiv N_2/\eta_0\dot{\gamma}^0$
Second normal stress difference, homogeneous part [Eq. (45)]	$\mathbb{N}_{2,h}$
Second normal stress difference, particular part [Eq. (45)]	$\mathbb{N}_{2,p}$

Second normal stress difference, small-amplitude oscillatory shear response [Eq. (64)]	$\mathbb{N}_2^{\text{SAOS}} \equiv \lim_{\text{Wi} \rightarrow 0} [\mathbb{N}_2 / \text{Wi}]$
Second normal stress difference, small-amplitude oscillatory shear response, m th term [Eqs. (105), (109), (113), (117), (121), (125) and (129)]	$\mathbb{N}_2^{(m)}$
Second normal stress difference, steady shear response	$\mathbb{N}_{2,ss}$
Shear strain amplitude	$\gamma_0 \equiv \dot{\gamma}^0 / \omega$
Shear stress [Eq. (40)]	$\mathbb{S} \equiv \tau_{yx} / \eta_0 \dot{\gamma}^0$
Shear stress, homogeneous part [Eq. (46)]	\mathbb{S}_h
Shear stress, particular part [Eq. (51)]	\mathbb{S}_p
Time	$\tau \equiv \omega t = \text{De}(t / \lambda_1)$
Trig function of trig function in Eq. (48)	$\mathbb{S} \equiv \sin(\alpha \sin \tau)$
Trig function of trig function in Eq. (49)	$\mathbb{C} \equiv \cos(\alpha \sin \tau)$
Weissenberg number [Eq. (4)]	$\text{Wi} \equiv \lambda_1 \dot{\gamma}^0$

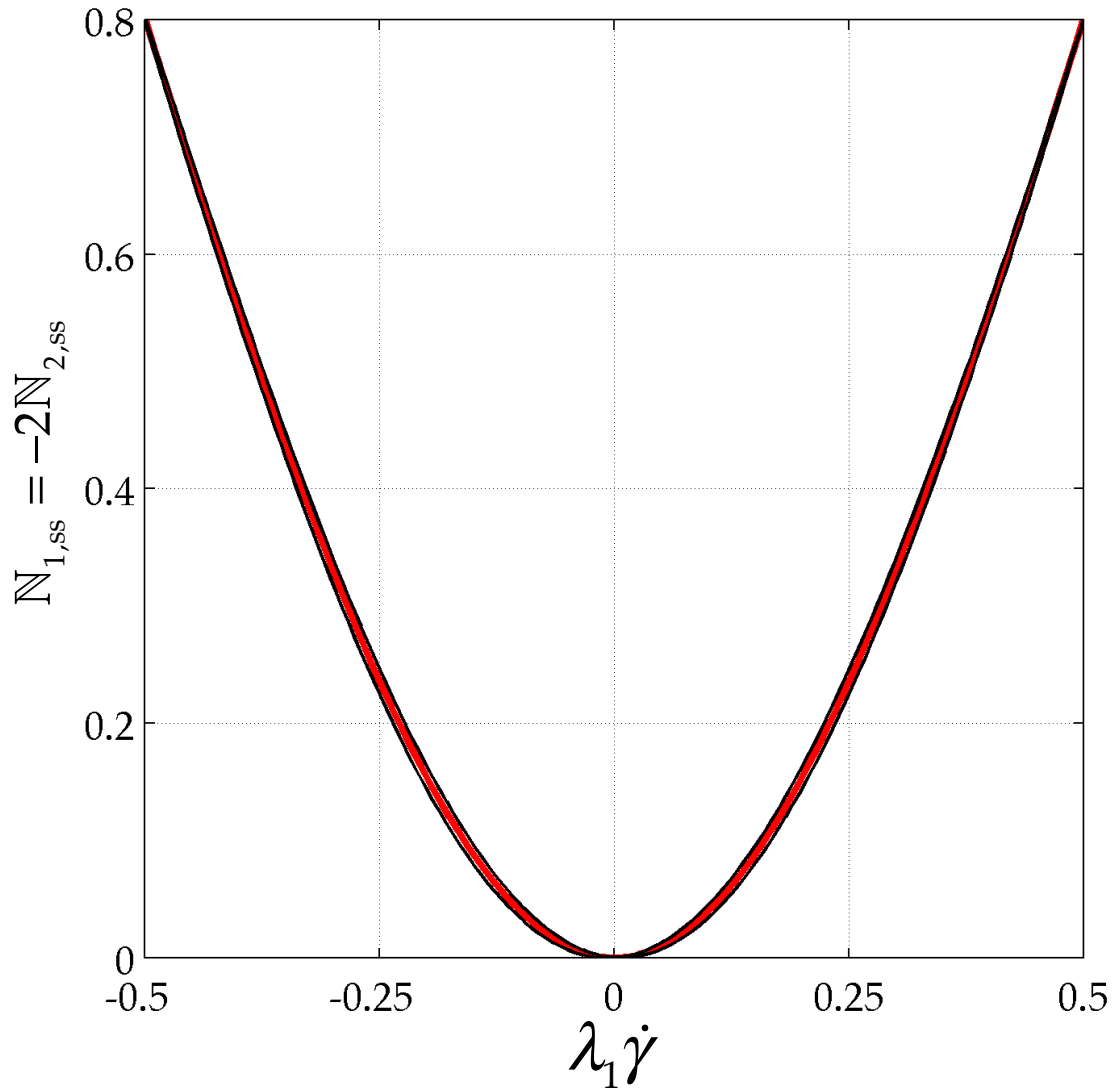


Figure 1: Close agreement between new exact solution for Oldroyd 8-constant framework (**black**) [Eqs. (54) and (55) [with Eqs. (56)–(62)] for the special case of the corotational Maxwell model ($\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for very low Deborah number [$Wi = \frac{1}{2}$, $De = \frac{1}{100}$] and the exact solution for steady shear flow (**red**) (Eq. (71); see also Figure 17 of [132]).

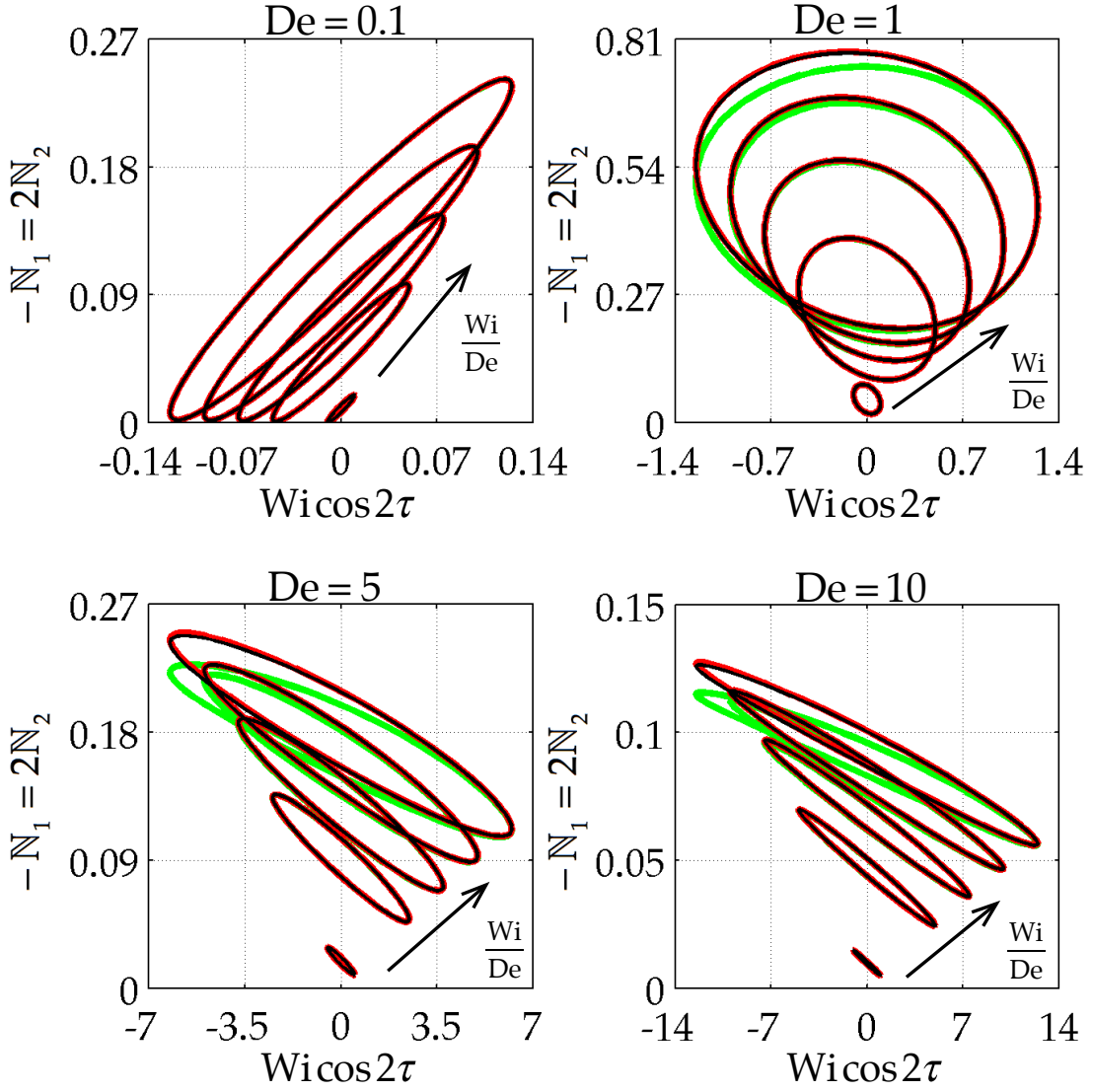


Figure 2: Counterclockwise frequency-matched loops of dimensionless minus the first, and plus twice the second, normal stress differences for the special case of the corotational Maxwell fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 0$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Our new exact solutions (Eqs. (54) and (55) [with Eqs. (56)–(62)], **black**) agree closely with the approximations [Eq. (75), **red**]. Prior approximations [Eq. (66) of [124], **green**] are improved importantly by including the next Wi^5 term [Eq. (75), **red**].

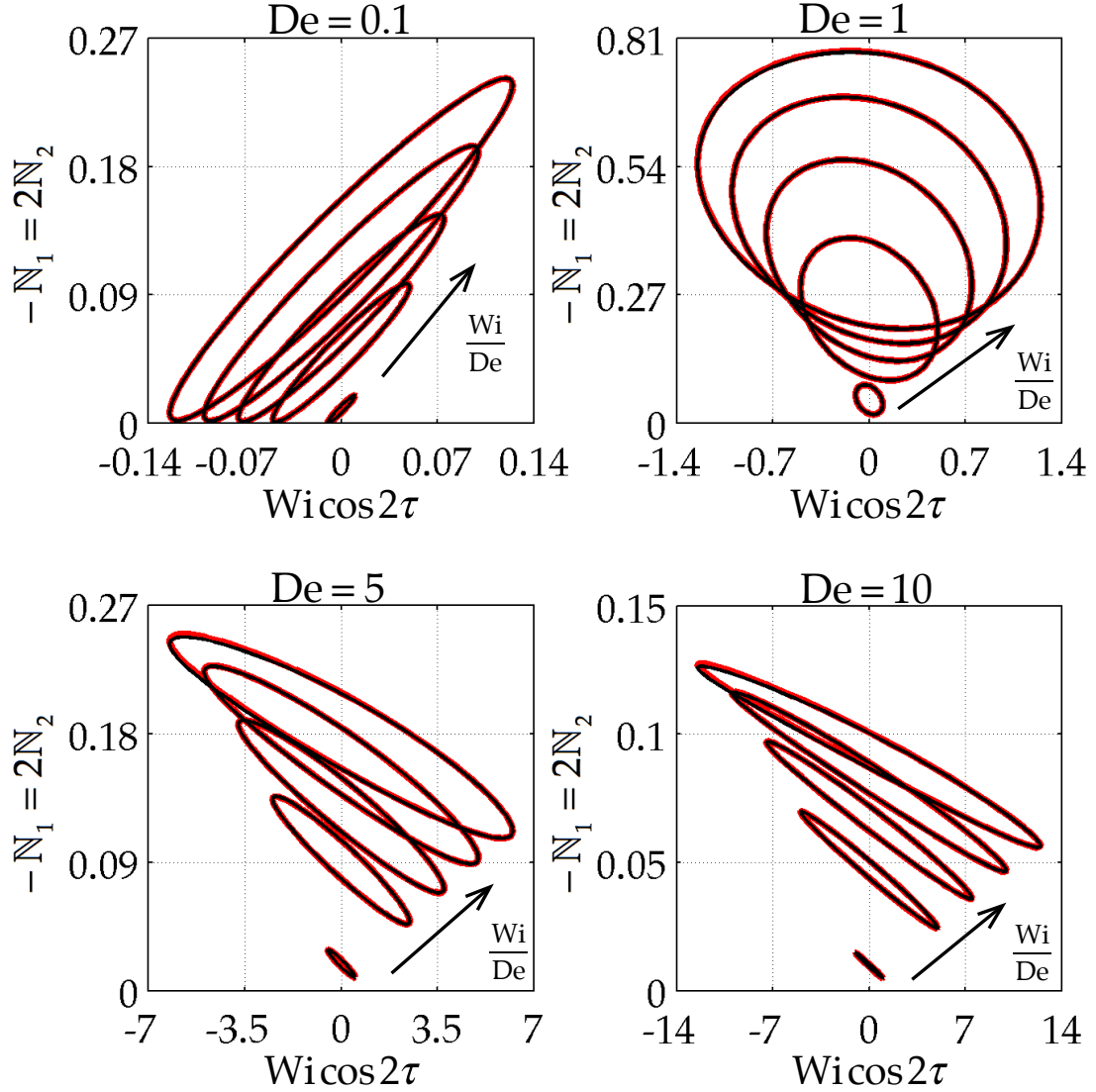


Figure 3: Counterclockwise frequency-matched loops of dimensionless minus the first, and plus twice the second, normal stress differences for the special case of the corotational Maxwell fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 0$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Our new exact solutions (Eqs. (54) and (55) [with Eqs. (56)–(62)], **black**) agree closely with the approximations [Eq. (75), **red**].

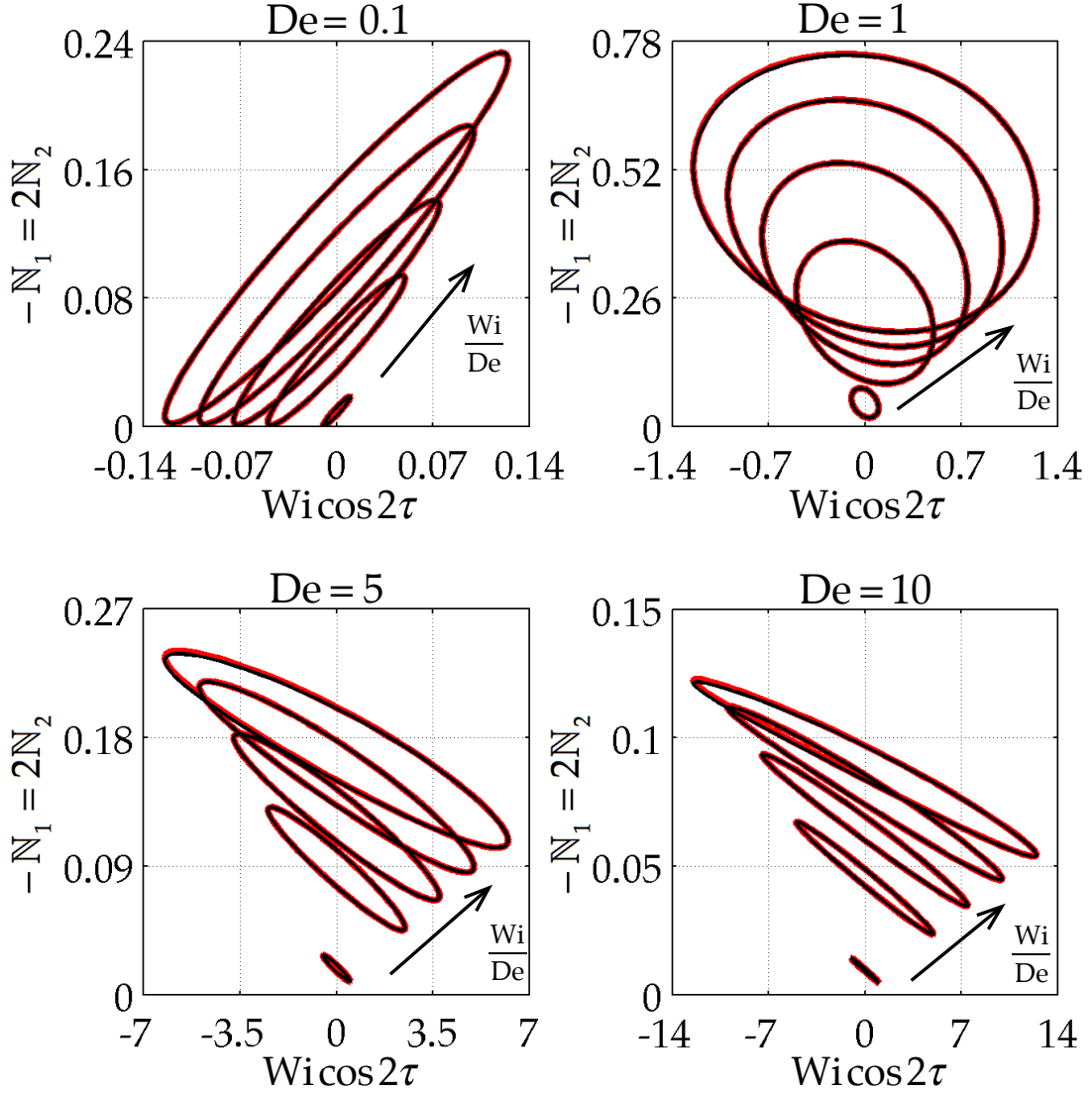


Figure 4: Counterclockwise frequency-matched loops of dimensionless minus the first, and plus twice the second, normal stress differences for the special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/27$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Our new exact solutions (Eqs. (54) and (55) [with Eqs. (56)–(62)], **black**) agree closely with the approximations [Eq. (75), **red**].

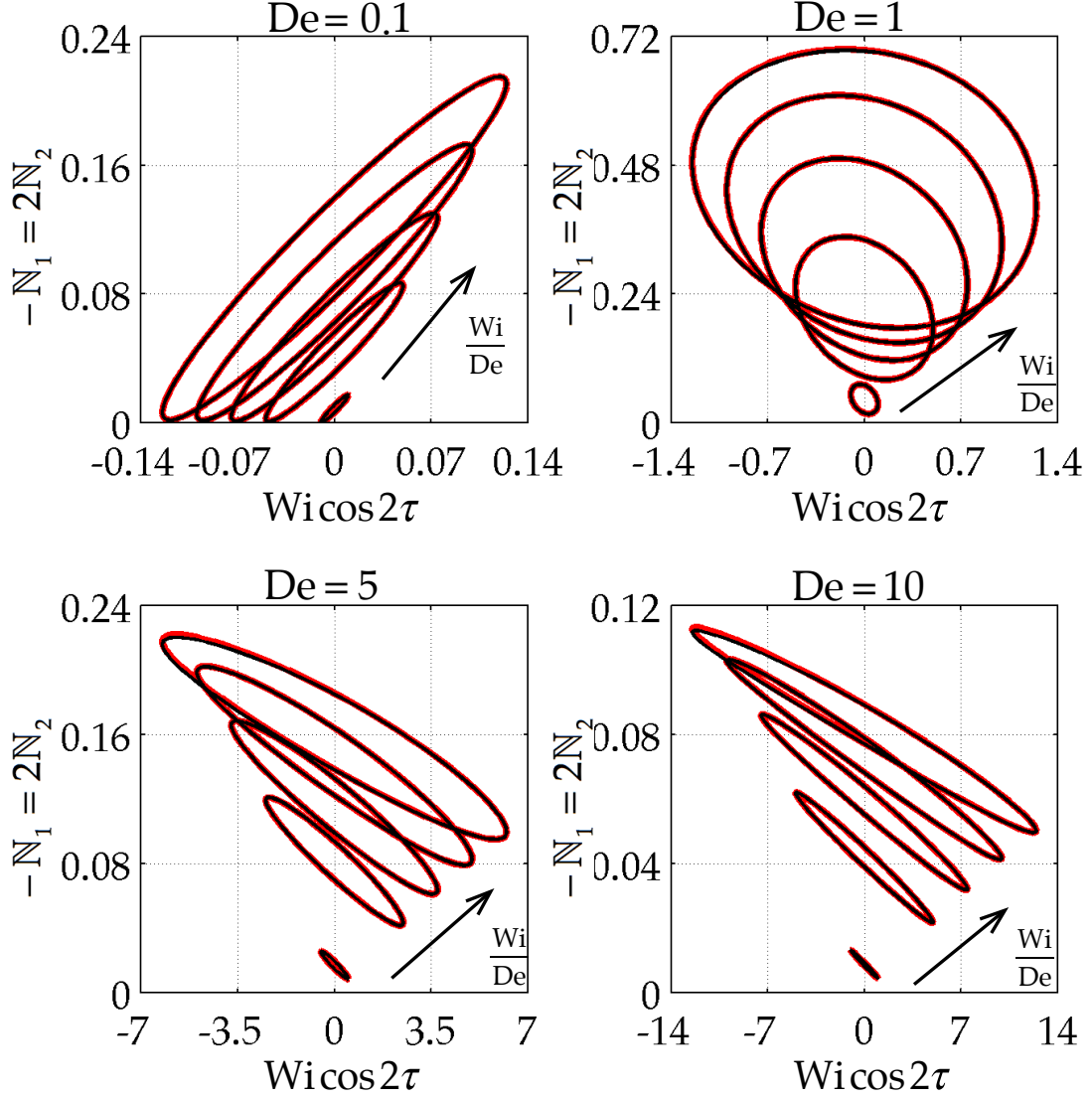


Figure 5: Counterclockwise frequency-matched loops of dimensionless minus the first, and plus twice the second, normal stress differences for the special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/9$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Our new exact solutions (Eqs. (54) and (55) [with Eqs. (56)–(62)], **black**) agree closely with the approximations [Eq. (75), **red**].

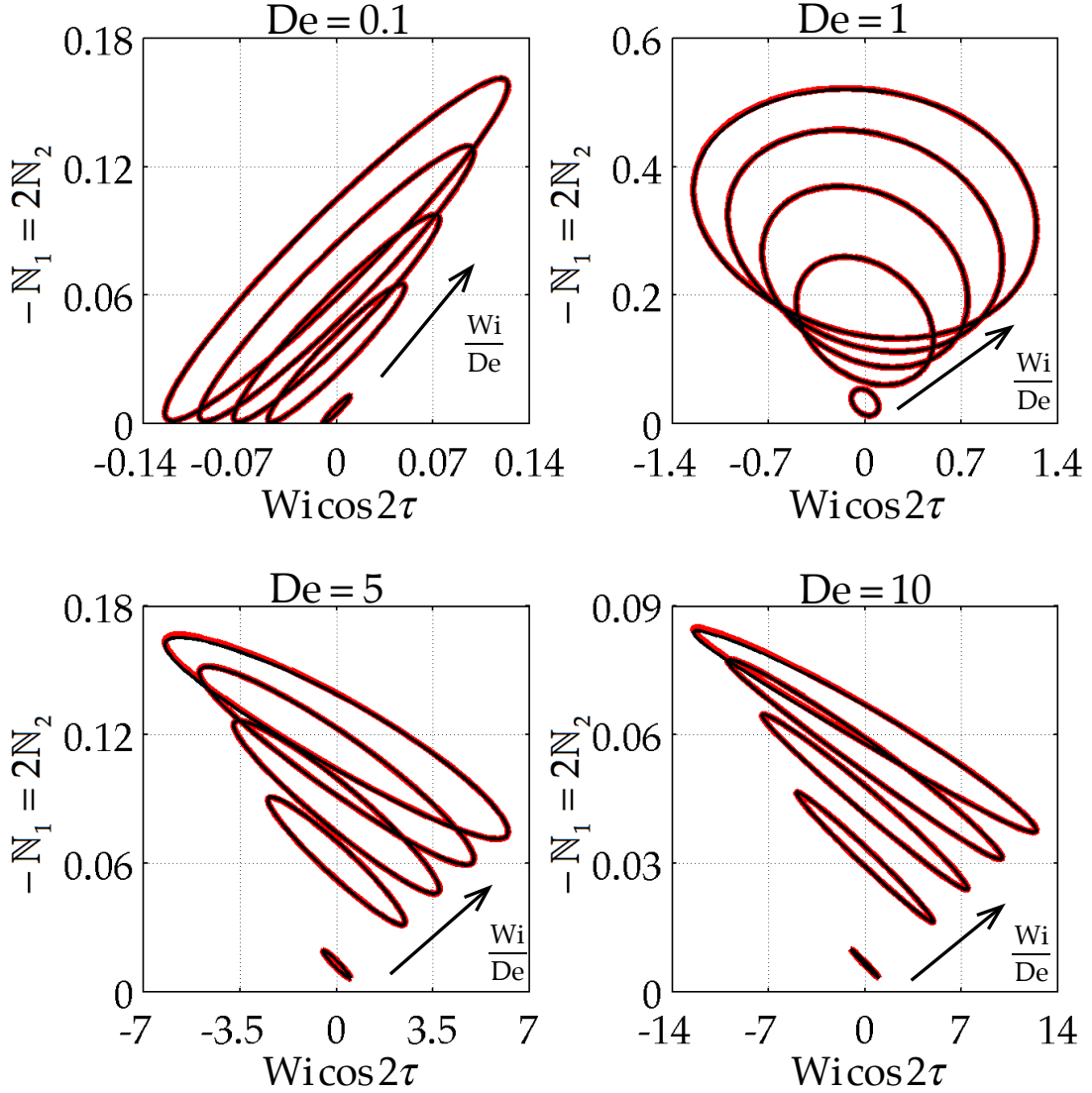


Figure 6: Counterclockwise frequency-matched loops of dimensionless minus the first, and plus twice the second, normal stress differences for the special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/3$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Our new exact solutions (Eqs. (54) and (55) [with Eqs. (56)–(62)], **black**) agree closely with the approximations [Eq. (75), **red**].

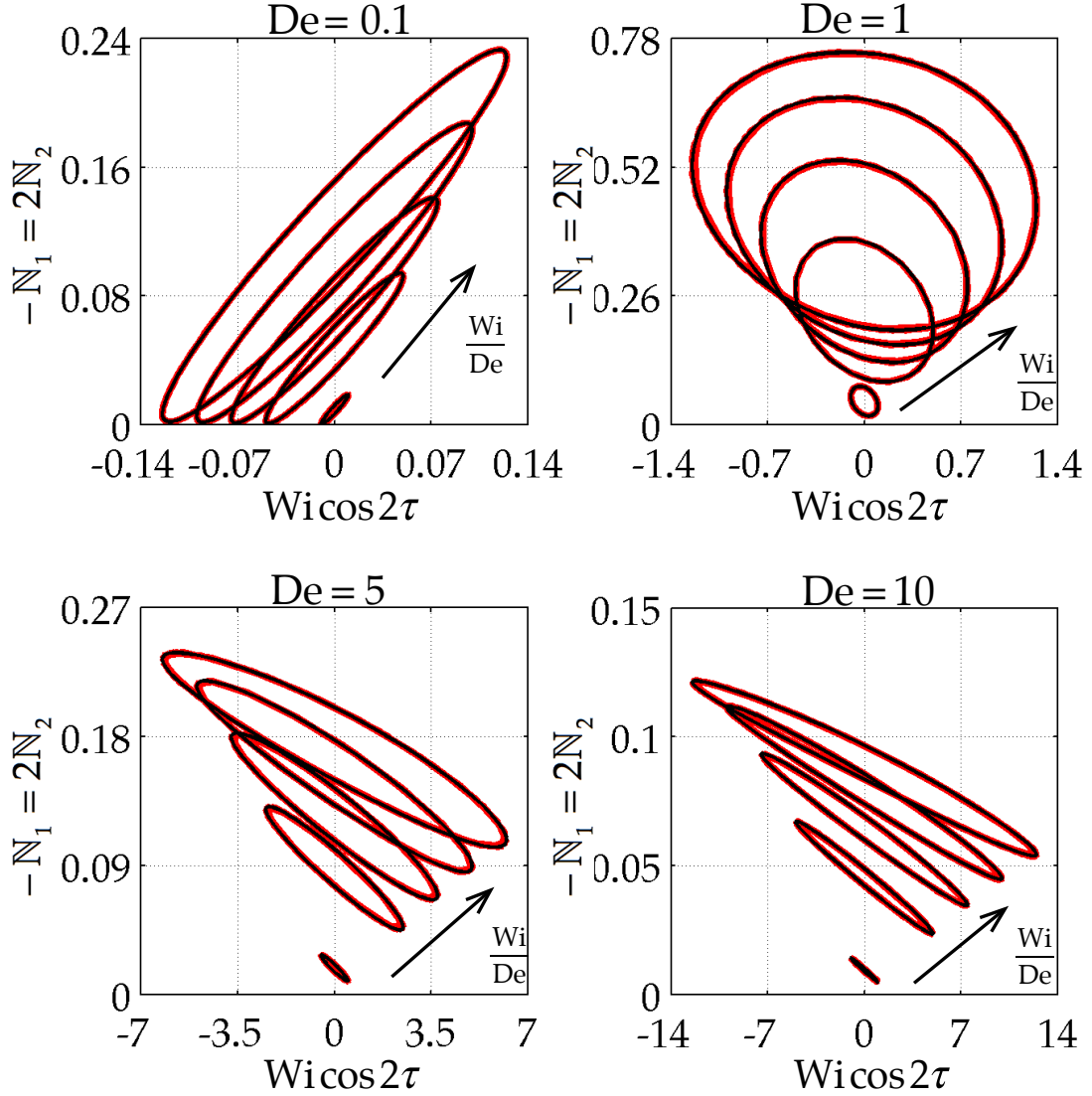


Figure 7: Counterclockwise frequency-matched loops of dimensionless minus the first, and plus twice the second, normal stress differences for the special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/27$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Our new exact solutions (Eqs. (54) and (55) [with Eqs. (56)–(62)], **black**) agree closely with the finite difference approximation [solving Eqs. (40)–(43) with Eq. (76), **red**].

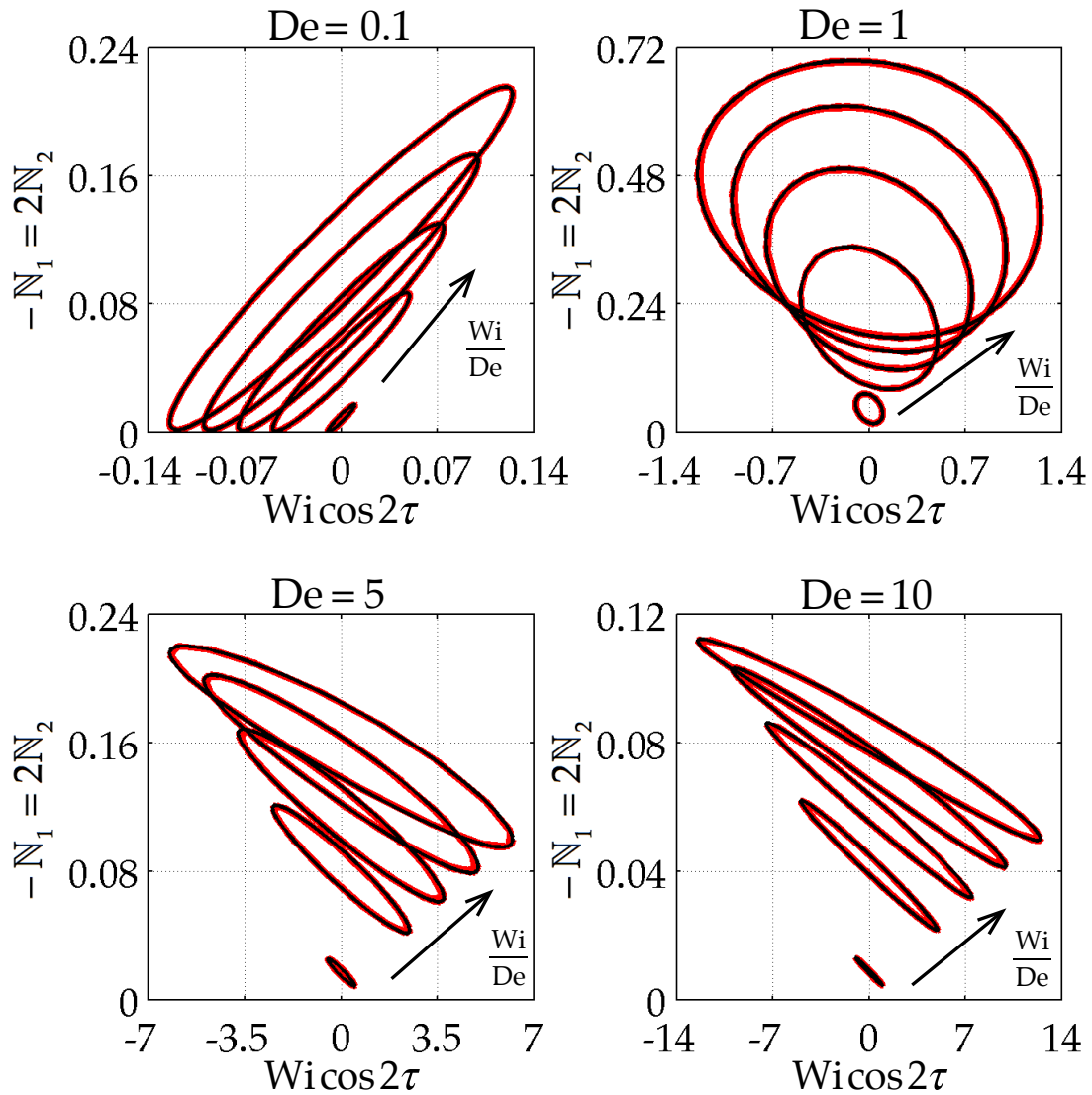


Figure 8: Counterclockwise frequency-matched loops of dimensionless minus the first, and plus twice the second, normal stress differences for the special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/9$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Our new exact solutions (Eqs. (54) and (55) [with Eqs. (56)–(62)], **black**) agree closely with the finite difference approximation [solving Eqs. (40)–(43) with Eq. (76), **red**].

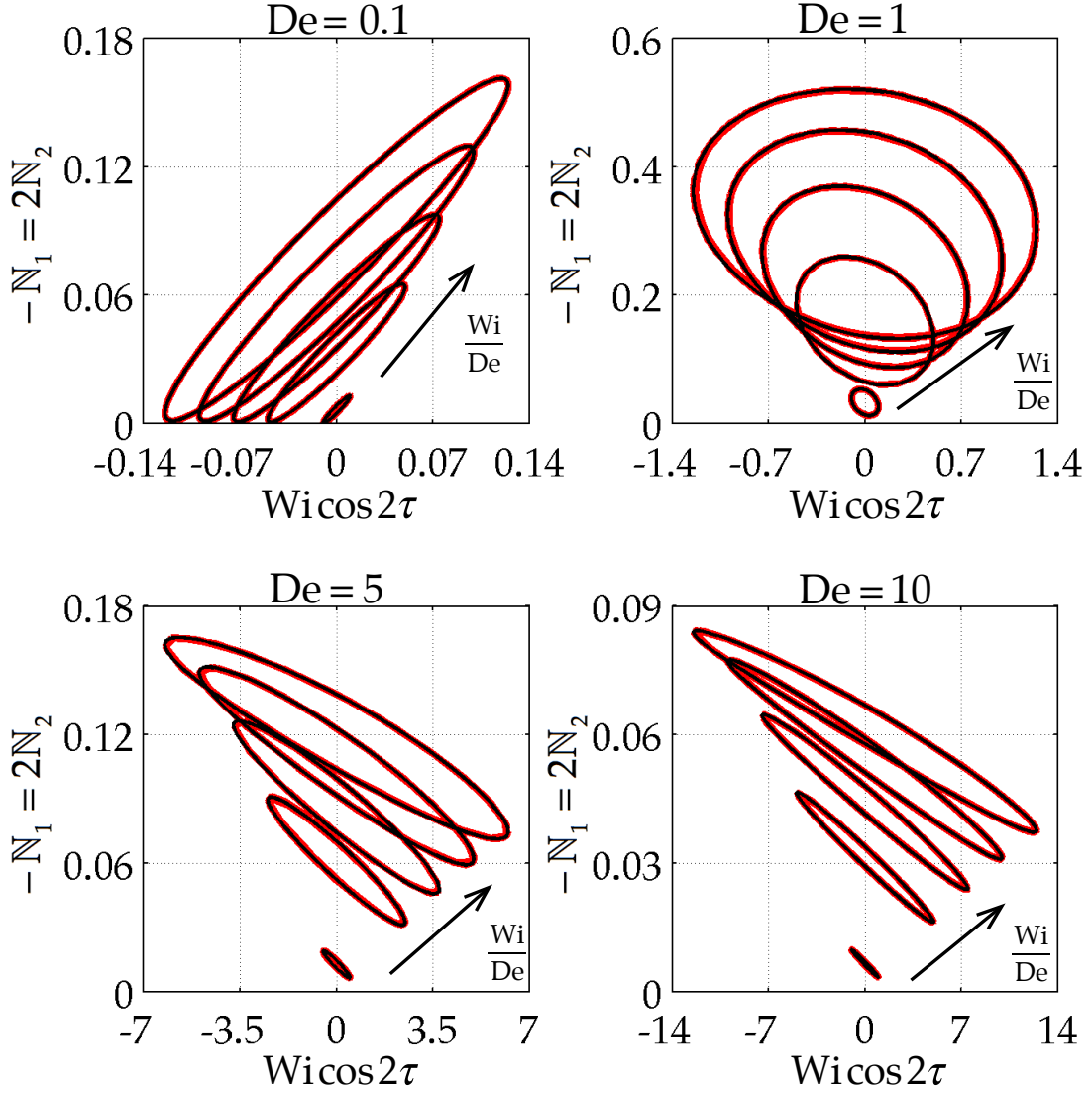


Figure 9: Counterclockwise frequency-matched loops of dimensionless minus the first, and plus twice the second, normal stress differences for the special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/3$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Our new exact solutions (Eqs. (54) and (55) [with Eqs. (56)–(62)], **black**) agree closely with the finite difference approximation [solving Eqs. (40)–(43) with Eq. (76), **red**].

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$s^2 = \alpha^2 \{1 + \cos 2\omega\tau \cos \omega\tau + \sin 2\omega t \sin \omega\tau\} (1 - \cos \omega\tau)$; Eq. (6.41a) should be
 $p_{11} - p_{22} = \alpha^2 \{A + B \cos 2\omega t + C \sin 2\omega t\}$; Eq. (6.41b) should be
 $p_{21} = \alpha \{D \cos \omega t + A \sin \omega t\}$; in line 4 of p. 113, $\alpha A \cos \omega t$ should be $\alpha D \cos \omega t$; in
the sentence preceding Eq. (6.43), and also in Eq. (6.43), “the out-of-phase part
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